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Surjective $L(3,1)$-labeling of circular-arc graphs

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Abstract
One of the important topics in the field of graph theory is graph labeling. $L(3,1)$-labeling problem has been widely studied in the last four decades due to its wide applications, specially in frequency assignment in mobile communication system, circuit design, radar, X-ray crystallography, coding theory, etc. Distance two surjective labeling is the surjective labeling satisfying the condition at distance one and two. Surjective $L(3,1)$-labeling is a distance two surjective labeling, which is now becomes a well studied problem. Motivated from the $L(3,1)$-labeling problem and the importance of surjective $L(3,1)$-labeling problem, we consider surjective $L(3,1)$-labeling problem for circular-arc graph (CAG), where CAG is a very important subclass of intersection graph. A surjective $L(3,1)$-labeling of a graph $G=(V,E)$ is a function ϑ from the node set $V(G)$ to the set of positive integers such that $|\vartheta(u)-\vartheta(v)|\geq 3$ if $d(u,v)=1$ and $|\vartheta(u)-\vartheta(v)|\geq 1$ if $d(u,v)=2$, and every label $1,2,\ldots,n$ is used exactly once, where $d(u,v)$ represents the distance between the nodes u and v and n is the number of nodes of the graph G. If a graph G with n nodes be label by surjective $L(3,1)$-labeling then the largest label used is obviously equal to n.
In this article, it is shown that for any CAG G with n nodes and degree $\Delta>2$ can be surjectively labeled using $L(3,1)$-labeling if $n=5\Delta-2$. Also, efficient algorithms are designed to label a CAG by surjective $L(3,1)$-labeling. This is the first result for the problems on $CAGs$.

Keywords
Frequency assignment, surjective $L(3,1)$-labeling, circular-arc graph.

AMS Subject Classification
05C85, 68R10.

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Contents

1. Introduction

One of the key topics in the field of graph theory is graph labeling (or coloring). Graph labeling are motivated by problems like channel assignment in wireless communications, traffic phasing, task assignment, fleet maintenance, etc. The channel assignment problem is that of assigning a frequency to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not a set of disallowed separations. This problem was formulated as a vertex coloring problem of graph by Hale [5]. In 1988, Roberts proposed a variation of the frequency assignment problem in which 'close' transmitters must receive different channel and 'very close' transmitters must receive channel at least two apart. To convert this problem into graph theory, the transmitters are represented by the nodes of a graph; two nodes u and v are 'very close' if the distance between them is 1 and 'close' if the distance between u and v is 2. The notion of $L(2,1)$-labeling was introduced by Griggs and Yeh [4] in connection with the problem of assigning frequencies in a multihop radio network.

$L(3,1)$-labeling of a graph $G=(V,E)$ is a mapping f from the vertex set V to the set of positive integers such that $|\vartheta(u)-\vartheta(v)|\geq 3$ if $d(u,v)=1$ and $|\vartheta(u)-\vartheta(v)|\geq 1$ if $d(u,v)=2$, where $d(u,v)$ represent the distance between the nodes u and v. The minimum span over all possible labeling functions of $L(3,1)$-labeling is denoted by $\lambda_{3,1}(G)$ and is called $\lambda_{3,1}$ number of G. Surjective $L(3,1)$-labeling is an extension of $L(3,1)$-labeling of graphs.

A surjective $L(3,1)$-labeling of a graph $G = (V,E)$ is a mapping f from the node set V to the set of positive integers such that $|\vartheta(u) - \vartheta(v)| \geq 3$ if $d(u,v) = 1$ and $|\vartheta(u) - \vartheta(v)| \geq 1$ if $d(u,v) = 2$, and every label $1, 2, \ldots, n$ is used exactly once. The largest label used in surjective $L(3,1)$-labeling is exactly equal to n, the number of nodes of G.

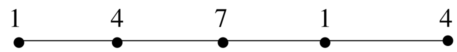

Figure 1. $L(3,1)$-labeling of a path with five nodes

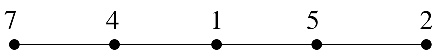

Figure 2. Surjective $L(3,1)$-labeling of a path with five nodes

In Figure 1, we have shown an $L(3,1)$-labeling of a path with 5 nodes and in Figure 2, surjective $L(3,1)$-labeling of a path with 5 nodes. Note that in figure 1 the same label used more than once but in figure 2 the labels 1 through 7 have all been used exactly once. So, in surjective $L(3,1)$-labeling there is a complicated task than $L(3,1)$-labeling.

Frequency assignment problem has been widely studied in the past [3–11, 13–18]. An advanced label research is going on [1, 2, 19, 20]. We focus our attention on surjective $L(3,1)$-labeling of *CAGs*.

$L(2,1)$-labeling problem has been widely studied in the last three decades due to its wide application, specially in frequency assignment in (mobile) communication system, X-ray crystallography, radar, coding theory, astronomy, circuit design, etc. Surjective $L(3,1)$-labeling is an extension of $L(3,1)$-labeling which is now becomes a well studied problem. Motivated from the $L(3,1)$-labeling problem and the importance of surjective $L(3,1)$-labeling problem, we consider surjective $L(3,1)$-labeling problem for *CAG*, where *CAG* is a very important subclass of intersection graph.

In this paper, for any *CAG* G, we have shown $\lambda_{3,1}(G) \leq 5\Delta - 2$ and also proved that a *CAG* can be surjectively labeled using $L(3,1)$-labeling if $n = 5\Delta - 2$, where n is the number of nodes and $\Delta > 2$ is the maximum degree of the graph G. Also an efficient algorithm is designed to label a *CAG* by surjective $L(3,1)$-labeling.

The remaining part of the paper is organized as follows. Some notations and definitions are presented in Section 2. In Section 3, some lemmas related to our work and an algorithm to surjective $L(3,1)$-label a *CAGs* are presented. In Section 4 conclusions are made.

2. Preliminaries and notations

The graphs used in this work are simple, finite with degree $\Delta > 2$. The class of *CAGs* is a very important subclass of intersection graphs [7]. A graph is a *CAG* if there exists a family A of arcs around a circle and a one-to-one correspondence between nodes of G and arcs A, such that two distinct nodes are adjacent in G if and only if there corresponding arcs intersect in A. Such a family of arcs is called an arc representation for G. A *CAG* and its corresponding circular-arc representation is shown in Figure 3.

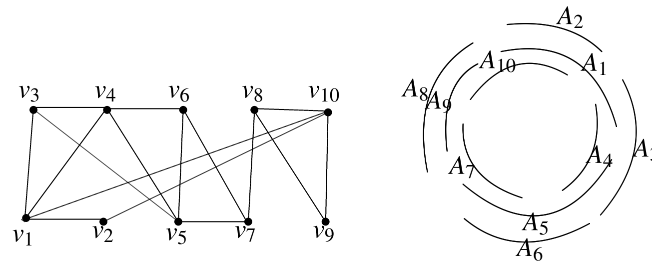

Figure 3. A circular arc graph and its corresponding circular-arc representation

It is assumed that all arcs must cover the circle, otherwise the *CAG* is nothing but an interval graph. The degree of the node v_k corresponding to the arc A_k is denoted by $d(v_k)$ and is defined by the maximum number of arcs which are adjacent to A_k. The maximum degree or the degree of a *CAG* G, denoted by $\Delta(G)$ or by Δ, is the maximum degree of all nodes corresponding to the arcs of G. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of arcs around a circle. Also, it is noted that an arc A_k of A and a node v_k of V are one and same thing.

For any *CAG* G with n nodes and corresponding arc set $A = \{A_1, A_2, \ldots, A_n\}$, we define the following objects:

1. $L^s(A_k)$: the set of used surjective $L(3,1)$-labels which are used before labeling the arc A_k, for any arc $A_k \in A$.

2. $L^i(A_k)$: the set of used surjective $L(3,1)$-labels at distance i $(i = 1,2)$ from the arc A_k, before labeling the arc A_k, for any arc $A_k \in A$.

3. $L^{vl}(1, A_k)$: the set of all valid labels to label the arc A_k before labeling A_k, satisfying the adjacency condition of $L(3,1)$-labeling, for any arc $A_k \in A$.

4. $L^{vl}(2, A_k)$: the set of all valid labels to label the arc A_k before labeling A_k, satisfying $L(3,1)$-labeling condition, for any arc $A_k \in A$.

5. $L^{svl}(A_k)$: the set of valid surjective $L(3,1)$-labels to label the arc A_k before labeling A_k, for any arc $A_k \in A$.

6. $\lambda^s_{3,1}(G)$: surjective $L(3,1)$-labeling number for the graph G.

7. f^s_j: the surjective $L(3,1)$-label of the arc A_j, for any arc $A_j \in A$.

8. L^s: the label set after completion of surjective $L(3,1)$-labeling of the graph G.

3. Surjective L(3,1)-labeling of circular-arc graphs

In this section, we present some lemmas related to the proposed problem. Also, an algorithm is designed for surjective $L(3,1)$-labeling of *CAGs*. The time complexity of the algorithm is also calculated.

Lemma 3.1. *For any CAG G, $|L^2(A_k)| \leq 2\Delta - 2$, for any arc $A_k \in A$.*

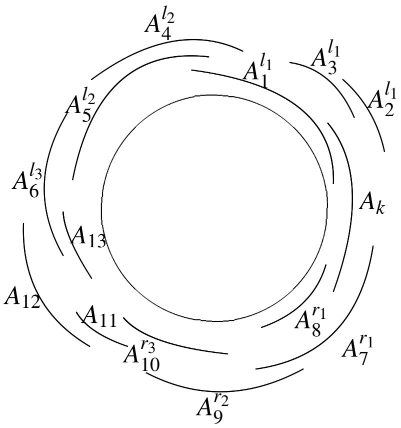

Figure 4. A *CAG*

Proof. Let G be a *CAG* and A_k be any arc of G. We consider a situation in which some arcs are already labeled and some arcs are unlabel. A_k is an unlabel arc of G and we want to label the arc A_k by surjective $L(2,1)$-labeling. Then two cases may arises.

Case 1: Let the arcs in the left side of A_k (i.e., $A_1^{l_1}, A_2^{l_1}, A_3^{l_1}, A_4^{l_2}, A_5^{l_2}, A_6^{l_3}$) are only labeled and the arcs in the right hand side of A_k (i.e., $A_7^{r_1}, A_8^{r_1}, A_9^{r_2}, A_{10}^{r_3}$) are not labeled. Since Δ is the degree of the graph, so, A_k is adjacent to at most Δ arcs. In figure 4, among all labeled arcs, A_k is adjacent to $A_1^{l_1}, A_2^{l_1}, A_3^{l_1}$. Again, among the labeled adjacent arcs of A_k, $A_1^{l_1}$ is the arc of maximum length. Since, G is a *CAG* so every 2-neighborhood arcs of A_k must be adjacent to $A_1^{l_1}$. Since Δ is the degree of the graph, so, except A_k, $A_1^{l_1}$ is adjacent to at most $\Delta - 1$ arcs of G. Hence, the cardinality of the 2-neighborhood arcs of A_k is at most $\Delta - 1$. Since A_k is arbitrary so, $|L^2(A_k)| \leq \Delta - 1$.

Case 2: Let some arcs in the left hand side of A_k (i.e., $A_1^{l_1}, A_2^{l_1}, A_3^{l_1}, A_4^{l_2}, A_5^{l_2}, A_6^{l_3}$) as well as right hand side of A_k (i.e., $A_7^{r_1}, A_8^{r_1}, A_9^{r_2}, A_{10}^{r_3}$) are labeled. Since, the degree of the graph is Δ, so, A_k is adjacent to at most Δ arcs. In Figure 4, among all arcs which are already labeled, A_k is adjacent to $A_1^{l_1}, A_2^{l_1}, A_3^{l_1}, A_7^{r_1}, A_8^{r_1}$. Again, among the arcs which are already labeled and also adjacent to the arc A_k, the largest length of all the left sided adjacent arcs of A_k is $A_1^{l_1}$ and that of right sided adjacent arcs of A_k is $A_7^{r_1}$. Since, G is a *CAG* so, every 2-neighborhood arcs of A_k must be adjacent to either $A_1^{l_1}$ or $A_7^{r_1}$ or both. Since, Δ is the degree of the graph, so, except A_k, $A_1^{l_1}$ is adjacent to at most $\Delta - 1$ arcs of G. Similarly, except A_k, $A_7^{r_1}$ is adjacent to at most $\Delta - 1$ arcs of G. Since A_k is arbitrary so, $|L^2(A_k)| \leq (\Delta - 1) + (\Delta - 1) = 2\Delta - 2$. i.e. $|L^2(A_k)| \leq 2\Delta - 2$. Combining all the cases we have $|L^2(A_k)| \leq 2\Delta - 2$. □

Observation 1: For any *CAG G*, $L^i(A_k) \subseteq L^s(A_k)$, for any arc A_k of G and $i = 1, 2$.

Theorem 3.2. *For any CAG G, $\lambda_{3,1}(G) \leq 5\Delta - 2$ and it can be surjectively $L(3,1)$-labeled if $n = 5\Delta - 2$, where Δ is the maximum degree of the graph and n is the number of nodes of G.*

Proof. Since the number of nodes of the graph G is n, so the number of arcs in the circular-arc representation of the graph is n. So, let $A = \{A_1, A_2, \ldots, A_n\}$. Since we label the arcs of a *CAG* by surjective $L(3,1)$-labeling, so no label is used more than once and also the labels are taken only from the set $\{1, 2, \ldots, n\}$. Therefore,

$$\begin{aligned} \lambda_{3,1}(G) &\leq 3|L^1(A_k)| + |L^2(A_k)| \\ &\leq 3\Delta + (2\Delta - 2), [byLemma1] \\ &\leq 5\Delta - 2 \end{aligned}$$

Again, since n is the number of nodes of the *CAG G*, so, for surjective $L(3,1)$-labeling n distinct labels must be required to label the whole graph. Also, $\lambda_{3,1}(G) \leq 5\Delta - 2$. So at most $5\Delta - 2$ labels must be required to label the graph G. Again, in surjective $L(3,1)$-labeling the largest label must be less than or equal to n, the number of nodes of the graph G. Hence, a *CAG G* can be surjectively label using $L(3,1)$-labeling if $n = 5\Delta - 2$. □

3.1 Algorithm for surjective L(3,1)-labeling

In this section, we design an algorithm to compute the set $L^{vl}(k, A_j)$ for $j = 2, 3, \ldots, n$; $k = 1, 2$ and also we design an algorithm to surjective $L(3,1)$-label a *CAG*. We consider a situation in which some arcs (the arcs A_k's with index $k < j$) are labeled by surjective $L(3,1)$-labeling and some arcs (the arcs A_k's with index $k \geq j$) are not labeled.

Algorithm *SLKVL*
Input: A_k, $k = 2, 3, \ldots, n$.
Output: $L^{vl}(p, A_k)$ for $p = 1, 2$; $k = 2, 3, \ldots, n$.

Step 1: Compute $L^1(A_k), L^2(A_k)$ and $L^s(A_k)$
  for $i = 1$ to r, where $r = max\{L^s(A_k)\} + 3$
    for $j = 1$ to $|L^1(A_k)|$
      let l_j be the jth element of $L^1(A_k)$
      if $|i - l_j| \geq 2$, then $L^{vl}(1, A_k) = \{i\}$
    end for;
  end for;
Step 2: for $m = 1$ to $|L^{vl}(1, A_k)|$
    for $n = 1$ to $|L^2(A_k)|$

if $|l_m - p_n| \geq 1$, then $L^{vl}(2,A_k) = \{l_m\}$
//where $l_m \in L^{vl}(1,A_k)$, $p_n \in L^2(A_k)$//
end for;
end for;
end *SLKVL*.

Lemma 3.3. *Algorithm SLKVL correctly compute $L^{vl}(p,A_k)$ for $p = 1,2$ and the running time for this algorithm is $O(\Delta^2)$.*

Proof. According to the algorithm *SLKVL* each element $i \in L^{vl}(1,A_j)$ is differ from l_r by at least 2 for each $l_r \in L^1(A_k)$. Therefore, $|i - l_r| \geq 2$ for all $i \in L^{vl}(1,A_k)$ and for all $l_r \in L^1(A_k)$. So, $L^{vl}(1,A_k)$ is correctly computed by the algorithm *SLKVL* for $A_k \in A$, $k = 2,3,\ldots,n$. Again, according to the above algorithm every element l_α of $L^{vl}(2,A_k)$ is differ from l_β by at least 1 for each $l_\beta \in L^2(A_k)$. Therefore, $|l_m - p_n| \geq 2$ for all $l_m \in L^{vl}(2,A_k)$ and for all $p_n \in L^1(A_k)$, and $|l_m - p_n| \geq 1$ for all $l_m \in L^{vl}(2,A_k)$ and for all $p_n \in L^2(A_k)$. Hence, $L^{vl}(p,A_k)$ is correctly computed by the algorithm *SLKVL* for each $k = 1,2$.

Since, L^s is the label set and $|L^s|$ be its cardinality, clearly $|L^i(A_k)| \leq |L^s|$ for $i = 1,2$ and for any $A_k \in A$, and also $r \leq 5\Delta$, where $r = max\{L^s(A_k)\} + 3$. So, $L^{vl}(1,A_k)$ is computed by using at most $5\Delta|L^s|$ times, i.e. using $O(\Delta|L^s|)$ times. Again, $|L^{vl}(2,A_n)| \leq 5\Delta$, so, $L^{vl}(2,A_k)$ is computed using at most $5\Delta|L^s|$ times, i.e. using $O(\Delta|L^s|)$ times. Since, $|L^s| \leq 5\Delta$ so, the over all running time for the algorithm *SLKVL* is $O(\Delta^2)$. □

Lemma 3.4. *For any CAG G, $L^{vl}(1,A_k)$ is the largest non empty set of labels satisfying the adjacency condition of $L(2,1)$-labeling, where $l \leq r$ for all $l \in L^{vl}(1,A_k)$, $r = max\{L^s(A_k)\} + 3$, for any $A_k \in A$.*

Proof. Since, $r = max\{L^s(A_k)\} + 3$ and $L^1(A_k) \subseteq L^s(A_k)$ (by Observation 3.1), so $|r - l_i| \geq 3$ for any $l_i \in L^1(A_k)$. Therefore, $r \in L^{vl}(1,A_k)$, so, $L^{vl}(1,A_k)$ is a non empty set. Again, let B be any set of labels satisfying the adjacency condition of $L(3,1)$-labeling, where $l \leq r$ for all $l \in B$, $r = max\{L^s(A_k)\} + 3$. Also, let $b \in B$. Then $|b - l_i| \geq 2$ for any $l_i \in L^1(A_k)$. Thus, $b \in L^{vl}(1,A_k)$. So, $b \in B$ implies $b \in L^{vl}(1,A_k)$. So, $B \subseteq L^{vl}(1,A_k)$. Since B is arbitrary, so, $L^{vl}(1,A_k)$ is the largest non empty set of labels satisfying the adjacency condition of $L(3,1)$-labeling such that $l \leq r$ for all $l \in L^{vl}(1,A_k)$, $r = max\{L^s(A_k)\} + 3$, for any $A_k \in A$. □

Lemma 3.5. *For any CAG G, $L^{vl}(2,A_k)$ is the largest non empty set satisfying $L(2,1)$-labeling condition, where $l \leq r$ for all $l \in L^{vl}(1,A_k)$, $r = max\{L^s(A_k)\} + 3$, for any $A_k \in A$.*

Proof. Since, $r = max\{L^s(A_k)\} + 3$ and $L^i(A_k) \subseteq L^s(A_k)$, for $i = 1,2$ (by Observation 3.1), so $|r - l_p| \geq 3$ for any $l_p \in L^i(A_k)$, $i = 1,2$, i.e., $|r - l_p| \geq 3$ for all $l_p \in L^1(A_k)$ and $|r - l_p| \geq 1$ for all $l_p \in L^2(A_k)$. Therefore, r is the valid $L(3,1)$-label of A_k, so, $r \in L^{vl}(2,A_k)$. This implies that $L^{vl}(2,A_k)$ is a non empty set. Again, let B be any set of labels satisfying $L(3,1)$-labeling conditions, where $l \leq r$ for all $l \in B$, $r = max\{L^s(A_k)\} + 3$. Also, let $b \in B$. Then $|b - l_p| \geq 3$ for any $l_p \in L^1(A_k)$ and $|b - l_q| \geq 1$ for any $l_q \in L^2(A_k)$. Thus, $b \in L^{vl}(2,A_k)$. Therefore, $b \in B$ implies $b \in L^{vl}(2,A_K)$. So, $B \subseteq L^{vl}(2,A_k)$. Since B is arbitrary, $L^{vl}(2,A_k)$ is the largest non empty set of labels satisfying $L(3,1)$-labeling, $l \leq r$ for all $l \in L^{vl}(2,A_k)$, $r = max\{L^s(A_k)\} + 3$, for any $A_k \in A$. □

Algorithm SL31

Input: The set of arcs of a *CAG* $A = \{A_1, A_2, \ldots, A_n\}$ and $L^{vl}(p,A_k)$ for $k = 2,3,\ldots,n$ and $p = 1,2$ where $n = 5\Delta - 2$.
Output: f_k^s, the surjective $L(3,1)$-label of A_k, $k = 1,2,\ldots,n$.

Step 1: Rearrange the arcs as follows:
$A_n = A_2$;
$A_{i+1} = A_{2i+1}$, for $i = 1,2,\ldots,n/2-1$;
$A_{i+n/2-1} = A_{2i}$, for $i = 2,3,\ldots,n/2$;
A_1 remains the same;
Step 2: (Initialization)
$f_1^s = 1$;
$L^s(A_2) = \{1\}$;
Step 3: for $k = 2$ to $n-1$
$L^{svl}(A_k) = L^{vl}(2,A_k) - L^s(A_k)$;
$f_k^s = min\{L^{svl}(A_k)\}$;
$L^s(A_{k+1}) = L^s(A_k) \bigcup \{f_k^s\}$;
end for;
Step 4: $L^{svl}(A_n) = L^{vl}(2,A_n) - L^s(A_n)$;
$f_n^s = min\{L^{svl}(A_n)\}$;
Step 5: $L^s = L^s(A_n) \bigcup \{f_n^s\}$;
end *SL*31.

Theorem 3.6. *The Algorithm SL31 correctly labels a CAG by surjective $L(3,1)$-labeling, where $n = 5\Delta - 2$.*

Proof. Let G be a *CAG* with n nodes such that $n = 5\Delta - 2$, where Δ is the maximum degree of the graph. After rearrangement of the arcs let $A = \{A_1, A_2, \ldots, A_n\}$ be the set of arcs of the *CAG* and let $f_1^s = 1$, $L^s(A_2) = \{1\}$.

We consider a circumstances in which the arcs $A_1, A_2, \ldots, A_{k-1}$ are already labeled for $k = 2,3,\ldots,n$ and the remaining arcs are unlabel. In this circumstances we want to label the arc A_k by surjective $L(3,1)$-labeling. According to lemma 4, $L^{vl}(2,A_k)$ is the largest non-empty set of labels satisfying $L(3,1)$-labeling, where $l \leq r$ for all $l \in L^{vl}(2,A_k)$, $r = max\{L^s(A_k)\} + 3$ for any $A_k \in A$.

Again, $L^{svl}(A_k) = L^{vl}(2,A_k) - L^s(A_k)$, so, $L^{svl}(A_k)$ is the largest non empty set of labels satisfying surjective $L(3,1)$-labeling, because the label in the set $L^{svl}(A_k)$ do not used earlier to label any arc and also satisfies $L(3,1)$-labeling. Since, we want to label the arc by using least possible label, so, $f_k^s = q$, where $q = min\{L^{svl}(2,A_k)\}$. Since, $L^{svl}(A_k)$ is the largest set of labels satisfying surjective $L(3,1)$-labeling, so, q is the least surjective label for the arc A_k. Again, by Theorem 1, $\lambda_{3,1}(G) = 5\Delta - 2$, so the label of A_k must be less than or equal to $5\Delta-2$. Again, since $n=5\Delta-2$, A_k is label using the label from the set $\{1,2,\ldots,n\}$ which is not previously used to label any arc. Since, A_k is arbitrary so, any *CAG* can be surjectively label by $L(3,1)$-labeling by Algorithm *SL*31 and $\lambda_{3,1}(G)=max\{L^s(A_n)\bigcup\{f_n^s\}\}=5\Delta-2=n$. □

Theorem 3.7. *The time complexity of Algorithm SL31 is $O(n\Delta^3)$, where n is the number of nodes of the graph and Δ is the degree of the graph such that $n=5\Delta-2$.*

Proof. By our proposed algorithm f_k^s, the surjective $L(3,1)$-label of A_k can be computed if $L^{svl}(A_k)$ is computed. Now by Lemma 2, algorithm *SLKVL* can compute $L^{vl}(p,A_k)$, $p=1,2$ using $O(\Delta^2)$ time. Using algorithm *A***diff***B*, $L^{vl}(2,A_k)-L^s(A_k)$ can be computed in $O(\Delta)$ time. So the total time required to compute $L^{svl}(A_k)$ is $O(\Delta^3)$. Since we need to find $L^{svl}(2,A_k)$ for $k=2,3,\ldots,n$, so the running time for Algorithm *SL*31 is $O((n-1)\Delta^3)$, i.e. $O(n\Delta^3)$. □

4. Conclusion

Although, the $L(3,1)$-labeling problem has been widely studied in the last four decades, there are only a few classes of graphs for which the result about surjective $L(2,1)$-labeling is available. For other classes of graphs surjective $L(3,1)$-labeling is clearly welcome. In this paper, we determine the upper bound $\lambda_{3,1}$ for a *CAG G*, and have shown that $\lambda_{3,1}(G)\leq 5\Delta-2$. Also, we have proved that a *CAG G* with n nodes can be surjectively labeled using $L(3,1)$-labeling if $n=5\Delta-2$. Also, we have presented an efficient algorithm to label a *CAG* by surjective $L(3,1)$-labeling. This is the first result for the problems on *CAGs*. The time complexity for the algorithms is $O(n\Delta^3)$. But we are unable to prove that whether surjective $L(3,1)$-labeling is possible or not for *CAG*, when $n\neq 5\Delta-2$. It is an open problem to the researchers.

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