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Syntactic semiring of l-fuzzy languages

K. Ajitha Kumari

Abstract

In this paper we study syntactic semiring of l-fuzzy languages.We prove that every finite idempotent semiring is a syntactic semiring of a l-fuzzy language. Also we study some properties of syntactic semiring of recognizable l-fuzzy languages. Here we prove that the syntctic semiring of a left singular l-fuzzy language is left singular.

Keywords

Generalized fuzzy languages, Syntactic Congruence, Syntactic Semiring, Left singular l-fuzzy languages.

AMS Subject Classification

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Department of Mathematics, B.J.M. Government College, Chavara, Kollam, Kerala, India.

*Corresponding author: [a]mbiliragamalika@gmail.com

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Contents

1. Introduction

Theory of formal languages is one of the fundamental concept in theoretical computer science.The study of regular languages using monoid was initiated by Kleene [7]. Zadeh [14] introduced the notion of a fuzzy subset of an ordinary set as a way of representing uncertainty. Zadeh and Lee [8] generalized the classical notion of languages to the concept of fuzzy languages in 1969. Semirings have been proved to be an important algebraic tool in theoretical computer science, in particular for studying automata and formal languages. Semiring recognizable languages was first studied by Polak. He introduced the concept of syntactic semiring of a language and studied its properties. Also he established a one-one correspondence between the lattices of all conjunctive varieties of languages and pseudovarieties of finite idempotent semiring. Here we consider fuzzy languages with membership values in complete complemented distributive lattice l (l-fuzzy languages). We study the syntactic semiring of l-fuzzy languages and show that every idempotent semiring is a syntactic semiring of a l-fuzzy language. We also discuss some properties of a syntactic semiring of recognizable l-fuzzy languages. We study about the syntactic semiring of left singular l-fuzzy languages.

2. Preliminaries

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms are as in [6, 7, 9].

A lattice is a partially ordered set in which every subset $\{a,b\}$ consisting of two elements has a least upper bound($a \vee b$) and a greatest lower bound($a \wedge b$).

A semiring is a nonempty set S together with two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that

1) $(S,+,0)$ is a commutative monoid.
2) $(S,\cdot,1)$ is a monoid
3) the distributive laws $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$ hold for every $a,b,c \in S$.
4) $0 \cdot a = a \cdot 0 = 0$ for every a.

A semiring S is called a partially ordered semiring if and only if S is a partially ordered set under the relation $\leq$ and the following conditions hold.

i) $0 \leq a$
ii) $a_1 \leq a_2$ implies $a_1 + a \leq a_2 + a$

ii) $a_1 \leq a_2$ implies $a_1 a \leq a_2 a$ and $aa_1 \leq aa_2$ for all $a, a_1, a_2 \in S$.

Theorem 2.1. *([13], Theorem 5.2) Each idempotent semiring is a partially ordered semiring under the relation $a_1 \leq a_2$ if and only if $a_1 + a_2 = a_2$ for all a_1, a_2 in the semiring.*

Let A be a nonempty finite set called an alphabet. Elements of A are called letters. A finite sequence of letters of A is called a word. A word of length zero is called empty word and is denoted by ε. A^+ denotes the set of all nonempty words over an alphabet A and $A^* = A^+ \cup \{\varepsilon\}$ is a monoid under the operation concatenation, called free monoid over A. A subset of A^* is called the language L over an alphabet A. Let $F(A^*)$ denote the set of all finite subsets of A^*. This set equipped with the operation multiplication $U \cdot V = \{uv \mid u \in U, v \in V\}$ and usual union form the free idempotent semiring over the alphabet A. Here empty set acted as identity element with respect to union and $\{\varepsilon\}$ acted as identity element with respect to $'\cdot'$.

Definition 2.2. *[12] A Language $L \subseteq A^*$ is recognizable by an idempotent semiring $(S, +, \cdot)$ with respect to its order ideal I if there exists a semiring morphism $\beta : (F(A^*), \cdot, \cup) \to (S, \cdot, +)$ such that $F(L) = \beta^{-1}(I)$.*

Let X be a nonempty set. A l-fuzzy set [or more precisely l-fuzzy subset] λ of X is a function from X into the complete complemented distributive lattice l. A l-fuzzy language λ over an alphabet A is a l-fuzzy subset of $A^*(resp. A^+)$. For l-fuzzy languages λ_1, λ_2 over an alphabet A, their meet is defined as

$(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u)$ for all $u \in A^*$.

Definition 2.3. *[10] A Fuzzy language λ over an alphabet A is recognizable by a monoid M if there is an onto homomorphism $\phi : A^* \to M$ and a fuzzy subset π of M such that $\lambda = \pi\phi^{-1}$ where $\lambda(u) = \pi\phi^{-1}(u) = \pi(\phi(u))$.*

Definition 2.4. *[5] A fuzzy subset λ of a semiring $(S, +, \cdot)$ is called a fuzzy left (right) ideal of S if*

1) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

2) $\lambda(x \cdot y) \geq \lambda(y)$ [$\lambda(x \cdot y) \geq \lambda(x)$] for all $x, y \in S$

Theorem 2.5. *[1] Let λ be a l-fuzzy language over an alphabet A. Then the following statements are equivalent*

(i) λ is recognizable.

(ii) $\sim_\lambda$ has finite index.

A fuzzy subset λ of a semiring S is called a fuzzy two-sided ideal or simply a fuzzy ideal of S if it is both a fuzzy left ideal or a fuzzy right ideal of S. A fuzzy ideal of a semiring S is called fuzzy ordered ideal if it satisfies the condition, $x \leq y$ implies $\lambda(x) \geq \lambda(y)$ for all $x, y \in S$.

Next result gives a characterisation of fuzzy ideals.

Theorem 2.6. *([5],Theorem.2.1) A fuzzy subset λ of a semiring S is a fuzzy ideal of S if and only if each nonempty level subset $\lambda_t = \{x \in S \mid \lambda(x) \geq t\}$ of λ is an ideal of S for all $t \in [0,1]$.*

3. Generalized Fuzzy Languages

Definition 3.1. *Let λ be a l-fuzzy language over A. The function $\lambda_{min} : F(A^*) \to l$ defined by $\lambda_{min}(U) = \bigwedge_{u \in U} \lambda(u), U \in F(A^*)$ is called the generalized fuzzy language determined by λ. If $|U| = 1$ then $\lambda_{min}(U) = \lambda(u)$. So we can view λ_{min} as a generalization of λ.*

Example 3.2. *Let $A = \{a, b\}$ and let $l = (\{\{c\}, \{d\}, \{c, d\}, \emptyset\}, \cup, \cap)$ be a complete distributive lattice. The function $\lambda : A^* \to l$ defined by*

$$\lambda(u) = \begin{cases} \{c\} & \text{if } u \in aA^* \\ \{d\} & \text{if } u \in bA^* \end{cases}$$

is a l-fuzzy language. Let $U \in F(A^*)$. Then the generalized fuzzy language determined by λ is

$$\lambda_{min}(U) = \begin{cases} \{c\} & \text{if } U \subseteq aA^* \\ \{d\} & \text{if } U \subseteq bA^* \\ \emptyset & \text{if } U \text{ contains elements of } aA^* \text{ and } bA^* \end{cases}$$

Definition 3.3. *[2] Let λ be a l-fuzzy language over A. We say that λ is recognized by an idempotent semiring S, if there exist a semiring homomorphism $\beta : F(A^*) \to S$ and a l-fuzzy ordered ideal γ of S such that $\lambda_{min} = \gamma\beta$. We also say that the idempotent semiring S recognizes λ by a homomorphism $\beta : F(A^*) \to S$.*

Let λ be a l-fuzzy language. Then λ is said to be semiring recognizable if it is recognized by a finite idempotent semiring.

Theorem 3.4. *Let λ be a l-fuzzy language over A. Then an idempotent semiring S recognizes λ by a homomorphism $\beta : F(A^*) \to S$ if and only if the congruence $ker\beta$ saturates λ_{min}.*

4. Syntactic Congruence

Let λ be a l-fuzzy language over A. Define a relation $\sim_{min}$ on $F(A^*)$ as follows
For $U, V \in F(A^*), U \sim_{min} V \Leftrightarrow \lambda_{min}(\bigcup_{u \in U} puq) = \lambda_{min}(\bigcup_{v \in V} pvq)$ for all $p, q \in A^*$. The restriction of λ_{min} to words (subsets of cardinality one) is λ. It is easily seen that the relation $\sim_{min}$ is a congruence on $F(A^*)$.

Thus the quotient structure $F(A^*)/\sim_{min}$ is a semiring under the binary operations

$$[U]_{\sim_{min}} \cup [V]_{\sim_{min}} = [U \cup V]_{\sim_{min}} \quad \text{and}$$

.

$$[U]_{\sim_{min}} \cdot [V]_{\sim_{min}} = [U \cdot V]_{\sim_{min}}$$

.

Since union is a commutative binary operation on $F(A^*)$, union is commutative on $F(A^*)/\sim_{min}$. Thus $F(A^*)/\sim_{min}$ is an idempotent semiring, called the syntactic semiring of

λ or λ_{min} . It is denoted by Syn(λ_{min}). The function η_{min} : $F(A^*) \to F(A^*)/\sim_{min}$ defined by $\eta_{min}(U) = [U]_{\sim_{min}}$ is a surjective semiring homomorphism, called the syntactic semiring homomorphism of λ (or λ_{min}).

Theorem 4.1. *The syntactic semiring Syn(λ_{min}) recognizes λ.*

Proof. We know that the map $\eta_{\min} : F(A^*) \to$ Syn(λ_{min}) is an onto homomorphism. We will show that the syntactic semiring Syn(λ_{min}) recognizes the l-fuzzy language λ by the syntactic homomorphism $\eta_{\min}$. By Theorem 3.4, it suffices to show that ker($\eta_{\min}$) saturate λ_{min}. Let $(U,V) \in$ ker $(\eta_{\min})$. Then $\eta_{\min}(U) = \eta_{\min}(V)$. That is, $[U]_{\sim_{min}} = [V]_{\sim_{min}}$. Thus $(U,V) \in \sim_{min}$. Hence $\lambda_{min}(\bigcup_{u\in U} puq) = \lambda_{min}(\bigcup_{v\in V} pvq)$ for all $p,q \in A^*$. In particular, when $p = q = \varepsilon$, we have

$$\begin{aligned}\lambda_{min}(U) &= \lambda_{min}(\bigcup_{u\in U} \varepsilon u \varepsilon))\\ &= \lambda_{min}(\bigcup_{v\in V} \varepsilon v \varepsilon)\\ &= \lambda_{min}(V).\end{aligned}$$

Thus ker($\eta_{\min}$) saturates λ_{min}. □

Theorem 4.2. *The idempotent semiring S recognizes a l-fuzzy language λ over an alphabet A if and only if the syntactic semiring Syn(λ_{min}) divides S.*

Proof. Suppose that an idempotent semiring S recognizes λ. Then there exist a homomorphism $\beta : F(A^*) \to S$ and a l-fuzzy ordered ideal μ of S such that $\lambda_{min} = \mu\beta$. Define a map θ from $\beta(F(A^*))$ to Syn(λ_{min}) by

$$\theta(\beta(U)) = [U]_{\sim_{min}},\ U \in F(A^*).$$

Let $U,V \in F(A^*)$ and let $\beta(U) = \beta(V)$. Then $(U,V) \in$ kerβ. By Theorem 3.4, we have $\lambda_{min}(U) = \lambda_{min}(V)$. So $(U,V) \in \sim_{min}$. Hence $[U]_{\sim_{min}} = [V]_{\sim_{min}}$. That is, $\theta(\beta(U)) = \theta(\beta(V))$. Thus θ is well defined.
For $U,V \in F(A^*)$, we have

$$\begin{aligned}\theta(\beta(U)\cdot\beta(V)) &= \theta(\beta(U\cdot V))\\ &= [U\cdot V]_{\sim_{min}}\\ &= [U]_{\sim_{min}}\cdot[V]_{\sim_{min}}\\ &= \theta(\beta(U))\cdot\theta(\beta(V))\end{aligned}$$

and

$$\begin{aligned}\theta(\beta(U)\cup\beta(V)) &= \theta(\beta(U\cup V))\\ &= [U\cup V]_{\sim_{min}}\\ &= [U]_{\sim_{min}}\cup[V]_{\sim_{min}}\\ &= \theta(\beta(U))\cup\theta(\beta(V)).\end{aligned}$$

Clearly $\theta(\beta(\emptyset)) = [\emptyset]_{\sim min}$ and $\theta(\beta\{\varepsilon\}) = [\{\varepsilon\}]_{\sim_{min}}$. These are the identities in Syn(λ_{min}). Thus θ is a homomorphism from $\beta(F(A^*))$ to Syn(λ_{min}). Since $\beta(F(A^*))$ is a subsemiring of S, we see that Syn(λ_{min}) divides idempotent semiring S.

Conversely assume that Syn(λ_{min}) divides S. So there exist a subsemiring S' of S and a surjective homomorphism $\varphi : S' \to$ Syn(λ_{min}). We define a map $\beta' : F(A^*) \to S'$ by

$$\beta'(U) = s \text{ if } \eta_{\min}(U) = \varphi(s),$$

for $U \in F(A^*)$ and $s \in S'$. Let $U_1,U_2 \in F(A^*)$ and let $\beta'(U_1) = s_1$ and $\beta'(U_2) = s_2$. Let $U_1 = U_2$ then $\eta_{\min}(U_1) = \eta_{\min}(U_2)$. Thus $\beta'(U_1) = \beta'(U_2)$. Hence β' is well defined. Also

$$\begin{aligned}\eta_{\min}(U_1\cdot U_2) &= \eta_{\min}(U_1)\cdot\eta_{\min}(U_2)\\ &= \varphi(s_1)\cdot\varphi(s_2)\\ &= \varphi(s_1\cdot s_2).\end{aligned}$$

Thus $\beta'(U_1\cdot U_2) = s_1\cdot s_2 = \beta'(U_1)\cdot\beta'(U_2)$ and

$$\begin{aligned}\eta_{\min}(U_1\cup U_2) &= \eta_{\min}(U_1)\cup\eta_{\min}(U_2)\\ &= \varphi(s_1)\cup\varphi(s_2)\\ &= \varphi(s_1+s_2).\end{aligned}$$

So $\beta'(U_1\cup U_2) = s_1+s_2 = \beta'(U_1)+\beta'(U_2)$, where + denotes the addition(commutative binary operation) on S' and $\cdot$ denotes the product on S'. Thus β' is a homomorphism from $F(A^*)$ to S' and $\varphi\beta' = \eta_{\min}$. Since S' is a subsemiring of the semiring S, there exists a homomorphism $\beta : F(A^*) \to S$.

Since Syn(λ_{min}) recognizes λ, there exists a l-fuzzy ordered ideal γ' of Syn(λ_{min}) such that $\lambda_{min} = \gamma'\eta_{\min}$. Define a map $\gamma : S \to l$ by $\gamma(s) = (\gamma'\varphi)(s)$, if $s \in S'$ and γ is defined arbitrarily if $s \in S\setminus S'$. Since γ' and φ are well defined, γ is also well defined.
Let $s_1,s_2 \in S$ and since γ' is an ordered ideal of Syn(λ_{min}), we have

$$\begin{aligned}&\gamma(s_1+s_2)\\ &= (\gamma'\varphi)(s_1+s_2)\\ &= \gamma'(\varphi(s_1+s_2))\\ &= \gamma'(\varphi(s_1)\cup\varphi(s_2))\\ &\geq \gamma'(\varphi(s_1))\wedge\gamma'(\varphi(s_2)) = (\gamma'\varphi)(s_1)\wedge(\gamma'\varphi)(s_2).\end{aligned}$$

That is, $\gamma(s_1+s_2) \geq \gamma(s_1)\wedge\gamma(s_2)$. Also we have

$$\begin{aligned}&\gamma(s_1\cdot s_2)\\ &= (\gamma'\varphi)(s_1\cdot s_2)\\ &= \gamma'(\varphi(s_1\cdot s_2))\\ &= \gamma'(\varphi(s_1)\cdot\varphi(s_2))\\ &\geq \gamma'(\varphi(s_1))\wedge\gamma'(\varphi(s_2)) = (\gamma'\varphi)(s_1)\wedge(\gamma'\varphi)(s_2).\end{aligned}$$

That is, $\gamma(s_1\cdot s_2) \geq \gamma(s_1)\wedge\gamma(s_2)$.

Let $s_1,s_2 \in S$ and $s_1 \leq s_2$. Then $s_1+s_2 = s_2$, since S is an idempotent semiring. Thus we get, $\varphi(s_1+s_2) = \varphi(s_2)$. That is, $\varphi(s_1)\cup\varphi(s_2) = \varphi(s_2)$. Hence $\varphi(s_1) \leq \varphi(s_2)$. Since γ' is an ordered ideal of Syn(λ_{min}), $\gamma'(\varphi(s_1)) \geq \gamma'(\varphi(s_2))$. That

is, $\gamma(s_1) \geq \gamma(s_2)$. Thus γ is a l-fuzzy ordered ideal of S.
For all $U \in F(A^*)$, we have

$$\begin{aligned}(\gamma\beta)(U) &= \gamma(\beta(U))\\ &= (\gamma'\varphi)(\beta(U))\\ &= \gamma'(\varphi(\beta(U)))\\ &= \gamma'(\varphi(\beta'(U)))\\ &= \gamma'(\eta_{\min}(U))\\ &= (\gamma'\eta_{\min})(U)\\ &= \lambda_{min}(U).\end{aligned}$$

So $\lambda_{min} = \gamma\beta$. Thus S recognizes λ. □

5. Ordered Syntactic Monoid

Let λ be a l-fuzzy language over an alphabet A. Now we consider a relation $\leq$ on syntactic monoid Syn(λ), defined by

$$[u]_{\sim_\lambda} \leq [v]_{\sim_\lambda} \Leftrightarrow \lambda(pvq) \leq \lambda(puq)$$

for all $p,q \in A^*$. (Syn $(\lambda), \leq)$ is called the ordered syntactic monoid and is denoted by $O(\lambda)$.

Theorem 5.1. *[3] Let λ be a l-fuzzy language over an alphabet A and λ_{min} be the generalized fuzzy language determined by λ. Then the function θ defined by $\theta([u]_{\sim_\lambda}) = [\{u\}]_{\sim_{min}}, u \in A^*$ is an injective semigroup homomorphism of ordered syntactic monoid $(O(\lambda),\cdot,\leq)$ into the syntactic semiring $(Syn(\lambda_{min}),\cup,\cdot)$.*

Theorem 5.2. *[3] The syntactic semiring $(Syn(\lambda_{min}),\cup,\cdot)$ of a l-fuzzy language λ over A is isomorphic to the semiring $(F(O(\lambda)),\cup,\cdot)$.*

In [1] we already proved that any monoid is a syntactic monoid of a l-fuzzy language. From the above theorems [5.1, 5.2] we get that there exists an injective semigroup homomorphism from an ordered syntactic monoid $(O(\lambda),\cdot,\leq)$ into the syntactic semiring $(\mathrm{Syn}(\lambda_{min}),\cup,\cdot)$ and an isomorphism from the syntactic semiring $(\mathrm{Syn}\lambda_{min}),\cup,\cdot)$ of a l-fuzzy language λ to the semiring $(F(O(\lambda)),\cup,\cdot)$. In [11] Polak proved that every finite idempotent semiring is isomorphic to the syntactic semiring of a recognizable language . From these results we get the following theorem.

Theorem 5.3. *Every finite idempotent semiring is a syntactic semiring of a l-fuzzy language.*

Example 5.4. *Let $S = (\{a,b\},+,*)$*

$+$	a	b
a	a	a
b	a	b

$*$	a	b
a	a	a
b	b	b

be an idempotent semiring. Let $l = (\{1,2,3,6\}, LCM, GCD)$ be the complete distributive lattice and $A = \{c,d\}$. Now we define a l-fuzzy language $\lambda : A^* \to l$ by

$$\lambda(u) = \begin{cases} 1 & if\ \ u \in cA^* \\ 2 & if\ \ u \in dA^*. \end{cases}$$

The generalized fuzzy language λ_{min} determined by λ is defined by

$$\lambda_{min}(U) = \begin{cases} 1 if\ \ U\ contains\ \ \ elements\ \ of\ cA^*\ \ and\ \ dA^* \\ or\ \ U \subseteq cA^* \\ 2\ if\ \ U \subseteq dA^*. \end{cases}$$

The syntactic semiring $Syn(\lambda_{min})$ of λ has two elements.
$[U]_{\sim_{min}} = \{w \in\ F(A^*)/w\ contains\ elements\ of cA^* and\ dA^*\}$
and
$[V]_{\sim_{min}} = \{w \in\ F(A^*)/w \subseteq\ dA^*\}$
Define a function γ from $Syn(\lambda_{min})$ to $(S,+,*)$ by $\gamma([U]_{\sim_{min}}) = a$ and $\gamma([V]_{\sim_{min}}) = b$. Then γ is a homomorphism from $Syn(\lambda_{min})$ to $(S,+,*)$. Also γ is one to one and onto. Hence γ is an isomorphism from $Syn(\lambda_{min})$ to $(S,+,*)$. Thus the semiring $(S,+,*)$ is the syntactic semiring of the l-fuzzy language λ.

The following theorem gives some properties of ordered syntactic monoid and syntactic semiring of a recognizable l-fuzzy language.

Theorem 5.5. *Let λ be a l-fuzzy language over A. Then the following are equivalent.*

(i) λ is recognizable.

(ii) The ordered syntactic monoid $O(\lambda)$ of λ is finite.

(iii) The syntactic semiring $Syn(\lambda_{min})$ of λ is finite.

(iv) λ is recognized by a finite idempotent semiring.

Proof. (**i**) $\Leftrightarrow$ (**ii**) Follows from Theorem 2.5.
(**iii**) $\Rightarrow$ (**ii**) Assume that (iii) holds. By Theorem 5.1, $O(\lambda)$ is finite.
(**ii**) $\Rightarrow$ (**iii**) Assume that (ii) holds. By Theorem 5.2, Syn(λ_{min}) is finite.
(**iii**) $\Leftrightarrow$ (**iv**) Follows from Theorem 4.1 and Theorem 4.2. □

6. Left Singular l-Fuzzy Languages

A semiring $(S,+,\cdot)$ is left singular under multiplication if $x \cdot y = x$ for all x,y in S.

Definition 6.1. *A l-fuzzy laguage $\lambda : A^+ \to l$ is said to be left singular if it satisfies the condition $\lambda(puvq) = \lambda(puq)$ for all p,q,u,v in A^+.*

Example 6.2. *Let $l = (\{c\},\{d\},\{c,d\},\emptyset\},\cup,\cap)$ be a complete distributive lattice and $A = \{a,b\}$. Let $\lambda : A^+ \to l$ be defined by*

$$\lambda(u) = \begin{cases} \{c\} & if\ u \in aA^+b \\ \{d\} & if\ u \in bA^+a \\ \emptyset & otherwise. \end{cases}$$

Then λ is a left singular l-fuzzy language.

Let λ be a left singular l-fuzzy language. Then

$$\begin{aligned}\lambda_{min}(pUVq) &= \bigwedge_{uv\in UV} \lambda(puvq)\\ &= \bigwedge_{u\in U} \lambda(puq)\\ &= \lambda_{min}(pUq).\end{aligned}$$

Thus a l-fuzzy language λ is left singular if and only if the generalized fuzzy language λ_{min} determined by λ satisfies the condition $\lambda_{min}(pUVq) = \lambda_{min}(pUq)$ for all $p,q \in A^+$ and $U,V \in F(A^+)$.

Theorem 6.3. *A l-fuzzy language $\lambda : A^+ \to l$ is left singular if and only if its syntactic semiring Syn(λ_{min}) is left singular under multiplication.*

Proof. Assume that the l-fuzzy language λ is left singular. Then $\lambda(puvq) = \lambda(puq)$ for all $u,v,p,q \in A^+$. The generalized fuzzy language λ_{min} determined by λ satisfies the condition for all $p,q \in A^+$ and $U,V \in F(A^+)$, $\lambda_{min}(pUVq) = \lambda_{min}(pUq)$. That is, for all $p,q \in A^+$ and $U,V \in F(A^+)$, $\lambda_{min}(\bigcup_{w\in UV} pwq) = \lambda_{min}(\bigcup_{u\in U} puq)$. Thus $UV \sim_{min} U$. Hence $[UV]_{\sim_{min}} = [U]_{\sim_{min}}$. That is, $[U]_{\sim_{min}}[V]_{\sim_{min}} = [U]_{\sim_{min}}$. Thus the syntactic semiring Syn(λ_{min}) of λ is left singular under multiplication.

Conversely assume that the syntactic semiring Syn(λ_{min}) of λ is left singular under multiplication. So $[U]_{\sim_{min}}[V]_{\sim_{min}} = [U]_{\sim_{min}}$ for all $[U]_{\sim_{min}}, [V]_{\sim_{min}} \in$ Syn(λ_{min}). That is, $[UV]_{\sim_{min}} = [U]_{\sim_{min}}$. Thus $UV \sim_{min} U$. Hence for all $p,q \in A^+$ and $U,V \in F(A^+)$, $\lambda_{min}(\bigcup_{w\in UV} pwq) = \lambda_{min}(\bigcup_{u\in U} puq)$. That is, $\lambda_{min}(pUVq) = \lambda_{min}(pUq)$ for all $p,q \in A^+$ and $U,V \in F(A^+)$. Therefore λ is a left singular l-fuzzy language. $\square$

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