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A convolution and product theorems for the *N*-dimensional fractional Fourier transform

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Abstract

The n-dimensional fractional Fourier transform, which is a generalization of the one dimensional fractional Fourier transform, has many applications in several areas, including signal processing and optics. In a recent paper, derived n-dimensional fractional Fourier transforms of a product and of a convolution of two functions. Unfortunately, their convolution formulas do not generalize very nicely the classical result for the Fourier transform and Laplace transform, which states that the Fourier transform of the convolution of two functions is the product of their Fourier transforms. The purpose of this note is to introduce a new convolution structure for the n-dimensional fractional Fourier transform that preserves the convolution theorem for the one dimensional fractional Fourier transform.

Keywords

Fourier transform, fractional Fourier transform, product, convolution, $n-$ dimension and kernel.

AMS Subject Classification

41A63, 42A38, 65R10, 46F99.

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Contents

1. Introduction

In general, integral transforms are useful tools for solving problems involving certain types of partial differential equations, mainly when their solutions on the corresponding domains of definition are difficult to deal with. For a given partial differential equation defined on a domain, the application of a suitable integral transform allows it to be expressed in such a form that its mathematical manipulation is easier than the original one. In this way, if a solution on the transformed domain is found, then an application of the inverse integral transform will give the solution of the original partial differential equation. Even though there are a number of integral transforms suitable for different differential equations, the most known in the applied mathematics community are the Laplace transform and the Fourier transform [5].

The Fourier transform has long been proved to be extremely useful as applied to signal and image processing and for analyzing quantum mechanics phenomena. Particularly, in genetics and medical areas, it helped in analyzing biosignals such as heart rate variation and in interpreting X-ray computed tomography images.

We know that the one dimensional Fourier transform is extended to one dimensional Fractional Fourier transform. Using this extension there are many results and application's obtained mainly by Alieva Tatiana in [3], [4], Almedia L.B.in [6], victor Namis in [9], Mcbride Kerr in [10] and precisely the product and convolution theorems are derived by Ahmed Zayed in [1], L.B.Almeida in [2] and Gaikwad and Chaudhary in [8] etc..

Due to many applications of Fourier transform, in this paper the n-dimensional fractional Fourier transform which is defined in [7] and its convolution are considered, n-dimensional fractional Fourier transform of usual product and convolution of two n-dimensional functions are obtained in [8] but they are not in usual form as like convolution and product theorem of Fourier and Laplace transform. So a new convolution is defined for two functions in $L^1(\mathbb{R}^n)\cap C^1(\mathbb{R}^n)$, by using usual convolution and convolution in convolution and product theorem of one dimensional Fractional Fourier transform given by Ahmed Zayed in [1]. For this convolution, a convolution and product theorem for n-dimensional fractional Fourier transform is obtained. A new convolution for the n-dimensional fractional Fourier transform that preserves the convolution theorem for the n-dimensional Fourier transform and one dimensional fractional Fourier transform. In section 1 involve introduction, in section 2 consist of some basic definitions. In section 3 we have the usual product and convolution of n-dimensional Fractional Fourier transform which was preciously explained in [8]. In section 4 we define a new convolution and obtain a convolution and product theorem for n-dimensional Fractional Fourier transform.

2. The fractional Fourier transform

Definition 2.1. *[5] If $h\in L^1(\mathbb{R})\cap C^1(\mathbb{R})$ then the pair of Fourier transform and its inverse Fourier transform are given by*

$$H(v_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(t_1)e^{-jt_1v_1}dt_1$$

and

$$h(t_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(v_1)e^{jt_1v_1}dv_1$$

where $t_1,\ v_1\in\mathbb{R}$.

Definition 2.2. *[5] If $\alpha_1\in\mathbb{R}$; with α_1 is a constant and $h\in L^1(\mathbb{R})\cap C^1(\mathbb{R})$, then the pair of one dimensional fractional Fourier transform and its inverse are given by*

$$R^{\alpha_1}[h(t_1)](v_1)=H_{\alpha_1}(v_1)=\int_{-\infty}^{\infty}h(t_1)K_{\alpha_1}(t_1,v_1)dt_1$$

where

$$K_{\alpha_1}(t_1,v_1)=\sqrt{\frac{1-j\cot\alpha_1}{2\pi}}\times \exp\left\{j\left[\frac{1}{2}(t_1^2+v_1^2)\cot\alpha_1-t_1v_1\csc\alpha_1\right]\right\};$$

if $\alpha_1\neq\pi m$, for all $m=0,1,2,\ldots$ and

$$h(t_1)=\int_{-\infty}^{\infty}\overline{K}_{\alpha_1}(t_1,v_1)H_{\alpha_1}(v_1)dv_1$$

where

$$\overline{K}_{\alpha_1}(t_1,v_1)=\sqrt{\frac{1+j\cot\alpha_1}{2\pi}}\times \exp\left\{-j\left[\frac{1}{2}(t_1^2+v_1^2)\cot\alpha_1-t_1v_1\csc\alpha_1\right]\right\};$$

if $\alpha_1\neq\pi m$, for all $m=0,1,2,\ldots$.

Definition 2.3. *[5] Using vector notation with $t=(t_1,t_2,\cdots,t_n)\in\mathbb{R}^n, v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$ and $h\in L^1(\mathbb{R}^n)\cap C^1(\mathbb{R}^n)$, the n-dimensional Fourier transform, denoted by $H(v)$, is defined by*

$$H(v)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h(t)e^{-j(v\cdot t)}dt$$

or simply by

$$H(v)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}h(t)\left[\prod_{p=1}^{n}e^{-jv_pt_p}\right]dt,$$

where $v\cdot t=v_1t_1+v_2t_2+\cdots+v_nt_n$ and $dt=dt_1dt_2\cdots dt_n$. It's inverse n-dimensional Fourier transform denoted by $h(t)$, is defined by

$$h(t)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}H(v)e^{j(v\cdot t)}dv$$

or simply by

$$h(t)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}H(v)\left[\prod_{p=1}^{n}e^{jv_pt_p}\right]dv,$$

where $v\cdot t=v_1t_1+v_2t_2+\cdots+v_nt_n$ and $dv=dv_1dv_2\cdots dv_n$.

Definition 2.4. *[7] Let $t=(t_1,t_2,\cdots,t_n)\in\mathbb{R}^n$, $v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$ and $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathbb{R}^n$; with all components of α are constants. The kernel $K_\alpha(t,v)$ is defined as follows*

$$K_\alpha(t,v)=\prod_{p=1}^{n}K_{\alpha_p}(t_p,v_p),$$

where

$$K_{\alpha_p}(t_p,v_p)=\sqrt{\frac{1-j\cot\alpha_p}{2\pi}}\times \exp\left\{j\left[\frac{1}{2}(t_p^2+v_p^2)\cot\alpha_p-t_pv_p\csc\alpha_p\right]\right\};$$

if $\alpha_p\neq\pi m$, for all $m=0,1,2,\ldots$ and $p=1,2,\ldots,n$.

Definition 2.5. *[7] Let $t=(t_1,t_2,\cdots,t_n)\in\mathbb{R}^n$, $v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$, $h\in L^1(\mathbb{R}^n)\cap C^1(\mathbb{R}^n)$ and $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathbb{R}^n$. The n-dimensional fractional Fourier transform with parameter $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ of $h(t)=h(t_1,t_2,\cdots,t_n)$ denoted by $R^\alpha[h(t)](v)$ or $H_\alpha(v)$ or $H_{\alpha_1,\alpha_2,\cdots,\alpha_n}(v_1,v_2,\cdots,v_n)$ and is given by the following integral transform*

$$\begin{aligned}&R^\alpha[h(t)](v)\\&=H_\alpha(v)\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h(t_1,t_2,\cdots,t_n)\times\\&\qquad\left[\prod_{p=1}^{n}K_{\alpha_p}(t_p,v_p)\right]dt_1dt_2\cdots dt_n\\&=\int_{\mathbb{R}^n}h(t)K_\alpha(t,v)dt,\end{aligned}\tag{2.1}$$

where all the components of α are constants, $h(t) = h(t_1, t_2, \cdots, t_n), dt = dt_1 dt_2 \cdots dt_n$ and

$$K_\alpha(t, v) = \prod_{p=1}^{n} \left[\sqrt{\frac{1 - j\cot\alpha_p}{2\pi}} \times \exp\left\{ j\left[\frac{1}{2}(t_p^2 + v_p^2)\cot\alpha_p - t_p v_p \csc\alpha_p \right] \right\} \right];$$

if $\alpha_p \neq \pi m$, for all $m = 0, 1, 2, \ldots$ and $p = 1, 2, \ldots, n$. Also if $\alpha = 0$ then $R^\alpha[h(t)](v) = h(t)$.

Note 2.6. *If $\alpha = \left(\frac{\pi}{2}, \frac{\pi}{2}, \cdots, \frac{\pi}{2} \right)$, then the equation (2.1) reduces to n-dimensional Fourier transform.*

Note 2.7. *If $n = 1$, then the equation (2.1) reduces to one-dimensional fractional Fourier transform.*

Theorem 2.8. *[7] Let $h \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. If $t, v, \alpha \in \mathbb{R}^n$, where all the components of α are constants, $R^\alpha[h(t)](v)$ or $H_\alpha(v)$ is the n-dimensional fractional Fourier transform of $h(t)$, then $h(t)$ is given by*

$$h(t) = \int_{\mathbb{R}^n} \overline{K}_\alpha(t, v) H_\alpha(v) dv,$$

where $\overline{K}_\alpha(t, v) = \prod_{p=1}^{n} \left[\sqrt{\frac{1 + j\cot\alpha_p}{2\pi}} \times \exp\left\{ -j\left[\frac{1}{2}(t_p^2 + v_p^2)\cot\alpha_p - t_p v_p \csc\alpha_p \right] \right\} \right]$ *and* $dv = dv_1 dv_2 \cdots dv_n$.

Definition 2.9. *[5] If $h, g \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, then denote their convolution f by $h * g$ and defined as*

$$f(y) = [h * g](y) = \int_{\mathbb{R}^n} h(t) g(y - t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(t_1, t_2, \cdots t_n) \times g(y_1 - t_1, y_2 - t_2, \cdots, y_n - t_n) dt_1 dt_2 \cdots dt_n.$$

3. The *n*-dimensional fractional Fourier transform of usual Product and convolution

Let $a_p = a_p(\alpha_p) = \frac{\cot\alpha_p}{2}, b_p = b_p(\alpha_p) = \sec\alpha_p$, $c_p = c_p(\alpha_p) = \sqrt{1 - j\cot\alpha_p} \; \forall \; p = 1, 2, \cdots, n$ $\Rightarrow 2a_p(\alpha_p) b_p(\alpha_p) = \csc\alpha_p; \forall \; p = 1, 2, \cdots, n$. Throughout this paper the constants a_p, b_p and $c_p \; \forall \; p = 1, 2, \cdots, n$, will denote these values.

3.1 The transform of a Product

Let us consider two functions $x, y \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and let $z(t) = x(t) y(t)$. The function z is in $L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and thus its n-dimensional fractional Fourier transform Z_α is obtained by the definition. The value of $Z_\alpha(u)$, expressing $x(t)$ in terms of its inverse n-dimensional fractional Fourier transform is given by [8] as follows:

$$Z_\alpha(u) = \left(\prod_{p=1}^{n} \frac{|\csc\alpha_p|}{\sqrt{2\pi}} e^{j\left(\frac{u_p^2}{2}\right)\cot\alpha_p} \right) \times \int_{\mathbb{R}^n} X_\alpha(v) \left(\prod_{p=1}^{n} e^{-j\left(\frac{v_p^2}{2}\right)\cot\alpha_p} \right) \times Y[(u - v)\csc\alpha] dv, \tag{3.1}$$

where Y is the n-dimensional Fourier transform of y.
Replacing v by $u - v$, in above equation, we get result in another form as follows.

$$Z_\alpha(u) = \left(\prod_{p=1}^{n} \frac{|\csc\alpha_p|}{\sqrt{2\pi}} \right) \int_{\mathbb{R}^n} X_\alpha(u - v) Y(v\csc\alpha) \times \left(\prod_{p=1}^{n} e^{j\left(u_p v_p - \left(\frac{v_p^2}{2}\right)\right)\cot\alpha_p} \right) dv, \tag{3.2}$$

We then make the further change $v\csc\alpha \to v$ or $v \to v\sin\alpha$ $\Rightarrow v_p\csc\alpha_p = v_p$ or $v_p \to v_p\sin\alpha_p \Rightarrow dv_p\csc\alpha_p = dv_p$ or $dv_p = dv_p\sin\alpha_p \Rightarrow dv\prod_{p=1}^{n}\csc\alpha_p = dv$ or $dv = dv\prod_{p=1}^{n}\sin\alpha_p$, resulting in

$$Z_\alpha(u) = \frac{1}{(2\pi)^{\left(\frac{n}{2}\right)}} \int_{\mathbb{R}^n} X_\alpha(u - v\sin\alpha) Y(v) \times \left(\prod_{p=1}^{n} e^{-j\left(\frac{v_p^2}{2}\right)\sin\alpha_p\cos\alpha_p + j u_p v_p \cos\alpha_p} \right) dv. \tag{3.3}$$

Equation (3.1)-(3.3)are valid if α_p is not multiple of π for all p, $p = 1, 2, \ldots, n$. Of course the role of x and y can be interchanged. These results can be extended in general form as in more general form subsection.

3.2 The transform of a convolution

Let us again take two functions $x, y \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, and their convolution $w(t) = (x * y)(t) = \int_{\mathbb{R}^n} x(u) y(t - u) du$ is in $L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$; then the n-dimensional Fourier transform of $w(t)$ is denoted by $W(w)$ and defined by $W(w) = (\sqrt{2\pi})^n X(w) Y(w)$, where X and Y are n-dimensional Fourier transform of $x(t)$ and $y(t)$ respectively in $L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. We know that W_α is the n-dimensional fractional Fourier transform of W, with angles $\alpha_1 - \frac{\pi}{2}, \alpha_2 - \frac{\pi}{2}, \ldots, \alpha_n - \frac{\pi}{2}$. We can therefore use (3.1) to obtain,

$$W_\alpha(u) = \left(\prod_{p=1}^{n} |\sec\alpha_p| e^{-j\left(\frac{u_p^2}{2}\right)\tan\alpha_p}\right) \int_{\mathbb{R}^n} X_\alpha(v)\times$$
$$y[(u-v)\sec\alpha]\left(\prod_{p=1}^{n} e^{j\left(\frac{v_p^2}{2}\right)\tan\alpha_p}\right)dv, \tag{3.4}$$

where y is inverse n-dimensional Fourier transform of Y. Replacing v by $u-v$, in above equation, we get,

$$W_\alpha(u) = \left(\prod_{p=1}^{n} |\sec\alpha_p|\right) \int_{\mathbb{R}^n} X_\alpha(u-v)y(v\sec\alpha)\times$$
$$\left(\prod_{p=1}^{n} e^{j\left[\frac{(v_p^2-2u_pv_p)}{2}\right]\tan\alpha_p}\right)dv. \tag{3.5}$$

Again put $v\sec\alpha = v$ or $v = v\cos\alpha \Rightarrow v_p\sec\alpha_p = v_p$ or $v_p = v_p\cos\alpha_p \Rightarrow dv_p\sec\alpha_p = dv_p$ or $dv_p = dv_p\cos\alpha_p$ $\Rightarrow dv(\prod_{p=1}^{n}\sec\alpha_p) = dv$ or $dv = dv(\prod_{p=1}^{n}\cos\alpha_p)$ obtaining

$$W_\alpha(u) = \int_{\mathbb{R}^n} X_\alpha(u-v\cos\alpha)y(v)\times$$
$$\left(\prod_{p=1}^{n} e^{j\left(\frac{v_p^2}{2}\right)\sin\alpha_p\cos\alpha_p - ju_pv_p\sin\alpha_p}\right)dv. \tag{3.6}$$

Equation (3.4)-(3.6) are valid if $\alpha_p - \frac{\pi}{2}$ is not a multiple of π for all p; $p = 1,2,\ldots,n$.

4. Product and convolution of *n*-dimensional Fractional Fourier transform

Let us define the following definition

Definition 4.1. *For any function $f \in \mathrm{L}^1(\mathbb{R}^n)\cap\mathrm{C}^1(\mathbb{R}^n)$, let us define the functions $\tilde{f}(x)$ and $\tilde{\tilde{f}}(x)$ by*
$\tilde{f}(x) = \left[\prod_{p=1}^{n} e^{ja_p(\alpha_p)x_p^2}\right] f(x)$ *and*
$\tilde{\tilde{f}}(x) = \left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)x_p^2}\right] f(x).$
For any two functions f and g, we define the convolution operation $\star$ by

$$(f\star g)(x) = \left[\prod_{p=1}^{n} \frac{c_p(\alpha_p)}{\sqrt{2\pi}} e^{-ja_p(\alpha_p)x_p^2}\right](\tilde{f}*\tilde{g})(x).$$

Where $*$ is the convolution operation for the Fourier transform as defined by definition 2.9. Likewise, we define the operation $\otimes$ by

$$(f\otimes g)(x) = \left[\prod_{p=1}^{n} \frac{e^{ja_p(\alpha_p)x_p^2}}{\sqrt{2\pi}}\right](\tilde{\tilde{f}}*\tilde{\tilde{g}})(x)$$

Theorem 4.2. *Let $h(x) = (f\star g)(x)$ and F_α, G_α and H_α denote the n-dimensional Fractional Fourier transform of f, g and h respectively. Then*

$$H_\alpha(u) = F_\alpha(u)G_\alpha(u)\prod_{p=1}^{n} e^{-ja_p(\alpha_p)u_p^2}. \tag{4.1}$$

Moreover

$$F_\alpha\left[f(x)g(x)\prod_{p=1}^{n} e^{ja_p(\alpha_p)x_p^2}\right](u)$$
$$= \prod_{p=1}^{n} c_p(-\alpha_p)(F_\alpha\otimes G_\alpha)(u). \tag{4.2}$$

Proof From the definition of the n-dimensional Fractional Fourier transform and first definition of convolution in the definition 4.1, we have

$$H_\alpha(u) = \int_{\mathbb{R}^n} h(t)\times$$
$$\prod_{p=1}^{n} \frac{c_p(\alpha_p)}{\sqrt{2\pi}} e^{j\left[a_p(\alpha_p)(t_p^2+u_p^2)-2a_p(\alpha_p)b_p(\alpha_p)u_pt_p\right]}dt;$$

where $u_p, t_p, \alpha_p \in \mathbb{R}^n$ with α_p is constant for $p = 1,2,\cdots n$ and
$dt = dt_1dt_2\cdots dt_n$.

$$H_\alpha(u)$$
$$= \int_{\mathbb{R}^n}\left[\prod_{p=1}^{n}\frac{c_p(\alpha_p)}{\sqrt{2\pi}}e^{-ja_p(\alpha_p)t_p^2}\right](\tilde{f}*\tilde{g})(t)\times$$
$$\left[\prod_{p=1}^{n}\frac{c_p(\alpha_p)}{\sqrt{2\pi}}e^{j\left[a_p(\alpha_p)(t_p^2+u_p^2)-2a_p(\alpha_p)b_p(\alpha_p)u_pt_p\right]}\right]dt$$
$$= \left[\prod_{p=1}^{n}\frac{c_p^2(\alpha_p)}{2\pi}\right]\times$$
$$\int_{\mathbb{R}^n}\left[\prod_{p=1}^{n}e^{j\left[a_p(\alpha_p)u_p^2-2a_p(\alpha_p)b_p(\alpha_p)u_pt_p\right]}\right](\tilde{f}*\tilde{g})(t)dt$$
$$= \left[\prod_{p=1}^{n}\frac{c_p^2(\alpha_p)}{2\pi}\right]\times$$
$$\int_{\mathbb{R}^n}\left[\prod_{p=1}^{n}e^{j\left[a_p(\alpha_p)u_p^2-2a_p(\alpha_p)b_p(\alpha_p)u_pt_p\right]}\right]dt\int_{\mathbb{R}^n}f(x)\times$$
$$\left[\prod_{p=1}^{n}e^{ja_p(\alpha_p)x_p^2}\right]g(t-x)\left[\prod_{p=1}^{n}e^{ja_p(\alpha_p)(t_p-x_p)^2}\right]dx$$
$$= \left[\prod_{p=1}^{n}\frac{c_p^2(\alpha_p)}{2\pi}\right]\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x)g(t-x)\times$$
$$\prod_{p=1}^{n} exp\left\{j\left[a_p(\alpha_p)(u_p^2+x_p^2+t_p^2+x_p^2-2t_px_p)\right.\right.$$
$$\left.\left.-2a_p(\alpha_p)b_p(\alpha_p)u_pt_p\right]\right\}dxdt.$$

By making the change of the variable, $t-x=v$
i.e. $(t_1-x_1, t_2-x_2,\cdots,t_n-x_n) = (v_1,v_2,\cdots,v_n)$

i.e. $t_p - x_p = v_p, \forall p = 1,2,\cdots,n.$
Or $t_p = v_p + x_p, \forall p = 1,2,\cdots,n.$
$\Rightarrow dt_p = dv_P, \forall p = 1,2,\cdots,n$
$\Rightarrow dt = dt_1 dt_2 \cdots dt_n = dv_1 dv_2 \cdots dv_n = dv.$
Therefore

$$\begin{aligned}
&H_\alpha(u)\\
&=\left[\prod_{p=1}^{n}\frac{c_p^2(\alpha_p)}{2\pi}\right]\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(v)\times\\
&\prod_{p=1}^{n} e^{j\left[a_p(\alpha_p)(u_p^2+x_p^2+v_p^2)-2a_p(\alpha_p)b_p(\alpha_p)u_p(x_p+v_p)\right]}dxdv\\
&=\left[\prod_{p=1}^{n}\frac{c_p^2(\alpha_p)}{2\pi}e^{-ja_p(\alpha_p)u_p^2}\right]\\
&\int_{\mathbb{R}^n} f(x)\prod_{p=1}^{n} e^{j\left[a_p(\alpha_p)(u_p^2+x_p^2)-2a_p(\alpha_p)b_p(\alpha_p)u_p x_p\right]}dx\\
&\int_{\mathbb{R}^n} g(v)\prod_{p=1}^{n} e^{j\left[a_p(\alpha_p)(u_p^2+v_p^2)-2a_p(\alpha_p)b_p(\alpha_p)u_p v_p\right]}dv\\
&=\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)u_p^2}\right]F_\alpha(u)G_\alpha(u)
\end{aligned}$$

and this complete the proof of equation (4.1). As for equation (4.2) we have from second definition of convolution in the definition 4.1.

$$\begin{aligned}
&(F_\alpha\otimes G_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}}{\sqrt{2\pi}}\right](\tilde{\tilde{F}}_\alpha * \tilde{\tilde{G}}_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}}{\sqrt{2\pi}}\right]\int_{\mathbb{R}^n}\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)x_p^2}\right]F_\alpha(x)\\
&\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)(u_p-x_p)^2}\right]G_\alpha(u-x)dx.
\end{aligned}$$

We know that

$$\begin{aligned}
F_\alpha(x) = &\left[\prod_{p=1}^{n}\frac{c_p(\alpha_p)}{\sqrt{2\pi}}\right]\int_{\mathbb{R}^n} f(z)\\
&\prod_{p=1}^{n} e^{j\left[a_p(\alpha_p)(z_p^2+x_p^2)-2a_p(\alpha_p)b_p(\alpha_p)z_p x_p\right]}dz;
\end{aligned}$$

where $z = (z_1, z_2, \cdots, z_n)$.

Therefore

$$\begin{aligned}
&(F_\alpha\otimes G_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}}{\sqrt{2\pi}}\right]\int_{\mathbb{R}^n}\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)(x_p^2+(u_p-x_p)^2)}\right]\times\\
&\left\{\left[\prod_{p=1}^{n}\frac{c_p(\alpha_p)}{\sqrt{2\pi}}\right]\int_{\mathbb{R}^n} f(z)\prod_{p=1}^{n} exp\left\{j\left[a_p(\alpha_p)(z_p^2+x_p^2)\right.\right.\right.\\
&\left.\left.\left.-2a_p(\alpha_p)b_p(\alpha_p)z_p x_p\right]\right\}dz\right\}G_\alpha(u-x)dx\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}c_p(\alpha_p)}{2\pi}\right]\int_{\mathbb{R}^n} f(z)\left[\int_{\mathbb{R}^n} G_\alpha(u-x)\times\right.\\
&\left.\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)\left[(u_p-x_p)^2-z_p^2+2b_p(\alpha_p)x_p z_p\right]}\right]dx\right]dz.
\end{aligned}$$

By making the change of variable, $u - x = v \Rightarrow x = u - v$.
That is $(x_1, x_2, \cdots, x_n) = (u_1 - v_1, u_2 - v_2, \cdots, u_n - v_n)$.
Or $x_p = u_p - v_p, \forall\ p = 1,2,\cdots,n$
$\Rightarrow dx_p = -dv_p, \forall\ p = 1,2,\cdots,n.$
Therefore

$$\begin{aligned}
&(F_\alpha\otimes G_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}c_p(\alpha_p)}{2\pi}\right]\int_{\mathbb{R}^n} f(z)\left[\int_{\mathbb{R}^n} G_\alpha(v)\times\right.\\
&\left.\left[\prod_{p=1}^{n} e^{-ja_p(\alpha_p)\left[v_p^2-z_p^2+2b_p(\alpha_p)(u_p-v_p)z_p\right]}\right]dv\right]dz\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}c_p(\alpha_p)}{c_p(-\alpha_p)\sqrt{2\pi}}\right]\times\\
&\int_{\mathbb{R}^n}\left[\prod_{p=1}^{n} e^{2ja_p(\alpha_p)\left[z_p^2-b_p(\alpha_p)u_p z_p\right]}\right]f(z)\times\\
&\left\{\left[\prod_{p=1}^{n}\frac{c_p(-\alpha_p)}{\sqrt{2\pi}}\right]\int_{\mathbb{R}^n} G_\alpha(v)\times\right.\\
&\left.\prod_{p=1}^{n} e^{-ja_p(\alpha_p)\left[(v_p^2+z_p^2)-2b_p(\alpha_p)v_p z_p\right]}dv\right\}dz.
\end{aligned}$$

Which, in view of the inversion formula for the n-dimensional Fractional Fourier transform,can be reduced to

$$\begin{aligned}
&(F_\alpha\otimes G_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{e^{ja_p(\alpha_p)u_p^2}c_p(\alpha_p)}{c_p(-\alpha_p)\sqrt{2\pi}}\right]\int_{\mathbb{R}^n} f(z)g(z)\times\\
&\prod_{p=1}^{n} exp\left\{2ja_p(\alpha_p)\left[z_p^2-b_p(\alpha_p)u_p z_p\right]\right\}dz
\end{aligned}$$

Which iplies that

$$
\begin{aligned}
&(F_\alpha \otimes G_\alpha)(u)\\
&=\left[\prod_{p=1}^{n}\frac{c_p(\alpha_p)}{c_p(-\alpha_p)\sqrt{2\pi}}\right]\int_{\mathbb{R}^n}\left[f(z)g(z)\prod_{p=1}^{n}e^{ja_p(\alpha_p)z_p^2}\right]\\
&\qquad\left[\prod_{p=1}^{n}e^{ja_p(\alpha_p)\left[z_p^2+u_p^2-2b_p(\alpha_p)u_pz_p\right]}\right]dz\\
&=\left[\prod_{p=1}^{n}\frac{1}{c_p(-\alpha_p)}\right]F_\alpha\left[f(z)g(z)\prod_{p=1}^{n}e^{ja_p(\alpha_p)z_p^2}\right](u)
\end{aligned}
$$

Therefore

$$
F_\alpha\left[f(z)g(z)\prod_{p=1}^{n}e^{ja_p(\alpha_p)z_p^2}\right](u)=\left[\prod_{p=1}^{n}c_p(-\alpha_p)\right]\times (F_\alpha \otimes G_\alpha)(u).
$$

Remark 4.3. *In this paper we have defined the kernel and integral transform for those* $\alpha=(\alpha_1,\ \alpha_2,\ \cdots,\alpha_n)\in\mathbb{R}^n$ *such that all* $\alpha_i's$ *are not multiple of* π*. We could not consider the case where all* $\alpha_i's$ *or some of* $\alpha_i's$ *are multiple of* π*.*

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