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Non-adjacent vertex sum polynomial of Umbrella graphs, Jahangir graph, Tadpole graph and Lollipop graph

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Abstract
Let $G=(V,E)$ be a graph. The non-adjacent vertex sum polynomial of the graph $G=(V,E)$ is defined as $NAVSP(G,x)=\sum_{j=0}^{\triangle(G)} n_{(\triangle(G)-j)}x^{\alpha_{\triangle(G)-j}}$ where $n_{\triangle(G)-j}$ is the sum of the number of non-adjacent vertices of all the vertices of degree $\triangle(G)-j$ and $\alpha_{\triangle(G)-j}$ is the sum of the degree of non-adjacent vertices of the vertices of degree $\triangle(G)-j$. In this paper we derived the non-adjacent vertex sum polynomial for Umbrella graph, Jahangir graph, Tadpole graph and Lollipop graph.

Keywords
Umbrella graph, Jahangir graph, Tadpole graph and Lollipop graph.

AMS Subject Classification
05C31.

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1. Introduction

In a graph $G=(V,E)$ we mean a finite undirected, non-trivial graph without loops and multiple edges. The vertex set is denoted by V and the edge set by E, for $v \in V$, $d(v)$ is the number of edges incident with v, the maximum degree of G is defined as $\triangle(G)=max\{d(v)/v \in V\}$ for terms not defined here, we refer to Frank Harary [3]. An Umbrella graph $U_{m,n}$ is the graph obtained by joining a path P_n with a central vertex of a fan F_m.[7]. Jahangir graph $J_{s,m}$ for $m \geq 3$ is a graph on $sm+1$ vertices that is a graph consisting of a cycle C_{sm} with one additional vertex which is adjacent to m vertices of C_{sm} at a distance $'s'$ to each other an C_{sm} [6]. A Tadpole $T_{n,k}$ is the graph obtained by appending a path P_k to cycle C_n [7]. A Lollipop graph denoted by $L_{m,n}$ is a graph which is constructed by appending a complete graph K_m, $m \geq 3$ to a pendent vertex of path graph P_n.

2. Main Results

Theorem 2.1. *Let $U_{m,n}$ be a Umbrella graph with $m+n$ vertices. Then the non-adjacent vertex sum polynomial of $U_{m,n}$ is* $NAVSP(U_{m,n},x)=$

$$\begin{cases} (2n-2)x^{6n-7}+(2n-3)x^{6n-9}+(2n^2-11n+12)x^{6n-10} \\ +(2n-3)x^{5n-9}+(4n-6)x^{5n-10}+(4n-8)x^{5n-13}+ \\ (2n^2-12n+16)x^{5n-14}+(n-2)x^{2n-5} \\ if\ m,n(m\geq 5, n\geq 5)\ are\ odd\ and \\ if m,n(m\geq 6, n\geq 6)\ are\ even \\ (m+n-2)x^{6m-5}+(m+n-3)x^{6m-7}+ \\ (2n^2-12n+16)x^{6m-8}+(m+n-3)x^{5m-7} \\ +(4n-8)x^{5m-8}+(4n-10)x^{5m-11}+ \\ (2n^2-15n+25)x^{5m-12}+(n-2)x^{2m-3} \\ if m(m\geq 6)\ is\ even\ and\ if\ n(n\geq 7)\ is\ odd \\ (m+n-2)x^{6m-9}+(m+n-3)x^{6m-11}+ \\ (2n^2-10n+8)x^{6m-12}+(m+n-3)x^{5m-11} \\ +(4n-4)x^{5m-12}+(4n-6)x^{5m-15}+ \\ (2n^2-9n+9)x^{5m-16}+(n-2)x^{2m-7} \\ if m(m\geq 7)\ is\ odd\ and\ if\ n(n\geq 6)\ is\ even \end{cases}$$

Proof. Let $V=\{\{u_i,v_j/1\leq i\leq m, 1\leq j\leq n\}$ be the vertex set and $E = \{u_i u_{i+1}, v_j v_{j+1} / 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_1 u_i / 1 \leq i \leq m\}$ be the edge set of the graph $U_{m,n}$. Then $U_{m,n}$ has $m+n$ vertices and $2m+n-2$ edges.

Case(i) Both m $(m \geq 5)$ and n $(n \geq 5)$ are odd.

The total number of non-adjacent vertices of v_n is $2n-2$, v_{n-1} is $2n-3$,$v_3, v_4, ..., v_{n-2}$ is $2n^2$-11n+12, v_2 is $2n-3$, u_1,u_m is $4n-6$, u_2, u_{m-1} is 4n-8,u_3,$u_4, ..., u_{m-2}$ is $2n^2-12n+16$ and v_1 is $n-2$ and sum of degree of these non-adjacent vertices is $6n-7, 6n-9, 6n-10, 5n-9, 5n-10, 5n-13, 5n-14$ and $2n-5$ respectively.

Case (ii) Both m $(m \geq 6)$ and n $(n \geq 6)$ are even.

The result is similar to case (i).

Case (iii) m $(m \geq 6)$ is even and n $(n \geq 7)$ is odd.

The total number of non-adjacent vertices of v_n is $m+n-2$, v_{n-1} is $m+n-3$,$v_3, v_4, ..., v_{n-2}$ is $2n^2-12n+16$, v_2 is $m+n-3, u_1, u_m$ is $4n-8, u_2, u_{m-1}$ is $4n-10, u_3, u_4, ..., u_{m-2}$ is $2n^2-15+25$ and v_1 is $n-2$ and sum of degree of these non-adjacent vertices is $6m-5, 6m-7, 6m-8, 5m-7, 5m-8, 5m-11, 5m-12$ and $2m-3$ respectively.

Case (iv) m $(m \geq 7)$ is odd and $(n \geq 6)$ is even.

The total number of non-adjacent vertices of v_n is $m+n-2$, v_{n-1} is $m+n-3$, $v_3, v_4, ..., v_{n-2}$ is $2n^2-10n+8$, v_2 is $m+n-3, u_1, u_m$ is $4n-4$,$u_2 u_{m-1}$ is $4n-6$,$u_3, u_4, ..., u_{m-2}$ is $2n^2-9n+9$ and v_1 is $n-2$ and sum of degree of these non-adjacent vertices is $6m-9, 6m-11, 6m-12, 5m-11, 5m-12, 5m-15, 5m-16$ and $2m-7$ respectively. Combining all these we get the required polynomial. ☐

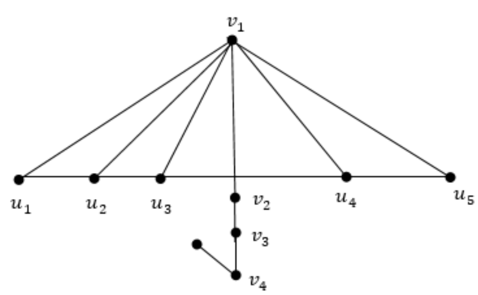

Figure 1

Here $NAVSP(U_{5,5}, x) = 8x^{23} + 7x^{21} + 7x^{20} + 77x^{16} + 14x^{15} + 12x^{12} + 6x^{11} + 3x^5$.

Example 2.2. *The non-adjacent vertex sum polynomial of the Umbrella $u_{5,5}$,$u_{6,7}$ and $u_{7,6}$ are given in Figure 1, Figure 2 and Figure 3 respectively.*

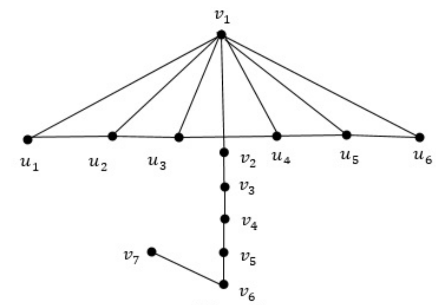

Figure 2

Here $NAVSP(U_{6,7}, x) = 11x^{31} + 10x^{29} + 30x^{28} + 10x^{23} + 20x^{22} + 18x^{19} + 18x^{18} + 5x^9$.

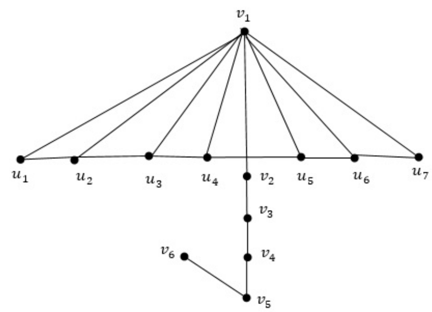

Figure 3

Here $NAVSP(U_{7,6}, x) = 11x^{33} + 10x^{31} + 20x^{30} + 10x^{24} + 20x^{23} + 18x^{20} + 27x^{19} + 4x^7$.

Theorem 2.3. *Let J_{sm} be a Jahangir graph with $sm+1$ vertices. Then the non-adjacent vertex sum polynomial of J_{sm} is*

$$NAVSP(J_{sm}, x)) = \begin{cases} (s^4-3s^3-2s^2+6s)x^{2s^2+2s-6} + (2s^3-4s)x^{2s^2+2s-7} + \\ (s^3-3s)x^{2s^2+s-7} + (s^2-s)x^{2(s^2-s)} if\ s, m\ is\ even \\ (s \geq 4, m \geq 4)\ if\ s, m\ is\ odd \\ (s \geq 5, m \geq 5) \\ (s^4+4s-37)(m^2+m-2)x^{2m^2+4m-6} + \\ (2s-2)(m^2+m-2)x^{2m^2+4m-7} + (s^2-s-3)mx^{2m^2+3m-7} \\ +(s^2-2s+1)x^{2m^2} \\ if\ s(s \geq 5)\ is\ odd\ and\ if\ m(m \geq 4)\ is\ even \\ (s^2-2s-3)(m^2-m-2)x^{2m^2-6} \\ +(2s+2)(m^2-m-2)x^{2m^2-7} \\ +(s^2+s-3)mx^{2m^2-m-7} + (2s^2-17)x^{2m^2-4m} \\ if\ s(s \geq 4)\ is\ even\ and\ if\ m\ (m \geq 5)\ is\ odd \end{cases}$$

Proof. Let J_{sm} be a Jahangir graph with $sm+1$ vertices namely $v_1, v_2, ..., v_{sm}, v_{sm+1}$. In a Jahangir graph J_{sm}, m vertices have degree 3, $d(v_{sm+1}) = m$ and the remaining vertices have degree 2.

Case (i) Both s $(s \geq 5)$ and m $(m \leq 5)$ are odd.

The total number of non-adjacent vertices of degree 3 vertices is s^2-3s, those vertices adjacent to degree 3 vertices except v_{sm+1} is $2s^2-3s$,v_{sm+1} is s^2-s and the remaining vertices in the cycle is $(s^4-3s^3-2s^2+6s)$ and sum of degree of these non-adjacent vertices is $2s^2+s-7$, $2s^2+2s-7$, $2(s^2-s)$ and $2s^2+2s-6$ respectively.

Case (ii) Both s $(s \geq 4)$ and m $(m \geq 4)$ are even. This proof is similar to case (i).

Case (iii) $s(s \geq 5)$ is odd and $m(m \geq 4)$ is even.

The total number of non-adjacent vertices of degree 3 vertices is $(s^2-s-3)m$, those vertices adjacent to degree 3 vertices except v_{sm+1} is $(2s-2)(m^2+m-2)$, v_{sm+1} is (s^2-2s+1), and the remaining inner vertices in the cycle is $(s^2+4s-37)$ (m^2+m-2) and sum of degree of these non-adjacent vertices is $2m^2+3m-7$, $2m^2+4m-7$, $2m^2$ and $2m^2+4m-6$ respectively.

Case (iv) s $(s \geq 4)$ is even and m $(m \geq 5)$ is odd.

The total number of non-adjacent vertices of degree 3 vertices is $m(s^2+s-3)$, those vertices adjacent to degree 3 vertices except v_{sm+1} is $(2s+2)(m^2-m-2)$, v_{sm+1} is $(2s^2-17)$ and the remaining inner vertices in the cycle is (s^2-2s-3) (m^2-m-2) and sum of degree of these non-adjacent vertices is $2m^2-m-7$, $2m^2-7$, $2m^2-4m$ and $2m^2-6$ respectively. □

Example 2.4. *Put $s=5, n=4$ in the above theorem we have the graph.*

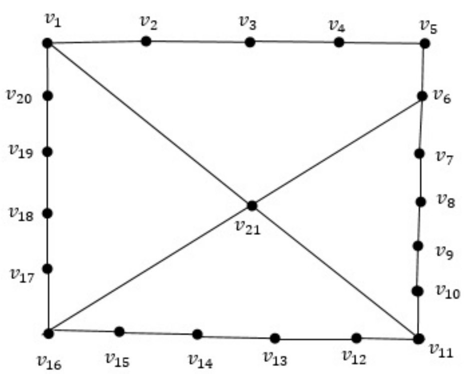

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Figure 4

Here $NAVSP(J_{5,4},x) = 144x^{42}+144x^{41}+68x^{37}+16x^{32}$.

Theorem 2.5. *Let $T_{n,k}$ be a Tadpole graph with $n+k$ vertices. Then the non-adjacent vertex sum polynomial of $T_{n,k}$ is* $NAVSP(T_{n,k},x) =$

$$\begin{cases} (2n-4)x^{4k+1}+(2n-5)x^{4k-1}+(2n-8)(2k-1)x^{4k-2} \\ +3(2n-5)x^{4k-3}+(2n-6)x^{4k-5} \\ if\, n(n\geq 6), k(k\geq 4) \text{ are even} \\ \text{and} \quad if\ n(n\geq 5), k(k\geq 3) \text{ are odd} \\ (2n-3)x^{4k-1}+(2n-4)x^{4k-3}+(2n-7)(2k-2)x^{4k-4} \\ +3(2n-4)x^{4k-5}+(2n-5)x^{4k-7} \\ if\, n(n\geq 5 \text{ is odd and } if\ k(k\geq 4) \text{ is even} \\ (2n-5)x^{4k+3}+(2n-6)x^{4k+1}+(2n-9)2kx^{4k} \\ +3(2n-6)x^{4k-1}+(2n-7)x^{4k-3} \\ if \quad n(n\geq 6) \text{ is even and } if\ k(k\geq 3) \text{ is odd} \end{cases}$$

Proof. Let $V=\{\{u_i,v_j/1\leq i\leq n, 1\leq j\leq k\}$ be the vertex set and $E=\{u_iu_{i+1},v_jv_{j+1}/1\leq i\leq n-1, 1\leq j\leq k-1\}\cup\{u_1v_1\}$ be the edge set of the graph $T_{n,k}$. Then $T_{n,k}$ has $n+k$ vertices and $n+k$ edges.

Case (i) Both n $(n\geq 6)$ and k $(k\geq 4)$ are even.

The total number of non-adjacent vertices of v_k is $(2n-4)$, v_{k-1} is $(2n-5), v_2, v_3, \dots, v_{k-2}$ and $u_3,u_4,\dots,u_{n-1}$ is $(2n-8)(2k-1)$, u_2,u_n and v_1 is $3(2n-5)$ and u_1 is $2n-6$ and sum of degree of these non-adjacent vertices is $4k+1, 4k-1, 4k-2, 4k-3$ and $4k-5$ respectively.

Case (ii) Both $n(n\geq 5)$ and $k(k\geq 3)$ are even. The result is similar to Case (i).

Case (iii) $n(n\geq 5)$ is odd and k $(k\geq 4)$ is even.

The total number of non-adjacent vertices of v_k is $2n-3, v_{k-1}$ is $(2n-4)$,$v_2,v_3,\dots,v_{k-2}$ and $u_3,u_4,\dots,u_{n-1}$ is $(2n-7)(2k-2), u_2, u_n, v_1$ is $3(2n-4)$ and u_1 is $2n-5$ and sum of degree of these non-adjacent vertices is $4k-1, 4k-3, 4k-4, 4k-5$ and $4k-7$ respectively.

Case (iv) n $(n\geq 6)$ is even and k $(k\geq 3)$ is odd. The total number of non-adjacent vertices of v_k is $(2n-5)$,v_{k-1} is $2n-6$,$v_2,v_3,\dots,v_{k-2}$ and $u_3,u_4,\dots,u_{n-1}$ is $2k(2n-9)$, u_2,u_n,v_1 is $3(2n-6)$ and u_1 is $2n-7$ and sum of degree of these non-adjacent vertices is $4k+3, 4k+1, 4k, 4k-1$ and $4k-3$ respectively. □

Example 2.6. *Take $n=6, k=4$ in the above theorem we have the graph.*

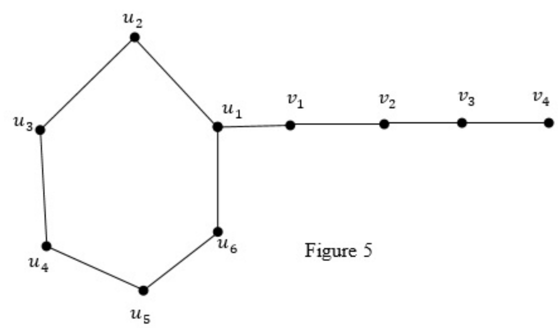

Figure 5

Here $NAVSP(T_{6,4},x) = 8x^{17}+7x^{15}+28x^{14}+21x^{13}+6x^{11}$

Theorem 2.7. *Let $L_{m,n}$ be a Lollipop graph with $m+n$ vertices. Then the non-adjacent vertex sum polynomial of $L_{m,n}$ is* $NAVSP(L_{m,n},x) =$

$$\begin{cases} (2m-4)x^{n^2+5n-1}+(2m-5)x^{n^2+5n-3}+(m-5)(2n-1) \\ x^{n^2+5n-4}+(2m-5)x^{n^2+4n-4}+(m-1)nx^{2n-1}+(m-3)x^{2n-3} \\ if\ m(m\geq 7), n(n\geq 5) \text{ are odd and} \\ if\ m(m\geq 6), n(n\geq 4) \text{ are even} \\ (2m-5)x^{n^2+7n+3}+(2m-6)x^{n^2+7n+1}+2(m-6)nx^{n^2+7n} \\ +(2m-6)x^{n^2+6n-1}+(m-1)nx^{2n-1}+(m-4)x^{2n-3} \\ if m(m\geq 8) \text{ is even and } if\ n(n\geq 5) \text{ is odd} \\ (2m-3)x^{n^2+3n-3}+(2m-4)x^{n^2+3n-5} \\ +(m-4)(2n-2)x^{n^2+3n-6}+(2m-4)x^{n^2+2n-5} \\ +(m-1)nx^{2n-1}+(m-2)x^{2n-3} \\ if\ m(m\geq 7) \text{ is odd and } if\ n(n\geq 6) \text{ is even} \end{cases}$$

Proof. Let $V=\{\{u_i,v_j/1\leq i\leq m, 1\leq j\leq n\}$ be the vertex set and $E = \{\{u_iu_{i+1}, v_jv_{j+1}/1 \le i \le m-1, 1 \le j \le n-1\} \cup \{u_1v_1\}$ be the edge set of the graph $L_{m,n}$. Then $L_{m,n}$ has $m+n$ vertices and $\binom{m}{2}+n$ edges.

Case (i) Both m $(m \ge 6)$ and $n(n \ge 4)$ are even.

The total number of non-adjacent vertices of v_n is $(2m-4)$,v_{n-1} is $2m-5, v_2, v_3, ..., v_{n-2}$ is $(m-5)(2n-1)$, v_1 is $(2m-5), u_2, u_3, ..., u_m$ is $n(m-1)$ and u_1 is $(m-3)$ and sum of degree of these non-adjacent vertices is n^2+5n-1, n^2+5n-3, n^2+5n-4, n^2+4n-4, $2n-1$ and $2n-3$ respectively.

Case (ii) Both $m(m \ge 7)$ and $n(n \ge 5)$ are odd. The result is similar to case(i).

Case (iii) m $(m \ge 8)$ is even and n $(n \ge 5)$ are odd.

The total number of non-adjacent vertices of v_n is $(2m-5), v_{n-1}$ is $2m-6, v_2, v_3, ..., v_{n-2}$ is $2n(m-6), v_1$ is $2m-5, u_3, u_4, ..., u_m$ is $n(m-1)$ and u_1 is $(m-4)$ and sum of degree of these non-adjacent vertices is $n^2+7n+3, n^2+7n+1, n^2+7n, n^2+6n-1, 2n-1$ and $2n-3$ respectively.

Case (iv) m $(m \ge 7)$ is odd and n $(n \ge 6)$ is even.

The total number of non-adjacent vertices of v_n is $(2m-3), v_{n-1}$ is $(2m-4), v_2, v_3, ..., v_{n-2}$ is $(m-4)(2n-2), v_1$ is $(2m-4), u_2, u_3, ..., u_m$ is $n(m-1)$ and u_1 is $m-2$ and sum of degree of these non-adjacent vertices is $n^2+3n-3, n^2+3n-5, n^2+3n-6, n^2+2n-5, 2n-1$ and $2n-3$ respectively. □

Example 2.8. *Take m=6,n=4 in the above theorem we have the graph.*

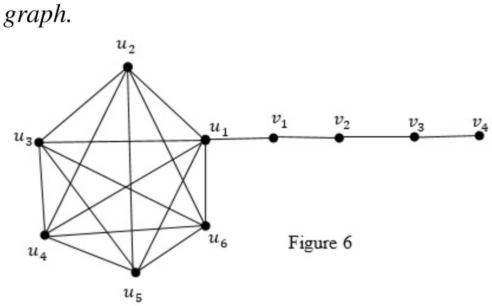

Figure 6

Here

$$NAVSP(L_{6,4}, x) = 8x^{35} + 7x^{33} + 7x^{32} + 7x^{28} + 20x^7 + 3x^5$$

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