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Functional inequalities for a generalized quadratic functional equation in various Banach spaces

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Abstract
In this paper, we study the stability of a generalized quadratic functional equation in sense of Ulam, Hyers and Rassias in Banach spaces, Quasi-β-2-Banach spaces and intuitionistic fuzzy-2-Banach spaces via two alternate methods.

Keywords
Quadratic functional equation, Generalized Hyers-Ulam stability, Banach space, Quasi-β-2-Banach space, intuitionistic fuzzy-2-Banach space, Hyers method, fixed point method.

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Contents

1. Introduction and Preliminaries

In 1940 Ulam [50] proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In 1941, Hyers [23] gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution were established in [3, 22, 38, 41, 43].

The general solution and generalized Ulam-Hyers stability of the following quadratic functional equations

$$q(x+y)+q(x-y)=2q(x)+2q(y) \tag{1.1}$$

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(x+z), \tag{1.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z) = f(x-y)+f(y+z)+f(z-x), \tag{1.3}$$

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 3f(x)+3f(y)+3f(z), \tag{1.4}$$

were investigated in [1, 7, 28, 29, 32]. Moreover, the stability of several other quadratic functional equations in the sense of Hyers, Ulam, Rassias were discussed in [13–15, 17, 18, 30, 42, 49].

In this paper, the authors introduce and establish the generalized Ulam-Hyers stability of a generalized quadratic functional equation

$$q(v+\eta w)-2q(v+(\eta-1)w)+q(v+(\eta-2)w)=2!\ q(w) \tag{1.5}$$

where $\eta \geq 1$ in Banach spaces, Quasi-β-2-Banach spaces and intuitionistic fuzzy-2-Banach spaces via two alternate methods.

Here, we provide the solution of the functional equation (1.5).

Theorem 1.1. *Assume A and B are real vector spaces. Suppose $q : A \to B$ is a function satisfying functional equation (1.5). Then the following assertions hold.*

(i) $q(0) = 0$;

(ii) $q(-w) = q(w)$, *that is q is even;*

(iii) $q(2w) = 4q(w)$.

Proof. Given $q : A \to B$ is a function satisfying equation (1.5). The proof of (i) follows by replacing (v,w) by $(0,0)$ in (1.5). Changing (v,w) by $(-\eta w, w)$ in (1.5), we have

$$q(-2w) = 2\,q(w) + 2q(-w), \quad \forall \quad w \in A. \tag{1.6}$$

Replacing w by $-w$ in (1.6), we arrive

$$q(2w) = 2\,q(-w) + 2q(w), \quad \forall \quad w \in A. \tag{1.7}$$

The proof of (ii) follows by comparing (1.6) and (1.7) with replacing w by $w/2$. The proof of (iii) follows by using (ii) in (1.6). □

Theorem 1.2. *Let A and B be real vector spaces. Suppose $q : A \to B$ is a function satisfying the functional equation (1.1) for all $x, y \in A$. Then $q : A \to B$ is a function satisfying equation (1.5), for all $v, w \in A$.*

Proof. Given $q : A \to B$ which is a function satisfying functional equation (1.1). Replacing (x,y) by $(0,0)$ in (1.1), we get $q(0) = 0$. Changing (x,y) by $(-0,w)$ in (1.1), we have $q(-w) = q(w)$ for all $w \in A$. In addition, switching (x,y) by (w,w) and $(2w,w)$ in (1.1), we arrive $q(2w) = 4q(w)$ and $q(3w) = 9q(w)$ for all $w \in A$. In general, for any positive integer a, we have $q(aw) = a^2 q(w)$ for all $w \in A$. Interchanging (x,y) by $(w, v + (\eta-1)w)$ in (1.1), we get

$$\begin{aligned} &q(w + (v + (\eta-1)w)) + q(w - (v + (\eta-1)w)) \\ &= 2q(w) + 2q(v + (\eta-1)w), \end{aligned} \tag{1.8}$$

for all $v, w \in A$. Using evenness of q, the above equation can be rewritten as

$$q(v + \eta w) + q(v + (\eta-2)w) = 2q(w) + 2q(v + (\eta-1)w), \tag{1.9}$$

for all $v, w \in A$. Rearrange the above equation we arrive our desired result. □

2. Stability Results In Banach Space

In this section, the generalized Ulam-Hyers stability of e functional equation (1.5) is discussed in Banach space using direct and fixed point methods.

2.1 Direct Method

Theorem 2.1. *Let $\delta \in \{-1, 1\}$. Assume A is a normed space and B is a Banach space. Suppose $\varpi : A^2 \to [0,\infty)$ and $q : A \to B$ are functions satisfying the condition and the inequality*

$$\lim_{\alpha\to\infty} \frac{\varpi\left(2^{\alpha\delta}v, 2^{\alpha\delta}w\right)}{4^{\alpha\delta}} = 0 \tag{2.1}$$

and

$$\begin{aligned} \Big\| &q(v + \eta w) - 2q(v + (\eta-1)w) + q(v + (\eta-2)w) \\ &- 2!\ q(w)\Big\| \le \varpi(v,w) \end{aligned} \tag{2.2}$$

for all $v, w \in A$. Then there exists a unique quadratic mapping $\mathscr{Q} : A \to B$ satisfying the functional equation (1.5) and the inequality

$$\|q(w) - \mathscr{Q}(w)\| \le \frac{1}{4}\sum_{i=\frac{1-\delta}{2}}^{\infty} \frac{\varpi(-\eta\cdot 2^{i\delta}w, 2^{i\delta}w)}{4^{i\delta}} \tag{2.3}$$

for all $w \in A$.

Proof. First, we give proof for $\delta = 1$. Interchanging (v,w) by $(-\eta w, w)$ in (2.2) and using evenness of q, one can find that

$$\|q(2w) - 4\,q(w)\| \le \varpi(-\eta w, w) \tag{2.4}$$

$$\left\|\frac{q(2w)}{4} - q(w)\right\| \le \frac{\varpi(-\eta w, w)}{4} \tag{2.5}$$

for all $w \in A$. Again, changing w by $2w$ and dividing by 4 in (2.5), one can observe that

$$\left\|\frac{q(2^2w)}{4^2} - \frac{q(2w)}{4}\right\| \le \frac{\varpi(-\eta\cdot 2w, 2w)}{4^2} \tag{2.6}$$

for all $w \in A$. Combining (2.5) and (2.6), one can arrive

$$\left\|\frac{q(2^2w)}{4^2} - q(w)\right\| \le \frac{1}{4}\left[\varpi(-\eta w, w) + \frac{\varpi(-\eta\cdot 2w, 2w)}{4}\right] \tag{2.7}$$

for all $w \in A$. In general for any positive integer α, one can verify that

$$\left\|\frac{q(2^\alpha w)}{4^\alpha} - q(w)\right\| \le \frac{1}{4}\sum_{i=0}^{\alpha-1} \frac{\varpi\left(-\eta\cdot 2^i w, 2^i w\right)}{4^i} \tag{2.8}$$

for all $w \in A$. Replacing w by $2^\beta w$ and dividing by 4^β in (2.8), for any $\beta, \alpha > 0$ and letting β tends to infinity we obtain the sequence $\left\{\frac{q(2^\alpha w)}{4^\alpha}\right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $\mathscr{Q} : A \to B$ such that

$$\mathscr{Q}(w) = \lim_{\alpha\to\infty} \frac{q(2^n w)}{4^\alpha}, \ \forall\ w \in A.$$

Letting $\alpha \to \infty$ in (2.8), we see that (2.3) holds for all $w \in A$ for $\delta = 1$. If we changing (v,w) by $(2^\alpha v, 2^\alpha w)$ and dividing by 4^α in (2.2), letting $\alpha \to \infty$ and using the definition of $\mathscr{Q}(w)$ one can see that $\mathscr{Q}$ satisfies (1.5) for all $v,w \in A$. In order to show $\mathscr{Q}$ is unique, let $\mathscr{R}$ be another quadratic mapping satisfying (1.5) and (2.3). Thus,

$$\begin{aligned}
&\|\mathscr{Q}(w)-\mathscr{R}(w)\| \\
&= \frac{1}{4^\beta}\left\|\mathscr{Q}(2^\beta w)-\mathscr{R}(2^\beta w)\right\| \\
&\le \frac{1}{4^\beta}\left\{\left\|Q(2^\beta w)-q(2^\beta w)\right\|+\left\|q(2^\beta w)-R(2^\beta w)\right\|\right\} \\
&\le \sum_{i=0}^{\infty}\frac{2\varpi\left(-\eta\cdot 2^{i+\beta}w, 2^{i+\beta}w\right)}{4^{i+\beta}} \to 0 \ \ as\ \beta\to\infty
\end{aligned}$$

for all $w \in A$. Hence, $\mathscr{Q}$ is unique. Thus the theorem holds for $\delta = 1$. Now, we give proof for $\delta = -1$. Replacing w by $w/2$ in (2.4), we get

$$\left\|q(w)-4\,q\left(\frac{w}{2}\right)\right\| \le \varpi\left(-\eta\frac{w}{2},\frac{w}{2}\right) \tag{2.9}$$

for all $w \in A$. Again replacing w by $w/2$ and multiply by 4 in (2.9), we obtain

$$\left\|4\,q\left(\frac{w}{2}\right)-4^2\,q\left(\frac{w}{2^2}\right)\right\| \le 4\varpi\left(-\eta\frac{w}{2^2},\frac{w}{2^2}\right) \tag{2.10}$$

for all $w \in A$. Combining (2.9) and (2.10) one can arrive

$$\begin{aligned}
&\left\|q(w)-4^2 f\left(\frac{w}{2^2}\right)\right\| \\
&\le \varpi\left(-\eta\frac{w}{2},\frac{w}{2}\right)+4\,\varpi\left(-\eta\frac{w}{2^2},\frac{w}{2^2}\right)
\end{aligned} \tag{2.11}$$

for all $w \in A$. In general for any positive integer α one can find that

$$\begin{aligned}
\left\|q(w)-4^\alpha f\left(\frac{x}{2^\alpha}\right)\right\| &\le \sum_{i=1}^{n}4^{i-1}\,\varpi\left(-\eta\frac{w}{2^i},\frac{w}{2^i}\right) \\
&= \frac{1}{4}\sum_{i=1}^{n}4^i\varpi\left(-\eta\frac{w}{2^i},\frac{w}{2^i}\right)
\end{aligned} \tag{2.12}$$

for all $w \in A$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.1 concerning the some stabilities for the functional equation (1.5).

Corollary 2.2. *Let m and d be nonnegative real numbers. Let a function $q : A \to B$ satisfies the inequality*

$$\begin{aligned}
&\Big\|q(v+\eta w)-2q(v+(\eta-1)w)+q(v+(\eta-2)w) \\
&\quad -2!\ q(w)\Big\| \le \begin{cases} m, \\ m\ \left(||v||^d+||w||^d\right), \\ m\ \left(||v||^{d_1}+||w||^{d_2}\right), \\ m\ ||v||^d||w||^d, \\ m\ ||v||^{d_1}||w||^{d_2}, \end{cases}
\end{aligned} \tag{2.13}$$

for all $v,w \in A$. Then there exists a unique quadratic function $\mathscr{Q} : A \to B$ satisfying the functional equation (1.5) and

$$\|q(w)-\mathscr{Q}(w)\| \le \begin{cases} \dfrac{m}{|3|}, \\ \dfrac{m\ (\eta^d+1)\ ||w||^d}{|2^d-4|}, & d\neq 2; \\ \dfrac{m\ \eta^{d_1}\ ||w||^{d_1}}{|2^{d_1}-4|}+\dfrac{m\ ||w||^{d_2}}{|2^{d_2}-4|}, & d_1,d_2\neq 2; \\ \dfrac{m\ \eta^d||w||^{2d}}{|2^{2d}-4|}, & 2d\neq 1; \\ \dfrac{m\ \eta^{d_1}||w||^{d_1+d_2}}{|2^{d_1+d_2}-4|}, & d_1+d_2\neq 2; \end{cases} \tag{2.14}$$

for all $w \in A$.

2.2 Fixed Point Method

We firstly recall the fundamental results in fixed point theory.

Theorem 2.3. *[39] (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
(F_1) $\qquad d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \ge 0,$
or
(F_2) there exists a natural number n_0 such that:
$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ for all $n \ge n_0$;
$(FPC2)$The sequence $(T^n x)$ is convergent to a fixed point y^* of T
$(FPC3)$ y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x,y) < \infty\}$;
$(FPC4)$ $d(y^*,y) \le \frac{1}{1-L}\ d(y,Ty)$ for all $y \in Y$.

For some applications of Theorem 2.3 in the stability of various functional equations in miscellaneous spaces, we refer to [8], [10] and [19].

Theorem 2.4. *Let A be a normed space and B be a Banach space. Suppose that $q : A \to B$ and $\varpi : A^2 \to [0,\infty)$ are functions satisfying the inequality (2.2) and the condition*

$$\lim_{\alpha\to\infty}\frac{1}{\theta_j^{2\alpha}}\varpi\left(\theta_j^\alpha v,\theta_j^\alpha w\right)=0 \tag{2.15}$$

for all $v,w \in A$, where

$$\theta_j = \begin{cases} 2 & j=0; \\ \frac{1}{2} & j=1. \end{cases} \tag{2.16}$$

If there exists $L = L(j)$ such that the function

$$\mathscr{W}(w,w) = \varpi\left(\frac{-\eta w}{2},\frac{w}{2}\right),$$

has the property

$$\frac{1}{\theta_j^2}\ \mathscr{W}\left(\theta_j w,\theta_j w\right) = L\,\mathscr{W}(w,w), \tag{2.17}$$

then there exists a unique quadratic mapping $\mathscr{Q} : A \to B$ satisfying functional equation (1.5) and

$$\|q(w) - \mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right)\mathscr{W}(w,w) \tag{2.18}$$

for all $w \in A$.

Proof. Define a set $\mathscr{X} = \{r/r : A \to B,\ r(0) = 0\}$ and introduce the generalized metric on the $\mathscr{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|r(w) - s(w)\| \leq K\ \ \mathscr{W}(w,w), w \in A\}$. It is easy to see that $(\mathscr{X}, d)$ is complete. Furthermore, define a function $T : \mathscr{X} \to \mathscr{X}$ by $Tr(w) = \frac{1}{\theta_j^2} r(\theta_j w)$, for all $w \in A$. This implies $d(Tr, Ts) \leq Ld(r,s)$, for all $r,s \in \mathscr{X}$. i.e., T is a strictly contractive mapping on $\mathscr{X}$ with Lipschitz constant L (see [39]). Using (2.17), it follows from (2.5) for the case $j = 0$ that

$$\begin{aligned} &\left\|\frac{q(2w)}{4} - q(w)\right\| \leq \frac{\varpi(-\eta w, w)}{4} \\ &\Rightarrow \left\|\frac{q(2w)}{4} - q(w)\right\| \leq \frac{\mathscr{W}(w,w)}{4} \\ &\Rightarrow d(Tq,q) \leq L < \infty. \end{aligned} \tag{2.19}$$

for all $w \in A$. Helping (2.17), it follows from (2.9) for the case $j = 1$ it reduces to

$$\begin{aligned} &\left\|q(w) - 4\, q\left(\frac{w}{2}\right)\right\| \leq \varpi\left(\frac{-\eta w}{2}, \frac{w}{2}\right) \\ &\Rightarrow \left\|q(w) - 4q\left(\frac{w}{2}\right)\right\| \leq \mathscr{W}(w,w) \\ &\Rightarrow d(q,Tq) \leq 1 < \infty. \end{aligned} \tag{2.20}$$

for all $w \in A$. Combining the above two cases, we arrive

$$d(q,Tq) \leq L^{1-j}.$$

Therefore (FPC1) of Theorem 2.3 holds. By (FPC2) of Theorem 2.3, it follows that there exists a fixed point $\mathscr{Q}$ of T in $\mathscr{X}$ such that

$$\mathscr{Q}(w) = \lim_{\alpha\to\infty} \frac{q(\theta_j^{\alpha} w)}{\theta_j^{2\alpha}}, \qquad \forall\ w \in A. \tag{2.21}$$

To order to prove $\mathscr{Q} : A \to B$ is quadratic the proof is similar ideas to that of Theorem 2.1. Again by (FPC3) of Theorem 2.3, $\mathscr{Q}$ is the unique fixed point of T in the set $\mathscr{Y} = \{A \in \mathscr{X} : d(q,\mathscr{Q}) < \infty\}$, $\mathscr{Q}$ is the unique function such that $\|q(w) - \mathscr{Q}(w)\| \leq K\,\mathscr{W}(w,w)$ for all $w \in A$ and $K > 0$. Finally by (FPC4) of Theorem 2.3, we obtain $d(q,\mathscr{Q}) \leq \frac{1}{1-L}\ d(q,Tq)$ this implies $d(q,\mathscr{Q}) \leq \frac{L^{1-j}}{1-L}$ which yields our desired result. □

The next corollary is a consequence of Theorem 2.4 concerning the stability of (1.5).

Corollary 2.5. *Let $q : A \to B$ be a mapping and there exists real numbers m and d such that*

$$\begin{aligned} &\Big\| q(v+\eta w) - 2q(v+(\eta-1)w) + q(v+(\eta-2)w) \\ &\quad - 2!\ \ q(w)\Big\| \leq \begin{cases} m, \\ m\ \left(||v||^d + ||w||^d\right), \\ m\ ||v||^d ||w||^d, \end{cases} \end{aligned} \tag{2.22}$$

for all $v,w \in A$. Then there exists a unique quadratic function $\mathscr{Q} : A \to B$ satisfying the functional equation (1.5) and

$$\|q(w) - \mathscr{Q}(w)\| \leq \begin{cases} \dfrac{m}{|3|}, \\ \dfrac{m\ (\eta^d+1)\ ||w||^d}{|2^d - 4|}, & d \neq 2; \\ \dfrac{m\ \eta^d ||w||^{2d}}{|2^{2d} - 4|}, & 2d \neq 1; \end{cases} \tag{2.23}$$

for all $w \in A$.

Proof. Take

$$\varpi(v,w) = \begin{cases} m, \\ m\ \left(||v||^d + ||w||^d\right), \\ m\ ||v||^d ||w||^d, \end{cases} \tag{2.24}$$

for all $v,w \in A$ in Theorem 2.4 for three cases. Replacing (v,w) by $(\theta_j^{\alpha}\ v, \theta_j^{\alpha}\ w)$ and dividing by $\theta_j^{2\alpha}$ in (2.24) one can see that (2.15) holds. Now, by definition of $\mathscr{W}(w,w)$ and its property, we have

$$\mathscr{W}(w,w) = \varpi\left(\frac{-\eta w}{2}, \frac{w}{2}\right) = \begin{cases} m \\ \dfrac{m\ (\eta^d+1)}{2^d}\ ||w||^d, \\ \dfrac{m\ \eta^d}{2^{2d}}\ ||w||^{2d}, \end{cases}$$

and

$$\begin{aligned} \frac{1}{\theta_j^2}\mathscr{W}(\theta_j w, \theta_j w) &= \begin{cases} \dfrac{m}{\theta_j^2} \\ \dfrac{m(\eta^d+1)}{2^d\theta_j^2}||\theta_j\ w||^d, \\ \dfrac{m\eta^d}{2^{2d}\theta_j^2}||\theta_j\ w||^{2d}, \end{cases} \\ &= \begin{cases} \theta_j^{-2}\ \mathscr{W}(w,w), \\ \theta_j^{d-2}\ \mathscr{W}(w,w), \\ \theta_j^{2d-2}\ \mathscr{W}(w,w), \end{cases} \end{aligned}$$

for all $w \in A$. Hence, the property (2.17) and the inequality (2.18) holds for the following cases.

$$L = \theta_j^{-2} = 2^{-2} \quad for \quad j = 0$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{2^{-2}}{1-2^{-2}}\right) m = \frac{m}{3}$$

$$L=\theta_j^{-2}=\frac{1}{2^{-2}}=2^2 \quad for \quad j=1$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{1}{1-2^{2}}\right) m = \frac{m}{-3}$$

$$L=\theta_j^{d-2}=2^{d-2} \quad for \quad j=0$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{2^{d-2}}{1-2^{d-2}}\right) \frac{m\,(\eta^d+1)}{2^d} ||w||^d = \frac{m\,(\eta^d+1)}{4-2^d} ||w||^d$$

$$L=\theta_j^{d-2}=\frac{1}{2^{d-2}}=2^{2-d} \quad for \quad j=1$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{1}{1-2^{2-d}}\right) \frac{m\,(\eta^d+1)}{2^d} ||w||^d = \frac{m\,(\eta^d+1)}{2^d-4} ||w||^d$$

$$L=\theta_j^{2d-2}=2^{2d-2} \quad for \quad j=0$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{2^{2d-2}}{1-2^{2d-2}}\right) \frac{m\,\eta^d}{2^{2d}} ||w||^{2d} = \frac{m\,\eta^d}{4-2^{2d}} ||w||^{2d}$$

$$L=\theta_j^{2d-2}=\frac{1}{2^{2d-2}}=2^{2-2d} \quad for \quad j=1$$

$$\|q(w)-\mathscr{Q}(w)\| \leq \left(\frac{L^{1-j}}{1-L}\right) \mathscr{W}(w,w) = \left(\frac{1}{1-2^{2-2d}}\right) \frac{m\,\eta^d}{2^{2d}} ||w||^{2d} = \frac{m\,\eta^d}{2^{2d}-4} ||w||^{2d}$$

Therefore, the proof is complete. □

3. Stability Results In Quasi-β-2-Banach Space

In this section, the generalized Ulam-Hyers stability of functional equation (1.5) is established in quasi-β-2-Banach space using direct and fixed point methods. Here, we give basic definitions and notations in quasi-β-2-Banach space [20, 21, 51, 52]; see also [35–37].

Definition 3.1. *Let X be a linear space of dimension greater than or equal to 2. Suppose $||(\bullet,\bullet)||$ is a real-valued function on $X\times X$ satisfying the following conditions:*

(QB2N1) *$||(x,y)||=0$ if and only if x,y are linearly dependent vectors,*

(QB2N2) *$||(x,y)||=||(y,x)||$ for all $x,y\in X$,*

(QB2N3) *$||(\lambda x,y)||=|\lambda|^\beta||(x,y)||$ for all $\lambda\in R$ and for all $x,y\in X$ where β is a a real number with $0<\beta\leq 1$*

(QB2N4) *If exists a constant $K\geq 1$ such that $||(x+y,z)||\leq K(||(x,z)||+||(y,z)||)$ for all $x,y,z\in X$.*

The pair $(X,||(\bullet,\bullet)||)$ is called quasi-β-normed space if $||(\bullet,\bullet)||$ is a quasi-β-2-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 3.2. *A quasi-β-2-Banach space is a complete quasi-β-normed space.*

3.1 Direct Method

Theorem 3.3. *Let $\delta\in\{-1,1\}$. Let A be a quasi-β-2-normed space and B be a quasi-β-2-Banach space. Suppose $\varpi: A^2\to[0,\infty)$ and $q:A\to B$ are functions satisfying the condition and the inequality*

$$\lim_{\alpha\to\infty}\frac{\varpi\left(2^{\alpha\delta}v,2^{\alpha\delta}w\right)}{4^{\alpha\delta}}=0 \tag{3.1}$$

and

$$\Big\|q(v+\eta w)-2q(v+(\eta-1)w)+q(v+(\eta-2)w) -2!\;q(w),z\Big\|\leq\varpi(v,w) \tag{3.2}$$

for all $v,w\in A$ and all $z\in B$. Then there exists a unique quadratic mapping $\mathscr{Q}:A\to B$ satisfying functional equation (1.5) and the inequality

$$\|(q(w)-\mathscr{Q}(w)),z\|\leq\frac{K^{\alpha-1}}{4^\beta}\sum_{i=\frac{1-\delta}{2}}^{\infty}\frac{\varpi(-\eta\cdot 2^{i\delta}w,2^{i\delta}w)}{4^{i\delta}} \tag{3.3}$$

for all $w\in A$ and all $z\in B$.

Proof. First, we give proof for $\delta=1$. Interchanging (v,w) by $(-\eta w,w)$ in (3.2), using evenness of q and (QB2N3), one can find that

$$\|(q(2w)-4\,q(w)),z\|\leq\varpi(-\eta w,w) \tag{3.4}$$

$$\left\|\left(\frac{q(2w)}{4}-q(w)\right),z\right\|\leq\frac{\varpi(-\eta w,w)}{4^\beta} \tag{3.5}$$

for all $w \in A$ and all $z \in B$. Again, changing w by $2w$ and dividing by 4 in (3.5), one can observe that

$$\left\|\left(\frac{q(2^2w)}{4^2}-\frac{q(2w)}{4}\right),z\right\| \leq \frac{\varpi(-\eta\cdot 2w,2w)}{4^{\beta}\cdot 4} \tag{3.6}$$

for all $w \in A$ and all $z \in B$. Combining (3.5) and (3.6), one can arrive

$$\left\|\left(\frac{q(2^2w)}{4^2}-q(w)\right),z\right\| \leq \frac{K}{4^{\beta}}\left[\varpi(-\eta w,w)+\frac{\varpi(-\eta\cdot 2w,2w)}{4}\right] \tag{3.7}$$

for all $w \in A$ and all $z \in B$. In general for any positive integer α, one can verify that

$$\left\|\left(\frac{q(2^{\alpha}w)}{4^{\alpha}}-q(w)\right),z\right\| \leq \frac{K^{\alpha-1}}{4^{\beta}}\sum_{i=0}^{\alpha-1}\frac{\varpi\left(-\eta\cdot 2^i w,2^i w\right)}{4^i} \tag{3.8}$$

for all $w \in A$ and all $z \in B$. The rest of the proof is similar lines to that of Theorem 2.1. This finishes the proof. □

Now, we have the upcoming corollary of Theorem 3.3 concerning the stability for functional equation (1.5).

Corollary 3.4. *Let λ and d be nonnegative real numbers. Let a function $q : A \to B$ satisfies the inequality*

$$\Big\|q(v+\eta w)-2q(v+(\eta-1)w)+q(v+(\eta-2)w) -2!\ q(w),z\Big\| \leq \begin{cases} m, \\ m\ \left(||v||^d+||w||^d\right), \\ m\ \left(||v||^{d_1}+||w||^{d_2}\right), \\ m\ ||v||^d||w||^d, \\ m\ ||v||^{d_1}||w||^{d_2}, \end{cases} \tag{3.9}$$

for all $v,w \in A$ and all $z \in B$. Then there exists a unique quadratic function $\mathscr{Q} : A \to B$ satisfying the functional equation (1.5) and

$$\|q(w)-\mathscr{Q}(w)\| \leq \begin{cases} \dfrac{4\ m\ K^{\alpha-1}}{4^{\beta}|3|}, \\ \dfrac{K^{\alpha-1}\ m\ (\eta^{\beta d}+1)\ ||w||^d}{4^{\beta}|2^{\beta d}-4|}, d \neq 2; \\ \dfrac{K^{\alpha-1}\ m\ \eta^{\beta d_1}\ ||w||^{d_1}}{4^{\beta}|2^{\beta d_1}-4|}+\dfrac{K^{\alpha-1}\ m\ ||w||^{d_2}}{4^{\beta}|2^{\beta d_2}-4|}, d_1,d_2 \neq 2; \\ \dfrac{K^{\alpha-1}\ m\ \eta^{\beta d}||w||^{2d}}{4^{\beta}|2^{\beta 2d}-4|}, 2d \neq 1; \\ \dfrac{K^{\alpha-1}\ m\ \eta^{\beta d_1}||w||^{d_1+d_2}}{4^{\beta}|2^{\beta(d_1+d_2)}-4|}, d_1+d_2 \neq 2; \end{cases} \tag{3.10}$$

for all $w \in A$ and all $z \in B$.

Theorem 3.5. *Let A be a quasi-β-2-normed space and B be a quasi-β-2-Banach space. Suppose that $q : A \to B$ and $\varpi : A^2 \to [0,\infty)$ are functions satisfying the inequality (3.2) and the condition*

$$\lim_{\alpha\to\infty}\frac{1}{\theta_j^{2\alpha}}\varpi\left(\theta_j^{\alpha}v,\theta_j^{\alpha}w\right)=0 \tag{3.11}$$

for all $v,w \in A$, where

$$\theta_j = \begin{cases} 2 & j=0; \\ \frac{1}{2} & j=1. \end{cases} \tag{3.12}$$

If there exists $L = L(j)$ such that the function

$$\mathscr{W}(w,w)=\varpi\left(\frac{-\eta w}{2},\frac{w}{2}\right),$$

has the property

$$\frac{1}{\theta_j^2}\mathscr{W}(\theta_j w,\theta_j w)=L\,\mathscr{W}(w,w). \tag{3.13}$$

Then there exists a unique quadratic mapping $\mathscr{Q} : A \to B$ satisfying functional equation (1.5) and

$$\|(q(w)-\mathscr{Q}(w)),z\| \leq \left(\frac{L^{1-j}}{1-L}\right)\mathscr{W}(w,w) \tag{3.14}$$

for all $w \in A$ and for all $z \in B$.

Proof. Define a set $\mathscr{X} = \{r/r : A \to B,\ r(0)=0\}$ and introduce the generalized metric on the $\mathscr{X}$ by, $d(r,s) = \inf\{K \in (0,\infty) : \|(r(w)-s(w)),z\| \leq K\mathscr{W}(w,w), w \in A, z \in B\}$. It is easy to see that $(\mathscr{X},d)$ is complete. In addition, define a function $T : \mathscr{X} \to \mathscr{X}$ by $Tr(w) = \frac{1}{\theta_j}r(\theta_j w)$, for all $w \in A$. This implies $d(Tr,Ts) \leq Ld(r,s)$, for all $r,s \in \mathscr{X}$. i.e., T is a strictly contractive mapping on $\mathscr{X}$ with Lipschitz constant L (see [39]).

With the help of (3.13), it follows from (3.5) for the case $j=0$ it reduces to

$$\begin{aligned} &\left\|\left(\frac{q(2w)}{4}-q(w)\right),z\right\| \leq \frac{\varpi(-\eta w,w)}{4^{\beta}} \\ &\Rightarrow \left\|\left(\frac{q(2w)}{4}-q(w)\right),z\right\| \leq \frac{\mathscr{W}(w,w)}{4^{\beta}} \\ &\Rightarrow d(Tq,q) \leq L < \infty. \end{aligned} \tag{3.15}$$

for all $w \in A$ and for all $z \in B$. Helping (3.13), it follows from (2.9) for the case $j=1$ it reduces to

$$\begin{aligned} &\left\|\left(q(w)-4\,q\left(\frac{w}{2}\right)\right),z\right\| \leq \varpi\left(\frac{-\eta w}{2},\frac{w}{2}\right) \\ &\Rightarrow \left\|\left(q(w)-4q\left(\frac{x}{2}\right)\right),z\right\| \leq \mathscr{W}(w,w) \\ &\Rightarrow d(q,Tq) \leq 1 < \infty. \end{aligned} \tag{3.16}$$

for all $w \in A$ and for all $z \in B$. Combining the above two cases, we arrive

$$d(q,Tq) \leq L^{1-j}.$$

The rest of the proof is similar to that of Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the some stabilities of (1.5). Since the proof is routine, is omitted.

Corollary 3.6. *Let $q : A \to B$ be a mapping and there exists real numbers m and d such that*

$$\Big\| q(v+\eta w) - 2q(v+(\eta-1)w) + q(v+(\eta-2)w) - 2!\ q(w), z\Big\| \le \begin{cases} m, \\ m\ \left(||v||^d + ||w||^d\right), \\ m\ ||v||^d ||w||^d, \end{cases} \tag{3.17}$$

for all $v,w \in A$ and for all $z \in B$. Then there exists a unique quadratic function $\mathscr{Q} : A \to B$ satisfying the functional equation (1.5) and

$$\|(q(w) - \mathscr{Q}(w)), z\| \le \begin{cases} \dfrac{m}{|3|}, \\ \dfrac{m\ (\eta^{\beta d}+1)\ ||w||^d}{|2^{\beta d}-4|}, & d \neq 2; \\ \dfrac{m\ \eta^{\beta d}||w||^{2d}}{|2^{\beta 2d}-4|}, & 2d \neq 1; \end{cases} \tag{3.18}$$

for all $w \in A$ and for all $z \in B$.

4. Stability Results In Intuitionistic Fuzzy -2-Banach Space

In this section, the generalized Ulam-Hyers stability of the functional equation (1.5) is established in intuitionistic fuzzy -2-Banach space using direct and fixed point methods. For more stability of the functional equation in intuitionistic fuzzy spaces, we refer to [9] and [11]. Here, we give basic definitions and notations in intuitionistic fuzzy -2-Banach space.

The basic definitions and notations in the setting of intuitionistic fuzzy normed space was introduced by Saadati and Park [47].

Definition 4.1. *A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:*

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 4.2. *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:*

(1)' $\diamond$ is commutative and associative;

(2)' $\diamond$ is continuous;

(3)' $a \diamond 0 = a$ for all $a \in [0,1]$;

(4)' $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 4.3. *The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy -2- normed space (for short, IF2NS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ,ν are fuzzy sets on $X \times X \times (0,\infty)$ satisfying the following conditions. For every $x,y,z \in X$ and $s,t > 0$;*
(IF2N1) $\mu(x,z,t) + \nu(x,z,t) \le 1$,
(IF2N2) $\mu(x,z,t) > 0$,
(IF2N3) $\mu(x,z,t) = 1$, if and only if $x = 0$.
(IF2N4) $\mu(\alpha x,z,t) = \mu\left(x,z,\frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IF2N5) $\mu(x,z,t) * \mu(y,z,s) \le \mu(x+y,z,t+s)$,
(IF2N6) $\mu(x,z,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(IF2N7)] $\lim_{t\to\infty} \mu(x,z,t) = 1$ and $\lim_{t\to 0} \mu(x,z,t) = 0$,
(IF2N8) $\nu(x,z,t) < 1$,
(IF2N9) $\nu(x,z,t) = 0$, if and only if $x = 0$.
(IF2N10) $\nu(\alpha x,z,t) = \nu\left(x,z,\frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IF2N11) $\nu(x,z,t) \diamond \nu(y,z,s) \ge \nu(x+y,z,t+s)$,
(IF2N12) $\nu(x,z,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(IF2N13) $\lim_{t\to\infty} \nu(x,z,t) = 0$ and $\lim_{t\to 0} \nu(x,z,t) = 1$. In this case, (μ,ν) is called an intuitionistic fuzzy -2- norm.

Example 4.4. *Let $(X,\|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \diamond b = \min\{a+b,1\}$ for all $a,b \in [0,1]$. For all $x,z \in X$ and every $t > 0$, consider*

$$\mu(x,z,t) = \begin{cases} \frac{t}{t+\|x,z\|} & if \quad t > 0; \\ 0 & if \quad t \le 0; \end{cases}$$

and

$$\nu(x,z,t) = \begin{cases} \frac{\|x,z\|}{t+\|x,z\|} & if \quad t > 0; \\ 0 & if \quad t \le 0. \end{cases}$$

Then $(X,\mu,\nu,*,\diamond)$ is an IF2N-space.

Definition 4.5. *Let $(X,\mu,\nu,*,\diamond)$ be an IF2NS. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy -2- convergent to a point $L \in X$ if*

$$\lim\ \ \mu(x_k - L,z,t) = 1 \quad and \quad \lim\ \ \nu(x_k - L,z,t) = 0$$

for all $t > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty$$

Definition 4.6. *Let $(X,\mu,\nu,*,\diamond)$ be an IF2NS. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy -2- Cauchy sequence if*

$$\mu\left(x_{k+p} - x_k,z,t\right) = 1 \qquad and \qquad \nu\left(x_{k+p} - x_k,z,t\right) = 0$$

for all $t > 0$, and $p = 1,2\cdots$.

Definition 4.7. *Let $(X,\mu,\nu,*,\diamond)$ be an IF2NS. Then $(X,\mu,\nu,*,\diamond)$ is said to be complete if every intuitionistic fuzzy -2- Cauchy sequence in $(X,\mu,\nu,*,\diamond)$ is intuitionistic fuzzy -2- convergent $(X,\mu,\nu,*,\diamond)$.*

4.1 Direct Method

Theorem 4.8. *Let $\delta \in \{1,-1\}$. Let A be a a linear space, (B,μ',ν') be a intuitionistic fuzzy -2-Banach space and (C,μ',ν') intuitionistic fuzzy -2-normed space. Let $\omega : X\times X \longrightarrow B$ and $q : A \to B$ are mappings satisfying the conditions and the inequality*

$$\left.\begin{array}{l}\mu'\left(\omega\left(2^{\alpha\delta}v,2^{\alpha\delta}w\right),z,t\right)\\ \qquad \geq \mu'\left(\gamma^{\alpha\delta}\omega(v,w),z,t\right)\\ \nu'\left(\omega\left(2^{\alpha\delta}v,2^{\alpha\delta}y\right),z,t\right)\\ \qquad \leq \nu'\left(\gamma^{\alpha\delta}\omega(v,w),z,t\right)\end{array}\right\} \tag{4.1}$$

$$\left.\begin{array}{l}\lim_{\alpha\to\infty}\mu'\left(\omega\left(2^{\delta n}v,2^{\delta\alpha}w\right),z,2^{2\delta\alpha}t\right)=1\\ \lim_{\alpha\to\infty}\nu'\left(\omega\left(2^{\delta n}v,2^{\delta\alpha}w\right),z,2^{2\delta\alpha}t\right)=0\end{array}\right\} \tag{4.2}$$

and

$$\left.\begin{array}{l}\mu\Big(q(v+\eta w)-2q(v+(\eta-1)w)\\ \quad +q(v+(\eta-2)w)-2!\ q(w),z,t\Big)\\ \geq \mu'(\omega(v,w),z,t)\\ \nu\Big(q(v+\eta w)-2q(v+(\eta-1)w)\\ \quad +q(v+(\eta-2)w)-2!\ q(w),z,t\Big)\\ \leq \nu'(\omega(v,w),z,t)\end{array}\right\} \tag{4.3}$$

for all $v,w\in A$ all $z\in B$ and all $t>0$ with $0<\left(\frac{\gamma}{a}\right)^{\delta}<1$. Then there exists a unique quadratic mapping $\mathscr{Q}:A\longrightarrow B$ satisfying (1.5) and

$$\left.\begin{array}{l}\mu(q(w)-\mathscr{Q}(w),z,t)\\ \geq \mu'(\omega(-\eta w,w),z,|4-p|t)\\ \nu(q(w)-\mathscr{Q}(w),z,t)\\ \leq \nu'(\omega(-\eta w,w),z,|4-p|t)\end{array}\right\} \tag{4.4}$$

for all $w\in A$ all $z\in B$ and all $t>0$.

Proof. **Case (i)** Let $\delta=1$.

Substituting (v,w) by $(-\eta w,w)$ in (4.3) and using evenness of q, one can find that

$$\left.\begin{array}{l}\mu(q(2w)-4q(w),z,t)\geq\mu'(\omega(-\eta w,w),z,t)\\ \nu(q(2w)-4q(w),z,t)\leq\nu'(\omega(-\eta w,w),z,t)\end{array}\right\} \tag{4.5}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Applying (IF2N4) and (IF2N10) in (4.5), we have

$$\left.\begin{array}{l}\mu\left(\frac{q(2w)}{4}-q(w),z,\frac{t}{4}\right)\geq\mu'(\omega(-\eta w,w),z,t)\\ \nu\left(\frac{q(2w)}{4}-q(w),z,\frac{t}{4}\right)\leq\nu'(\omega(-\eta w,w),z,t)\end{array}\right\} \tag{4.6}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Setting w by $2^{\alpha}w$ in (4.6), we obtain

$$\left.\begin{array}{l}\mu\left(\frac{q(2^{\alpha+1}w)}{4}-q(2^{\alpha}w),z,\frac{t}{4}\right)\\ \geq\mu'(\omega(-\eta 2^{\alpha}w,2^{\alpha}w),z,t)\\ \nu\left(\frac{q(2^{\alpha+1}w)}{4}-q(2^{\alpha}w),z,\frac{t}{4}\right)\\ \leq\nu'(\omega(-\eta 2^{\alpha}w,2^{\alpha}w),z,t)\end{array}\right\} \tag{4.7}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Using (4.1), (IF2N4), (IF2N10) in (4.7), we get

$$\left.\begin{array}{l}\mu\left(\frac{q(2^{\alpha+1}w)}{4^{(\alpha+1)}}-\frac{q(2^{\alpha}w)}{4^{\alpha}},z,\frac{t}{4\cdot4^{\alpha}}\right)\\ \geq\mu'\left(\omega(-\eta w,w),z,\frac{t}{\gamma^{\alpha}}\right)\\ \nu\left(\frac{q(2^{\alpha+1}w)}{4^{(\alpha+1)}}-\frac{q(2^{\alpha}w)}{4^{\alpha}},z,\frac{t}{4\cdot4^{\alpha}}\right)\\ \leq\nu'\left(\omega(-\eta w,w),z,\frac{t}{\gamma^{\alpha}}\right)\end{array}\right\} \tag{4.8}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Changing t into $\gamma^{\alpha}t$ in (4.8), we get

$$\left.\begin{array}{l}\mu\left(\frac{q(2^{\alpha+1}w)}{4^{(\alpha+1)}}-\frac{q(2^{\alpha}w)}{4^{\alpha}},z,\frac{t\cdot\gamma^{\alpha}}{4\cdot4^{\alpha}}\right)\\ \geq\mu'(\omega(-\eta w,w),z,t)\\ \nu\left(\frac{q(2^{\alpha+1}w)}{4^{(\alpha+1)}}-\frac{q(2^{\alpha}w)}{4^{\alpha}},z,\frac{t\cdot\gamma^{\alpha}}{4\cdot4^{\alpha}}\right)\\ \leq\nu'(\omega(-\eta w,w),z,t)\end{array}\right\} \tag{4.9}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Relations (IF2N4), (IF2N10) and equation (4.9) imply that

$$\left.\begin{array}{l}\mu\left(\frac{q(2^{\alpha}w)}{4^{\alpha}}-q(w),z,\sum_{i=0}^{\alpha-1}\frac{\gamma^{i}t}{4\cdot4^{i}}\right)\\ \geq\prod_{i=0}^{\alpha-1}\mu\left(\frac{q(2^{i+1}w)}{4^{(i+1)}}-\frac{q(2^{i}w)}{4^{i}},z,\frac{\gamma^{i}\ tr}{4\cdot4^{i}}\right)\\ \nu\left(\frac{q(2^{\alpha}w)}{4^{\alpha}}-q(w),z,\sum_{i=0}^{\alpha-1}\frac{\gamma^{i}t}{4\cdot4^{i}}\right)\\ \leq\prod_{i=0}^{\alpha-1}\nu\left(\frac{q(2^{i+1}w)}{4^{(i+1)}}-\frac{q(2^{i}w)}{4^{i}},z,\frac{\gamma^{i}\ t}{4\cdot4^{i}}\right)\end{array}\right\} \tag{4.10}$$

where

$$\prod_{i=0}^{\alpha-1}c_j=c_1*c_2*\cdots*c_n$$

and

$$\prod_{i=0}^{\alpha-1}d_j=d_1\diamond d_2\diamond\cdots\diamond d_n$$

for all $w\in A$ all $z\in B$ and all $t>0$. Hence, from (4.8) and (4.10), we find

$$\left.\begin{array}{l}\mu\left(\frac{q(2^{\alpha}w)}{4^{\alpha}}-q(w),z,\sum_{i=0}^{\alpha-1}\frac{\gamma^{i}\ t}{4\cdot4^{i}}\right)\\ \geq\prod_{i=0}^{\alpha-1}\mu'(\omega(-\eta w,w),z,t)\\ =\mu'(\omega(-\eta w,w),z,t)\\ \nu\left(\frac{q(2^{\alpha}w)}{4^{\alpha}}-q(w),z,\sum_{i=0}^{\alpha-1}\frac{\gamma^{i}\ t}{4\cdot4^{i}}\right)\\ \leq\prod_{i=0}^{\alpha-1}\nu'(\omega(-\eta w,w),z,t)\\ =\nu'(\omega(-\eta w,w),z,t)\end{array}\right\} \tag{4.11}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Switching w by $2^m w$ in (4.11) and using (4.1), (IF2N4), (IF2N10), we obtain

$$\left.\begin{array}{l} \mu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, \sum_{i=0}^{\alpha-1} \frac{\gamma^i\, t}{4\cdot 4^{(i+m)}}\right) \\ \geq \mu'(\omega(2^m w), z, t) = \mu'\left(\omega(-\eta w, w), z, \frac{t}{\gamma^m}\right) \\ \\ \nu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, \sum_{i=0}^{\alpha-1} \frac{\gamma^i\, t}{4\cdot 4^{(i+m)}}\right) \\ \leq \nu'(\omega(2^m w), z, t) = \nu'\left(\omega(-\eta w, w), z, \frac{t}{\gamma^m}\right) \end{array}\right\} \tag{4.12}$$

for all $w \in A$ all $z \in B$ and all $t > 0$ and all $m, n \geq 0$. Replacing t by $\gamma^m t$ in (4.12), we get

$$\left.\begin{array}{l} \mu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, \sum_{i=0}^{\alpha-1} \frac{\gamma^{i+m}\, t}{4\cdot 4^{(i+m)}}\right) \\ \geq \mu'(\omega(-\eta w, w), z, t) \\ \\ \nu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, \sum_{i=0}^{\alpha-1} \frac{\gamma^{i+m}\, t}{4\cdot 4^{(i+m)}}\right) \\ \leq \nu'(\omega(-\eta w, w), z, t) \end{array}\right\} \tag{4.13}$$

for all $w \in A$ all $z \in B$ and all $t > 0$ and all $m, n \geq 0$. The relation (4.12) implies that

$$\left.\begin{array}{l} \mu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, t\right) \\ \geq \mu'\left(\omega(-\eta w, w), z, \frac{t}{\sum_{i=m}^{\alpha-1} \frac{\gamma^i}{4\cdot 4^i}}\right) \\ \\ \nu\left(\frac{q(2^{\alpha+m}w)}{4^{(n+m)}} - \frac{q(2^m w)}{4^m}, z, t\right) \\ \leq \nu'\left(\omega(-\eta w, w), z, \frac{t}{\sum_{i=m}^{\alpha-1} \frac{\gamma^i}{4\cdot 4^i}}\right) \end{array}\right\} \tag{4.14}$$

holds for all $w \in A$ all $z \in B$ and all $t > 0$ and all $m, n \geq 0$. Since $0 < p < 4$ and $\sum_{i=0}^{\alpha} \left(\frac{p}{4}\right)^i < \infty$, the Cauchy criterion for the convergence in IF2NS shows that the sequence $\left\{\frac{q(2^\alpha w)}{4^\alpha}\right\}$ is Cauchy in (B, μ, ν). Since (B, μ, ν) is a complete IF2NS this sequence converges to some point $\mathscr{Q}(w) \in B$. So, one can define the mapping $\mathscr{Q} : A \longrightarrow B$

$$\lim_{\alpha\to\infty} \mu\left(\frac{q(2^\alpha w)}{4^\alpha} - \mathscr{Q}(w), z, t\right) = 1,$$

$$\lim_{\alpha\to\infty} \nu\left(\frac{q(2^\alpha w)}{4^\alpha} - \mathscr{Q}(w), z, t\right) = 0$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Hence

$$\frac{q(2^\alpha w)}{4^\alpha} \xrightarrow{IF} \mathscr{Q}(w), \quad as \quad n \to \infty.$$

Letting $m = 0$ in (4.14), we arrive

$$\left.\begin{array}{l} \mu\left(\frac{q(2^\alpha w)}{4^\alpha} - q(w), z, t\right) \geq \mu'\left(\omega(-\eta w, w), z, \frac{t}{\sum_{i=0}^{\alpha-1} \frac{\gamma^i}{4\cdot 4^i}}\right) \\ \nu\left(\frac{q(2^\alpha w)}{4^\alpha} - q(w), z, t\right) \leq \nu'\left(\omega(-\eta w, w), z, \frac{t}{\sum_{i=0}^{\alpha-1} \frac{\gamma^i}{4\cdot 4^i}}\right) \end{array}\right\} \tag{4.15}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Taking α tend to infinity in (4.15), we have

$$\left.\begin{array}{l} \mu\left(\mathscr{Q}(w) - q(w), z, t\right) \geq \mu'(\omega(-\eta w, w), z,\ t(4-p)) \\ \\ \nu\left(\mathscr{Q}(w) - q(w), z, t\right) \leq \nu'(\omega(-\eta w, w), z,\ t(4-p)) \end{array}\right\} \tag{4.16}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. To prove $\mathscr{Q}$ satisfies (1.5), Interchiging (v, w) into $(2^\alpha v, 2^\alpha w)$ in (4.3) respectively, we obtain

$$\left.\begin{array}{l} \mu\left(\frac{1}{2^{2\alpha}} q(2^\alpha(v+\eta w)) - 2q(2^\alpha(v+(\eta-1)w)) \right. \\ \left. +q(2^\alpha(v+(\eta-2)w)) - 2!\ q(2^\alpha w), z, t\right) \\ \geq \mu'(\omega(2^\alpha v, 2^\alpha w), z, 2^{2\alpha} t) \\ \\ \nu\left(\frac{1}{2^{2\alpha}} q(2^\alpha(v+\eta w)) - 2q(2^\alpha(v+(\eta-1)w)) \right. \\ \left. +q(2^\alpha(v+(\eta-2)w)) - 2!\ q(2^\alpha w), z, t\right) \\ \leq \nu'(\omega(2^\alpha v, 2^\alpha w), z, 2^{2\alpha} t) \end{array}\right\} \tag{4.17}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Now,

$$\begin{aligned} &\mu\Big(\mathscr{Q}(v+\eta w) - 2\mathscr{Q}(v+(\eta-1)w) \\ &\qquad + \mathscr{Q}(v+(\eta-2)w) - 2!\ \mathscr{Q}(w), z, t\Big) \\ &\geq \mu\left(\mathscr{Q}(v+\eta w) - \frac{1}{2^{2\alpha}} q(v+\eta w), z, \frac{t}{5}\right) * \\ &\quad \mu\left(-2\mathscr{Q}(v+(\eta-1)w) + \frac{2}{2^{2\alpha}} q(v+(\eta-1)w), z, \frac{t}{5}\right) * \\ &\quad \mu\left(\mathscr{Q}(v+(\eta-2)w) - \frac{1}{2^{2\alpha}} q(v+(\eta-2)w), z, \frac{t}{5}\right) * \\ &\quad \mu\left(-2!\ \mathscr{Q}(w) + \frac{2!}{2^{2\alpha}} q(w), z, \frac{t}{5}\right) * \\ &\quad \mu\Big(\frac{1}{2^{2\alpha}} q(v+\eta w) - \frac{2}{2^{2\alpha}} q(v+(\eta-1)w) \\ &\qquad + \frac{1}{2^{2\alpha}} q(v+(\eta-2)w) - \frac{2!}{2^{2\alpha}} q(w), z, \frac{t}{5}\Big) \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\nu\Big(\mathscr{Q}(v+\eta w) - 2\mathscr{Q}(v+(\eta-1)w) \\ &\qquad + \mathscr{Q}(v+(\eta-2)w) - 2!\ \mathscr{Q}(w), z, t\Big) \\ &\geq \nu\left(\mathscr{Q}(v+\eta w) - \frac{1}{2^{2\alpha}} q(v+\eta w), z, \frac{t}{5}\right) \diamond \\ &\quad \nu\left(-2\mathscr{Q}(v+(\eta-1)w) + \frac{2}{2^{2\alpha}} q(v+(\eta-1)w), z, \frac{t}{5}\right) \diamond \\ &\quad \nu\left(\mathscr{Q}(v+(\eta-2)w) - \frac{1}{2^{2\alpha}} q(v+(\eta-2)w), z, \frac{t}{5}\right) \diamond \\ &\quad \nu\left(-2!\ \mathscr{Q}(w) + \frac{2!}{2^{2\alpha}} q(w), z, \frac{t}{5}\right) \diamond \\ &\quad \nu\Big(\frac{1}{2^{2\alpha}} q(v+\eta w) - \frac{2}{2^{2\alpha}} q(v+(\eta-1)w) \\ &\qquad + \frac{1}{2^{2\alpha}} q(v+(\eta-2)w) - \frac{2!}{2^{2\alpha}} q(w), z, \frac{t}{5}\Big) \end{aligned} \tag{4.19}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Moreover,

$$\left.\begin{array}{l}\lim\limits_{\alpha\to\infty}\mu\Big(\frac{1}{2^{2\alpha}}q\left(2^{\alpha}(v+\eta w)\right)-2q\left(2^{\alpha}(v+(\eta-1)w)\right)\\ +q\left(2^{\alpha}(v+(\eta-2)w)\right)-2!\ \ q\left(2^{\alpha}w\right),z,t\Big)=1\\ \lim\limits_{\alpha\to\infty}\nu\Big(\frac{1}{2^{2\alpha}}q\left(2^{\alpha}(v+\eta w)\right)-2q\left(2^{\alpha}(v+(\eta-1)w)\right)\\ +q\left(2^{\alpha}(v+(\eta-2)w)\right)-2!\ \ q\left(2^{\alpha}w\right),z,t\Big)=0\end{array}\right\} \tag{4.20}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Letting $n \to \infty$ in (4.18), (4.19) and using (4.20), we observe that $\mathscr{Q}$ fulfills (1.5). Therefore, $\mathscr{Q}$ is a quadratic mapping. In order to prove $\mathscr{Q}(w)$ is unique, let $\mathscr{Q}'(w)$ be another quadratic functional equation satisfying (1.5) and (4.4). Hence,

$$\begin{aligned}&\mu(\mathscr{Q}(w)-\mathscr{Q}'(w),z,t)\\ &\geq \mu\left(\mathscr{Q}(2^{\alpha}w)-q(2^{\alpha}w),z,\frac{t.2^{\alpha}}{2}\right)*\\ &\qquad\qquad \mu\left(q(2^{\alpha}w)-\mathscr{Q}'(2^{\alpha}w),z,\frac{t.2^{\alpha}}{2}\right)\\ &\geq \mu'\left(\omega(2^{\alpha}w),z,\frac{t\,2^{\alpha}(4-p)}{2}\right)\\ &\geq \mu'\left(\omega(-\eta w,w),z,\frac{t\,2^{\alpha}(4-p)}{2\cdot\gamma^{\alpha}}\right)\\ &\nu(\mathscr{Q}(w)-\mathscr{Q}'(w),z,t)\\ &\leq \nu\left(\mathscr{Q}(2^{\alpha}w)-q(2^{\alpha}w),z,\frac{t.2^{\alpha}}{2}\right)\diamond\\ &\qquad\qquad \nu\left(q(2^{\alpha}w)-\mathscr{Q}'(2^{\alpha}w),z,\frac{t.2^{\alpha}}{2}\right)\\ &\leq \nu'\left(\omega(2^{\alpha}w),z,\frac{t\,2^{\alpha}(4-p)}{2}\right)\\ &\leq \nu'\left(\omega(-\eta w,w),z,\frac{t\,2^{\alpha}(4-p)}{2\cdot\gamma^{\alpha}}\right)\end{aligned}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Since $\lim\limits_{\alpha\to\infty}\dfrac{t\,2^{\alpha}(4-p)}{2\,\gamma^{\alpha}}=\infty$, we obtain

$$\left.\begin{array}{l}\lim\limits_{\alpha\to\infty}\mu'\left(\omega(-\eta w,w),z,\frac{t\,2^{\alpha}(4-p)}{2\cdot\gamma^{\alpha}}\right)=1\\ \lim\limits_{\alpha\to\infty}\nu'\left(\omega(-\eta w,w),z,\frac{t\,2^{\alpha}(4-p)}{2\cdot\gamma^{\alpha}}\right)=0\end{array}\right\}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Thus

$$\left.\begin{array}{l}\mu(\mathscr{Q}(w)-\mathscr{Q}'(w),z,t)=1\\ \nu(\mathscr{Q}(w)-\mathscr{Q}'(w),z,t)=0\end{array}\right\}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Hence, $\mathscr{Q}(w) = \mathscr{Q}'(w)$. Therefore, $\mathscr{Q}(w)$ is unique.

Case 2: For $\delta = -1$. Putting w by $\frac{w}{2}$ in (4.5), we get

$$\left.\begin{array}{l}\mu\left(q(w)-4f\left(\frac{w}{2}\right),z,t\right)\geq \mu'\left(\omega\left(\frac{-\eta w}{2},\frac{w}{2}\right),z,t\right)\\ \nu\left(q(w)-4f\left(\frac{w}{2}\right),z,t\right)\leq \nu'\left(\omega\left(\frac{-\eta w}{2},\frac{w}{2}\right),z,t\right)\end{array}\right\} \tag{4.21}$$

for all $v, y \in A$ all $z \in B$ and all $t > 0$. The rest of the proof is similar to that of Case 1. This completes the proof. □

The following corollary is an immediate consequence of Theorem 4.8, regarding the stability of (1.5).

Corollary 4.9. *Suppose that a function $q : A \longrightarrow B$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\Big(q\left(v+\eta w\right)-2q\left(v+(\eta-1)w\right)+q\left(v+(\eta-2)w\right)\\ \quad -2!\ \ q\left(w\right),z,t\Big)\geq\begin{cases}\mu'\left(m,z,t\right),\\ \mu'\left(m,z\left(||v||^{d}+||w||^{d}\right),t\right),\\ \mu'\left(m,z\left(||v||^{d_1}+||w||^{d_2}\right),t\right),\\ \mu'\left(m,z||v||^{d}||w||^{d},t\right),\\ \mu'\left(m,z||v||^{d_1}||w||^{d_2},t\right),\end{cases}\\ \nu\Big(q\left(v+\eta w\right)-2q\left(v+(\eta-1)w\right)+q\left(v+(\eta-2)w\right)\\ \quad -2!\ \ q\left(w\right),z,t\Big)\leq\begin{cases}\nu'\left(m,z,t\right),\\ \nu'\left(m,z,\left(||v||^{d}+||w||^{d}\right),t\right),\\ \nu'\left(m,z,\left(||v||^{d_1}+||w||^{d_2}\right),t\right),\\ \nu'\left(m,z,||v||^{d}||w||^{d},t\right),\\ \nu'\left(m,z,||v||^{d_1}||w||^{d_2},t\right),\end{cases}\end{array}\right\} \tag{4.22}$$

for all $v, w \in A$ all $z \in B$ and all $t > 0$, where m, d, d_1, d_2 are constants with $m > 0$. Then there exists a unique quadratic mapping $\mathscr{Q} : X \longrightarrow B$ such that

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq\begin{cases}\mu'\left(m,|4-p|t\right),\\ \mu'\left(m||w||^{d}(|\eta|^{d}+1),z,|4-p|t\right),\\ \mu'\Big(m\eta^{d_1}||w||^{d_1}+m||w||^{d_2},\\ \qquad z,|4-p|t+|4-p|t\Big),\\ \mu'\left(m||w||^{2d}\eta^{d},z,|4-p|t\right)\\ \mu'\left(m||w||^{d_1+d_2}\eta^{d_1},z,|4-p|t\right)\end{cases}\\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq\begin{cases}\nu'\left(m,|4-p|t\right),\\ \nu'\left(m||w||^{d}(|\eta|^{d}+1),z,|4-p|t\right),\\ \nu'\Big(m\eta^{d_1}||w||^{d_1}+m||w||^{d_2},\\ \qquad z,|4-p|t+|4-p|t\Big),\\ \nu'\left(m||w||^{2d}\eta^{d},z,|4-p|t\right)\\ \nu'\left(m||w||^{d_1+d_2}\eta^{d_1},z,|4-p|t\right)\end{cases}\end{array}\right\} \tag{4.23}$$

for all $w \in A$ all $z \in B$ and all $t > 0$.

4.2 Fixed Point Method

Theorem 4.10. *Let A be a a linear space, (B, μ', ν') be a intuitionistic fuzzy -2-Banach space and (C, μ', ν') intuitionistic fuzzy -2-normed space. Suppose that $q : A \to B$ and $\varpi : A^2 \to [0, \infty)$ are functions satisfying inequality (4.3) and* *the condition*

$$\left.\begin{aligned} \lim_{n\to\infty}\mu'\left(\omega\left(\theta_j^\alpha x,\theta_j^\alpha y\right),z,\theta_j^\alpha t\right)=1\\ \lim_{n\to\infty}\nu'\left(\omega\left(\theta_j^\alpha x,\theta_j^\alpha y\right),z,\theta_j^\alpha t\right)=0\end{aligned}\right\} \tag{4.24}$$

for all $v,w\in A$ all $z\in B$ and all $t>0$ where

$$\theta_j=\begin{cases}2 & if \quad j=0\\ \frac{1}{2} & if \quad j=1.\end{cases} \tag{4.25}$$

If there exists $L=L(j)$ such that the functions

$$\mu'(\mathscr{W}(w,w),z,t)=\mu'\left(\omega\left(\frac{-\eta w}{2},\frac{w}{2}\right),z,t\right),$$

$$\nu'(\mathscr{W}(w,w),z,t)=\nu'\left(\omega\left(\frac{-\eta w}{2},\frac{w}{2}\right),z,t\right)$$

has properties

$$\left.\begin{aligned} L\mu'\left(\frac{1}{\theta_j^2}\mathscr{W}(\theta_j w,\theta_j w),z,t\right)=\mu'(\mathscr{W}(w,w),z,Lt)\\ \nu'\left(\frac{1}{\theta_j^2}\mathscr{W}(\theta_j w,\theta_j w),z,t\right)=\nu'(\mathscr{W}(w,w),z,Lt)\end{aligned}\right\} \tag{4.26}$$

for all $w\in A$ all $z\in B$ and all $t>0$, then there exists a unique quadratic function $\mathscr{Q}:A\longrightarrow B$ satisfying the functional equation (1.5) and

$$\left.\begin{aligned} \mu(q(w)-\mathscr{Q}(w),z,t)\geq\mu'\left(\mathscr{W}(w,w),z,\frac{L^{1-i}}{1-L}t\right)\\ \nu(q(w)-\mathscr{Q}(w),z,t)\leq\nu'\left(\mathscr{W}(w,w),z,\frac{L^{1-i}}{1-L}t\right)\end{aligned}\right\} \tag{4.27}$$

for all $w\in A$ all $z\in B$ and all $t>0$.

Proof. Define a set $\mathscr{X}=\{r/r:A\to B,\ r(0)=0\}$ and introduce the generalized metric on the $\mathscr{X}$ by,

$$d(r,s)=\inf\left\{M\in(0,\infty):\left\{\begin{aligned}&\mu(r(w)-s(w),z,t)\\ &\geq\mu'(\mathscr{W}(w,w),z,Mt),\\ &\nu(r(w)-s(w),z,t)\\ &\leq\nu'(\mathscr{W}(w,w),z,Mt),\end{aligned}\right\}\right\}. \tag{4.28}$$

for all $w\in A$. It is easy to see that $(\mathscr{X},d)$ is complete. Besides, define a function $T:\mathscr{X}\to\mathscr{X}$ by $Tr(w)=\frac{1}{\theta_j^2}r(\theta_j w)$, for all $w\in A$. This implies $d(Tr,Ts)\leq Ld(r,s)$, for all $r,s\in\mathscr{X}$. i.e., T is a strictly contractive mapping on $\mathscr{X}$ with Lipschitz constant L (see [39]).

With the help of (4.26), (4.28), it follows from (4.6) for the case $j=0$ it reduces to

$$\left.\begin{aligned} \mu\left(\frac{q(2w)}{4}-q(w),z,t\right)\geq\mu'(\omega(-\eta w,w),z,4t)\\ \nu\left(\frac{q(2w)}{4}-q(w),z,t\right)\leq\nu'(\omega(-\eta w,w),z,4t)\end{aligned}\right\}$$
$$\left.\begin{aligned} \mu\left(Tq(w)-q(w),z,t\right)\geq\mu'(\mathscr{W}(w,w),z,Lt)\\ \nu\left(Tq(w)-q(w),z,t\right)\leq\nu'(\mathscr{W}(w,w),z,Lt)\end{aligned}\right\}$$
$$\Rightarrow d(Tq,q)\leq L<\infty. \tag{4.29}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Using (4.26), (4.28), it follows from (4.21) for the case $j=1$ that

$$\left.\begin{aligned} \mu\left(q(w)-4f\left(\tfrac{w}{2}\right),z,t\right)\geq\mu'\left(\omega\left(\tfrac{-\eta w}{2},\tfrac{w}{2}\right),z,t\right)\\ \nu\left(q(w)-4f\left(\tfrac{w}{2}\right),z,t\right)\leq\nu'\left(\omega\left(\tfrac{-\eta w}{2},\tfrac{w}{2}\right),z,t\right)\end{aligned}\right\}$$
$$\left.\begin{aligned} \mu\left(q(w)-Tq(w),z,t\right)\geq\mu'(\mathscr{W}(w,w),z,t)\\ \nu\left(q(w)-Tq(w),z,t\right)\leq\nu'(\mathscr{W}(w,w),z,t)\end{aligned}\right\}$$
$$\Rightarrow d(q,Tq)\leq 1<\infty. \tag{4.30}$$

Thus, from (4.29) and (4.30), we arrive

$$d(Jf,f)=\inf\left\{L^{1-i}\in(0,\infty):\left\{\begin{aligned}&\mu(Jq(w)-q(w),z,t)\\ &\geq\mu'(\mathscr{W}(w,w),z,L^{1-i}t),\\ &\nu(Jq(w)-q(w),z,t)\\ &\leq\nu'(\mathscr{W}(w,w),z,L^{1-i}t),\end{aligned}\right\}\right\}$$

for all $w\in A$ all $z\in B$. Therefore, (FPC1) of Theorem 2.3 holds. By (FPC2) of Theorem 2.3, it follows that there exists a fixed point $\mathscr{Q}$ of T in $\mathscr{X}$ such that

$$\lim_{\alpha\to\infty}\mu\left(\frac{q(\theta_j^\alpha x)}{\theta_j^\alpha}-\mathscr{Q}(w),z,t\right)=1,$$
$$\lim_{\alpha\to\infty}\nu\left(\frac{q(\theta_j^\alpha x)}{\theta_j^\alpha}-\mathscr{Q}(w),z,t\right)=0$$

for all $w\in A$ and all $t>0$. To order to prove $\mathscr{Q}:A\to B$ is quadratic the proof is similar ideas to that of Theorem 4.8. Again by (FPC3) of Theorem 2.3, $\mathscr{Q}$ is the unique fixed point of T in the set $\mathscr{Y}=\{\mathscr{Q}\in\mathscr{X}:d(q,\mathscr{Q})<\infty\}$, $\mathscr{Q}$ is the unique function such that

$$\left.\begin{aligned} \mu(q(w)-\mathscr{Q}(w),z,t)\geq\mu'(\mathscr{W}(w,w),z,L^{1-i}t),w\in A\\ \nu(q(w)-\mathscr{Q}(w),z,t)\leq\nu'(\mathscr{W}(w,w),z,L^{1-i}t),w\in A\end{aligned}\right\}$$

for all $w\in A$ all $z\in B$ and all $t>0$. Finally by (FP4), we obtain

$$\left.\begin{aligned} \mu(q(w)-\mathscr{Q}(w),z,t)\geq\mu'\left(\mathscr{W}(w,w),z,\frac{L^{1-i}}{1-L}t\right)\\ \nu(q(w)-\mathscr{Q}(w),z,t)\leq\nu'\left(\mathscr{W}(w,w),z,\frac{L^{1-i}}{1-L}t\right)\end{aligned}\right\}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Hence the proof is complete. □

Corollary 4.11. *Suppose that a function $q : A \longrightarrow B$ satisfies the double inequality*

$$\left.\begin{array}{l} \mu\Big(q(v+\eta w) - 2q(v+(\eta-1)w) \\ \quad +q(v+(\eta-2)w) - 2!\ q(w), z, t\Big) \\ \geq \begin{cases} \mu'(m,z,t), \\ \mu'\left(m\left(||x||^d + ||y||^d\right), z, t\right), \\ \mu'\left(m||x||^d||y||^d, z, t\right), \end{cases} \\ \nu\Big(q(v+\eta w) - 2q(v+(\eta-1)w) \\ \quad +q(v+(\eta-2)w) - 2!\ q(w), z, t\Big) \\ \leq \begin{cases} \nu'(m,z,t), \\ \nu'\left(m\left(||x||^d + ||y||^d\right), z, t\right), \\ \nu'\left(m||x||^d||y||^d, z, t\right), \end{cases} \end{array}\right\} \tag{4.31}$$

for all $w, y \in A$ and all $t > 0$, where m, a are constants with $m > 0$. Then there exists a unique quadratic mapping $\mathscr{Q}$: $X \longrightarrow B$ such that the double inequality

$$\left.\begin{array}{l} \mu\left(q(w) - \mathscr{Q}(w), z, t\right) \\ \geq \begin{cases} \mu'\left(m, z, \frac{t}{|3|}\right), \\ \mu'\left(m||w||^d(|\eta|^d+1), z, \frac{t}{|4-2^d|}\right), \\ \mu'\left(m||w||^{2d}\eta^d, z, \frac{t}{|4-2^{2d}|}\right) \end{cases} \\ \nu\left(q(w) - \mathscr{Q}(w), z, t\right) \\ \leq \begin{cases} \nu'\left(m, z, \frac{t}{|3|}\right), \\ \nu'\left(m||w||^d(|\eta|^d+1), z, \frac{t}{|4-2^d|}\right), \\ \nu'\left(m||w||^{2d}\eta^d, z, \frac{t}{|4-2^{2d}|}\right) \end{cases} \end{array}\right\} \tag{4.32}$$

holds for all $w \in A$ and all $t > 0$.

Proof. Set

$$\omega(v,w) = \begin{cases} m, \\ m\ \left(||v||^d + ||w||^d\right), \\ m\ ||v||^d||w||^d, \end{cases} \tag{4.33}$$

for all $v, w \in A$ in Theorem 4.8. Replacing (v,w) by $(\theta_j^{\alpha}\ v, \theta_j^{\alpha}\ w)$ and dividing by $\theta_j^{2\alpha}$ in (4.33) and taking α tends to infinity, one can see that (4.24) holds. Now, by definition of $\mathscr{W}(w,w)$ and its property, we have

$$\begin{aligned} &\mu'\left(\mathscr{W}(w,w), z, t\right) \\ &= \mu'\left(\omega\left(\frac{-\eta w}{2}, \frac{w}{2}\right), z, t\right) \\ &= \begin{cases} \mu'(m,z,t) \\ \mu'\left(\dfrac{m\ (\eta^d+1)}{2^d}\ ||w||^d, z, t\right) \\ \mu'\left(\dfrac{m\ \eta^d}{2^{2d}}\ ||w||^{2d}, z, t\right) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \nu'\left(\mathscr{W}(w,w), z, t\right) &= \nu'\left(\omega\left(\frac{-\eta w}{2}, \frac{w}{2}\right), z, t\right) \\ &= \begin{cases} \nu'(m,z,t) \\ \nu'\left(\dfrac{m\ (\eta^d+1)}{2^d}\ ||w||^d, z, t\right) \\ \nu'\left(\dfrac{m\ \eta^d}{2^{2d}}\ ||w||^{2d}, z, t\right) \end{cases} \end{aligned}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. In addition, from (4.26), we have

$$\begin{aligned} &\mu'\left(\frac{1}{\theta_j^2}\mathscr{W}(\theta_j w, \theta_j w), z, t\right) \\ &= \begin{cases} \mu'\left(\dfrac{m}{\theta_j^2}, z, t\right) \\ \mu'\left(\dfrac{m\ (\eta^d+1)}{2^d\theta_j^2}\ ||\theta_j w||^d, z, t\right) \\ \mu'\left(\dfrac{m\ \eta^d}{2^{2d}\theta_j^2}\ ||\theta_j w||^{2d}, z, t\right) \end{cases} \\ &= \begin{cases} \mu'\left(\mathscr{W}(w,w), z, \theta_j^{-2}t\right) \\ \mu'\left(\mathscr{W}(w,w), z, \theta_j^{d-2}t\right) \\ \mu'\left(\mathscr{W}(w,w), z, \theta_j^{2d-2}t\right) \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\nu'\left(\frac{1}{\theta_j^2}\mathscr{W}(\theta_j w, \theta_j w), z, t\right) \\ &= \begin{cases} \nu'\left(\dfrac{m}{\theta_j^2}, z, t\right) \\ \nu'\left(\dfrac{m\ (\eta^d+1)}{2^d\theta_j^2}\ ||\theta_j w||^d, z, t\right) \\ \nu'\left(\dfrac{m\ \eta^d}{2^{2d}\theta_j^2}\ ||\theta_j w||^{2d}, z, t\right) \end{cases} \\ &= \begin{cases} \nu'\left(\mathscr{W}(w,w), z, \theta_j^{-2}t\right) \\ \nu'\left(\mathscr{W}(w,w), z, \theta_j^{d-2}t\right) \\ \nu'\left(\mathscr{W}(w,w), z, \theta_j^{2d-2}t\right) \end{cases} \end{aligned}$$

for all $w \in A$ all $z \in B$ and all $t > 0$. Hence, inequality (4.27) is true for following:

$$L = \theta_j^{-2} = 2^{-2} \quad for \quad j = 0$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{2^{-2}}{1-2^{-2}}t\right)\\ =\mu'\left(m,z,\frac{t}{3}\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{2^{-2}}{1-2^{-2}}t\right)\\ =\nu'\left(m,z,\frac{t}{3}\right)\end{array}\right\}$$

$$L=\theta_j^{-2}=\frac{1}{2^{-2}}=2^2 \quad for \quad j=1$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^2}t\right)\\ =\mu'\left(m,z,\frac{t}{-3}\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^2}t\right)\\ =\nu'\left(m,z,\frac{t}{-3}\right)\end{array}\right\}$$

$$L=\theta_j^{d-2}=2^{d-2} \quad for \quad j=0$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{2^{d-2}}{1-2^{d-2}}\ \frac{1}{2^d}\ t\right)\\ =\mu'\left(m\ (\eta^d+1)||w||^d,z,\frac{1}{4-2^d}t\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{2^{d-2}}{1-2^{d-2}}\ \frac{1}{2^d}\ t\right)\\ =\nu'\left(m\ (\eta^d+1)||w||^d,z,\frac{1}{4-2^d}t\right)\end{array}\right\}$$

$$L=\theta_j^{d-2}=\frac{1}{2^{d-2}}=2^{2-d} \quad for \quad j=1$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^{2-d}}\ \frac{1}{2^d}t\right)\\ =\mu'\left(m\ (\eta^d+1)||w||^d,z,\frac{t}{2^d-4}\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^{2-d}}\ \frac{1}{2^d}t\right)\\ \nu'\left(m\ (\eta^d+1)||w||^d,z,\frac{t}{2^d-4}\right)\end{array}\right\}$$

$$L=\theta_j^{2d-2}=2^{2d-2} \quad for \quad j=0$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{2^{2d-2}}{1-2^{2d-2}}\ \frac{1}{2^{2d}}\ t\right)\\ =\mu'\left(m\ \eta^d||w||^{2d},z,\frac{1}{4-2^{2d}}t\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{2^{2d-2}}{1-2^{2d-2}}\ \frac{1}{2^{2d}}\ t\right)\\ =\nu'\left(m\ \eta^d||w||^{2d},z,\frac{1}{4-2^{2d}}t\right)\end{array}\right\}$$

$$L=\theta_j^{2d-2}=\frac{1}{2^{2d-2}}=2^{2-2d} \quad for \quad j=1$$

$$\left.\begin{array}{l}\mu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \geq \mu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^{2-2d}}\ \frac{1}{2^{2d}}t\right)\\ =\mu'\left(m\eta^d||w||^{2d},z,\frac{t}{2^{2d}-4}\right)\\ \\ \nu\left(q(w)-\mathscr{Q}(w),z,t\right)\\ \leq \nu'\left(\mathscr{W}(w,w),z,\frac{1}{1-2^{2-2d}}\ \frac{1}{2^{2d}}t\right)\\ =\nu'\left(m\eta^d||w||^{2d},z,\frac{t}{2^{2d}-4}\right)\end{array}\right\}$$

The the proof is complete. □

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