Capacity constraint EOQ model for deteriorating items with two-level trade credit under financial environment

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Abstract

The present model has been originated a capacity constraint EOQ for deteriorating items with two levels of trade credit policy under the financial environment. In this study, the trade credit policy of two-level has been considered. In this policy, the wholesaler got a permissible delay time period M offered by the supplier, and the retailer also gets the delay period $N(M > N)$ offered by the wholesaler. All possible cases in the trade credit period are developed under the financial environment. Further, this model has been considering the deteriorating rate constant in nature. In addition to this study, find the optimal solution, several theoretical results have been established. Finally, three hypothetical numerical examples have been considered and the sensitivity analysis on the optimal case has been done on different parameters.

Keywords

Trade credit financing; Trade Credit Policy of Two-level; Deterioration; Limited storage capacity; Inventory.

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1. Introduction

An inventory control system is a vital tool for the better performance of any business. Because this system provides the solution of two essential questions of "when and how much order". Harris [16] derived the formula for the inventory system which is known as the EOQ formula. This result developed under the assumption that cash on delivery policy. However, in current scenarios, if the wholesaler and retailers both have an option to settle your accounts without any charges after some. The benefits of this policy, supplier attract their customers to buy more and reduce the holding cost. In inventory theory, this policy is called permissible delay in payment or trade credit period policy. In the last few decades, several studies have been done on inventory theory with this policy. The first model under this policy was developed by Goyal [14]. Teng [30] make improve Goyal's paper by differentiating between unit cost and unit price cost. Goyal's work extended by Aggarwal and Jaggi [1] allowing for deteriorating items, Numerous models were developed under different situations i.e. allowable shortages, deterioration, non-instantaneous deterioration, credit-linked demand, price dependent demand, time-dependent demand, different kind trade credit policies, and these policies under financial environment. Based on Goyal and Teng models, Jaggi et al. [14] formulated an EOQ model for the retailers with credit linked demand under shortages. A review article on that policy is done by Soni et al. [27]. Lou and Wang [20] contribute in literature by the seller's choice on the situation of the trade credit period under constant demand rate. Cheng et al.[9] contributed in this literature of trade credit period by considering trade credit period in different financial environment. Further, Jaggi et al. [15]

[16] and [17] extended Cheng et al. paper by price-dependent demand, allowable shortages and non–instantaneous deteriorating items. Other related articles by Chang et al. [4], [5], [6], Chung [8], Jamal et al. [18], Lou et al. [19].

This model is also focused on radial environmental changes, due to this effect most of the products sufferers their efficiency in due course, this phenomenon is known as deterioration. Ghare [11] was the first inventory model considering this effect on his model. Aggarwal and Jaggi [2] formulated an EOQ model with this effect under delay in payments. In this literature some other interesting work is done by Geeta and Uthaykumar [9], Maihami and Kamalabadi [21], Shah and Jani [24], Shaikh and Cardenas-Barron [25], Sharma et al. [26], Taleizadeh et al. [28], Tayakoli with Taleizadeh [29], Wu et al. [31]. Shah with Shah [23], Rafat [22], Goyal and Giri [12], Bakker et al. [2] also contributed to this literature by review articles.

Our Contribution: In the existing literature on the inventory model with trade credit period, some researchers have considering only one level trade credit period with the financial environment. But none of the researchers considering a two-level trade credit policy in the financial environment. Based on these scenarios, the present study considers that policy for capacity constraint and deteriorating items with two-level trade credit period under financial environment. This model has been considering the deteriorating rate constant in nature. In addition to this study, to find the optimal solution, several theoretical results have been established. Finally, three hypothetical numerical examples have been considered and the sensitivity analysis on the optimal case has been done on different parameters.

2. Model Formulation

2.1 Assumption

The present model is based on some realistic assumptions which are followed as

1. The rate of replenishment is instantaneous, and Lead-time is constant.

2. This model has been considering an infinite planning horizon.

3. Shortages are not allowed.

4. The deteriorating items are taken into this study with a constant fraction of the inventory level $I(t)$

5. The constant rate of demand is considered.

6. Two-level trade credit period is considered. In this policy, the supplier proffers permissible delay time period M to the trader (wholesaler) and the trader also gives that period $N(M < N)$ for his customers (retailers) who comes first day. After this time period will be reduced and it will be zero after N time period.

7. The time of settlement of account of trader is lies between $[M, T]$.

2.2 Notations

For the mathematical description of this model, the following notations are used. In addition, the following notations are used throughout this paper.

K	Demand rate per year
a	Cost of replenishment per order
c	Cost of purchasing per item
p	Price of selling per items
h_l	Cost of holding per item per unit time in OW
h_2	Cost of holding per item per unit time in RW
θ	Deterioration rate per unit time
M	Trader permissible delay in payment which is offered by the manufacturer (supplier)
N	Trader customer's permissible delay in payment which is offered by the trader
I_e	Interest earned rate
I_p	Interest paid rate
w	OW storage capacity
w^*	The generated amount by the wholesaler up to M
t_w	The time on horizon plane at which inventory level reduces to w
t_a	Time at which OW capacity is exhausted and equal to $\frac{1}{\theta} \ln \left(\frac{\theta w}{k} + 1 \right)$
T	Replenishment cycle length
$TC(T)$	Total cost per unit per unit time

2.3 Mathematical Description

During the time $t \in [0,T]$, the inventory level $I(t)$ is obtained from the following governing differential equation

$$\frac{dI(t)}{dt} + \theta I(t) = -K; \quad t \in [0, T]$$

To calculate $I(t)$ with the help of $I(T) = 0$

$$I(t) = \frac{K}{\theta} \left(e^{\theta(T-t)} - 1 \right); \quad t \in [0,T]$$

But $Q = I(0)$ at initial time $t = 0$.

$$Q = \frac{K}{\theta} \left(e^{\theta T} - 1 \right)$$

$$\Rightarrow \frac{dQ}{dT} = Ke^{\theta T}$$

Now we have to find the relation between t_a and w, which is given by this equation

$$w = \frac{k}{\theta} \left(e^{\theta t_e} - 1 \right)$$

$$\Rightarrow t_a = \frac{1}{\theta} \ln \left(\frac{\theta w}{K} + 1 \right)$$

The component of total cost per unit time $TC(T)$ are of the following

1. Cost of ordering per order per unit time $= \frac{a}{T}$

2. Deteriorating cost per unit time
$$= \frac{c(Q-KT)}{T} = \frac{cK}{\theta T}\left\{e^{\theta T} - \theta T - 1\right\}$$

3. Holding cost per unit time is determine in two cases as follows

Case 1: $T \le T_a$ i.e. $Q \le W$. In this case, no needs to rented warehouse. Therefore, no stock holding cost for items in RW.

Holding cost in OW is $= \frac{h_1 K}{\theta T} \int_0^T \left(e^{\theta(T-t)} - 1\right) dT$

$$= \frac{h_1 K}{\theta^2 T}\left\{e^{\theta T} - \theta T - 1\right\}$$

Case 2: $T > T_a$ or $Q > w$. In this case total holding cost per unit time is calculated in two parts.
Holding cost per unit time in OW is

$$= \frac{h_1}{\theta^2 T}\left\{K\left(e^{\theta T_e} - \theta T_a - 1\right) + w\theta^2 (T - T_a)\right\}$$

Holding cost per unit time in RW is

$$= \frac{h_2}{\theta^2 T}\left\{K\left(e^{\theta T} - e^{\theta T_e}\right) - \left(K\theta + \theta^2 w\right)(T - T_a)\right\}$$

Therefore, total holding cost when $T > T_a$ i.e. $Q > W$

$$= \frac{h_1\left\{K(e^{\theta T_e} - \theta T_a - 1) + \theta^2 w(T - T_a)\right\}}{\theta^2 T}$$
$$+ \frac{h_2\left\{K(e^{T\theta} - e^{\theta T_a}) - (K\theta + \theta^2 w)(T - T_a)\right\}}{\theta^2 T}$$

4. The opportunity cost i.e. interest earned and paid per unit time are calculated in three cases as:

   1. $T \le N$,
   2. $N \le T \le M$,
   3. $M \le T$.

Case 1: If $T \le N$
Interest earned $= pI_e K(M - N)$.
Interest paid $= 0$
Case 2: $N \le T \le M$

Interest earned

$$= \frac{KpI_e(T^2 - N^2) + (2KpT + KpI_e(T^2 - N^2)I_e)(M - T)}{2T}$$
$$= \frac{KpI_e(T^2 - N^2) + (2KTp + DpI_e(T^2 - N^2)I_e)(M - T)}{2T}$$

Interest paid $= 0$.
Case 3: When $M \le T$. In this case, both interest are determine in two sub cases

   (a) $cQ > w^*$ and
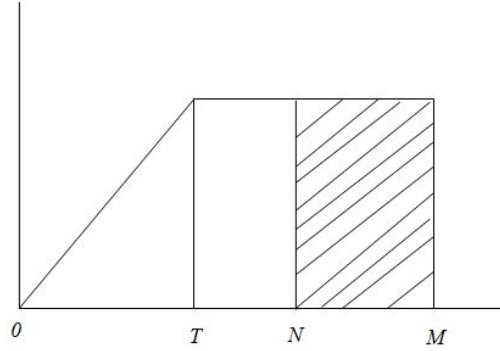   (b) $cQ \le w^*$ where $w^* = KMp + \frac{1}{2}KpI_e\left(M^2 - N^2\right)$
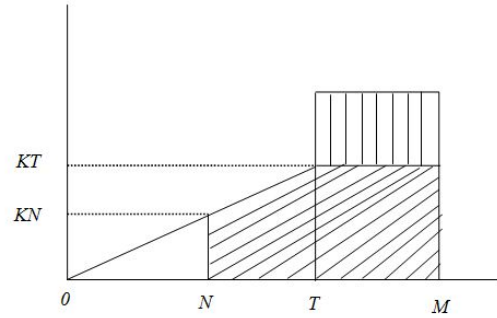

Figure 1

Figure 2

Sub Case 3.1: If $cQ > w^*$. In this retailer gradually reduces the finance loan from constant sales and revenue. The settlement point B after M is obtained from this equation

$$\left(Kp - \frac{1}{2}(cQ - w^*)I_p\right)(B - M) = (cQ - w^*)$$

$$B = M + \frac{2(cQ - w^*)}{2Kp - (cQ - W^*)I_p}$$

After B, the trader(wholesaler) generated the sale revenue and interest earned on that revenue up to T.
Thus, the annual interest earned

$$= \frac{1}{2T}KpI_e(T - B)^2$$

and the interest paid in this sub case is

$$= \frac{1}{2T}(cQ - w^*)(B - M)I_p$$

Sub Case 3.2: If $cQ \le w^*$. In this sub-case, trader has w^* amount at M. But he pays cQ amount to his supplier for settlement your account with supplier. After settlement his account, he has $(w^* - cQ)$ balance amount at M. On this amount, he earns the interest from M to T is $= (w^* - cQ)(T - M)I_e$. On the other hand, the trader constantly sells the products and generates the revenue up to T. The interest earned on that revenue during the period $[M, T] = \frac{1}{2}KpI_e(T - M)^2$.
Thus, the total interest earned per unit time in this sub case

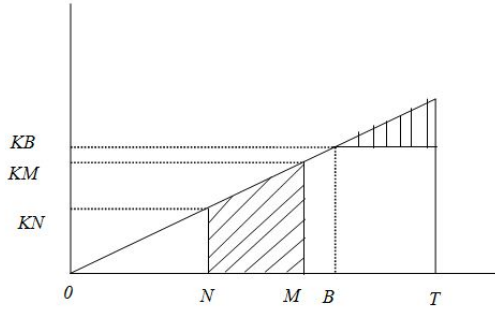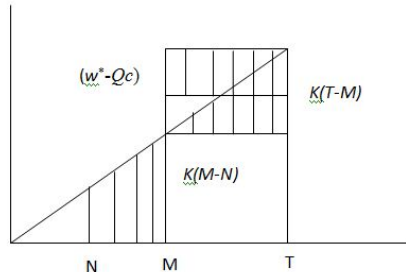$$= \frac{I_e}{T}(w^* - cQ)(T - M) + \frac{1}{2T}KpI_e(T - M)^2$$

Figure 3

Figure 4

In this sub case, the interest charge is zero. The total cost per unit time can be determine by following formula

$TC(T) =$ [ordering cost + cost of deterioration + cost of holding in (OW + RW) + interest paid - interest earned]$/T$

$TC(T)$ can be calculated on the positional values of $T_a, N, M,$ and T are as follows

1. If $T_a < N < M$
   The total cost per unit time as follows:

$$TC(T) = \begin{cases} TC_1(T); & \text{where, } 0 < T \leq T_a \\ TC_2(T); & \text{where, } T_a < T \leq N \\ TC_3(T); & \text{where, } N < T \leq M \\ TC_4(T); & \text{where, } M < T \ \& \ cQ \leq w* \\ TC_5(T); & \text{where, } M < T \ \& \ cQ > w* \end{cases}$$

Where,

$$TC_1(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right) - pKI_e(M - N) \tag{2.1}$$

$$TC_2(T) = \frac{a}{T} + \frac{K(h_2 + c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2 - h_1)}{\theta^2 T}$$
$$\left\{K\left(e^{T_e\theta} - T\theta_a - 1\right) + w\theta^2(T - T_a)\right\}$$
$$- pKI_e(M - N) \tag{2.2}$$

$$TC_3(T) = \frac{a}{T} + \frac{K(h_2 + c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right)$$
$$- \frac{(h_2 - h_1)}{\theta^2 T}\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2(T - T_a)\right\}$$
$$- \frac{1}{2T}\left\{KpI_e\left(T^2 - N^2\right)\right.$$
$$\left. + \left(2KTp + Kp\left(T^2 - N^2\right)I_e\right)I_e(M - T)\right\} \tag{2.3}$$

$$TC_4(T) = \frac{a}{T} + \frac{K(h_2 + c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2 - h_1)}{T\theta^2}$$
$$\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2(T - T_a)\right\}$$
$$- \frac{1}{T}I_e(w^* - cQ)(T - M) - \frac{1}{2T}KpI_e(T - M)^2 \tag{2.4}$$

$$TC_5(T) = \frac{a}{T} + \frac{K(h_2 + c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2 - h_1)}{T\theta^2}$$
$$\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2(T - T_a)\right\}$$
$$- \frac{1}{2T}KpI_e(T - B)^2 + \frac{1}{2T}(cQ - w^*)(B - M)I_p \tag{2.5}$$

2. If $N < T_a < M$
   The total cost per unit time as follows:

$$TC(T) = \begin{cases} TC_1(T); & \text{where, } 0 < T \leq N \\ TC_6(T); & \text{where, } N < T \leq T_a \\ TC_3(T); & \text{where, } T_a < T \leq M \\ TC_4(T); & \text{where, } M < T \ \& \ cQ \leq w^* \\ TC_5(T); & \text{where, } M < T \ \& \ cQ > w^* \end{cases}$$

Where,

$$TC_6(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right)$$
$$- \frac{1}{2T}\left\{KpI_e\left(T^2 - N^2\right)\right.$$
$$\left. + \left(2KTp + Kp\left(T^2 - N^2\right)I_e\right)I_e(M - T)\right\} \tag{2.6}$$

3. If $N < M < T_a$
   The total cost per unit time as follows:

$$TC(T) = \begin{cases} TC_1(T); & \text{where, } 0 < T \leq N \\ TC_6(T); & \text{w here, } N < T \leq M \\ TC_7(T); & \text{where, } M < T \leq T_a \ \& \ cQ \leq w^* \\ TC_8(T); & \text{where, } M < T \leq T_a \ \& \ cQ > w^* \\ TC_4(T); & \text{where, } M < T_a \leq T \ \& \ cQ \leq w^* \\ TC_5(T); & \text{where, } M < T_a \leq T \ \& \ cQ > w^* \end{cases}$$

Where,

$$TC_7(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{\theta^2 T}\left(e^{\theta T} - \theta T - 1\right)$$
$$- \frac{1}{2T}KpI_e(T - B)^2 + \frac{1}{2T}(cQ - w^*)(B - M)I_p \tag{2.7}$$

$$TC_8(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right)$$
$$- \frac{1}{T}(w^* - cQ)(T - M)I_e + \frac{1}{2T}KpI_e(T - M)^2 \tag{2.8}$$

3. The Convexities of $TC_i(T)$ and i=1,2, … 8

$$TC_1'(T) = -\frac{a}{T^2} - \frac{K(h_1+c\theta)}{T^2\theta^2}\left(e^{T\theta}-T\theta-1\right)$$
$$+\frac{D(h_1+c\theta)}{T\theta^2}\left(\theta e^{T\theta}-1\right)$$
$$=\frac{1}{(T\theta)^2}\left[-\theta^2 A + K(h_1+c\theta)\left\{T\theta e^{T\theta}-e^{T\theta}+1\right\}\right] \quad (3.1)$$

$$TC_1''(T) = \frac{2a}{T^3} + \frac{K(h+c\theta)}{T^3\theta^2}\left\{(T\theta)^2 e^{T\theta}-2T\theta e^{T\theta}+2e^{T\theta}-2\right\} \quad (3.2)$$

$$TC_2'(T) = \frac{1}{(T\theta)^2}\left[-\theta^2 A + K(h_2+c\theta)\left(T\theta e^{T\theta}-e^{T\theta}+1\right)\right.$$
$$\left.-K(h_2-h_1)\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)\right] \quad (3.3)$$

$$TC_2''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta}-2T\theta e^{T\theta}+2e^{T\theta}-2\right\}$$
$$+\frac{2K(h_2-h_1)}{T^3\theta^2}\left(T_a\theta e^{T_e\theta}-e^{\tau_\theta\theta}+1\right) \quad (3.4)$$

$$TC_3'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+\theta c)\left(T\theta e^{T\theta}-e^{T\theta}+1\right)\right.$$
$$\left.-K(h_2-h_1)\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)\right]$$
$$-\frac{KpI_e}{2T^2}\left\{\left(N^2-T^2\right)+(1+TI_e)+\left(N^2+T^2\right)I_e\right\} \quad (3.5)$$

$$TC_3''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta}-2T\theta e^{T\theta}+2e^{T\theta}-2\right\}$$
$$+\frac{2K(h_2-h_1)}{\theta^2 T^3}\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)$$
$$+\frac{KpI_e}{2T^3}\left\{T^2+N^2+I_e\left(N^2-T^2+2T^3\right)\right\} \quad (3.6)$$

$$TC_4'(T) = \frac{1}{(T\theta)^2}\left[-\theta^2 a + K(h_2+c\theta)\left(\theta T e^{T\theta}-e^{T\theta}+1\right)\right.$$
$$\left.-K(h_2-h_1)\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)\right]$$
$$+\frac{1}{T^2}\left[I_e(w^*-cQ)(T-M)+\frac{KpI_e}{2}(T-M)^2\right]$$
$$-\frac{1}{T}\left[-I_e c\frac{dQ}{dT}(T-M)+I_e(w^*-cQ)+KpI_e(T-M)\right] \quad (3.7)$$

$$TC_4'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+c\theta)\left(T\theta e^{T\theta}-e^{T\theta}+1\right)\right.$$
$$\left.-K(h_2-h_1)\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)\right]$$
$$+\frac{1}{T^2}\left[I_e(w^*-cQ)(T-M)+\frac{KpI_e}{2}(T-M)^2\right]$$
$$-\frac{1}{T}\left[-cKe^{T\theta}(T-M)I_e+(w^*-cQ)I_e+Kp(T-M)I_e\right] \quad (3.8)$$

$$TC_4''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left[(T\theta)^2 e^{T\theta}-2T\theta e^{T\theta}\right.$$
$$\left.+2e^{T\theta}-2\right]+\frac{2K(h_2-h_1)}{T^3\theta^2}\left[T_a\theta e^{T_z\theta}-e^{T_e\theta}+1\right]$$

$$+\frac{1}{T^3}\left[\begin{array}{c} 2MI_e(w^*-cQ)+(T-M) \\ \left\{cKI_e e^{T\theta}\left(T^2\theta-2\right)+KpTI_e\right\} \\ +2cKT^2 I_e e^{T\theta}+KpT^2 I_e \end{array}\right] \quad (3.9)$$

$$TC_5(T) = \frac{a}{T} + \frac{K(h_2+c\theta)}{T\theta^2}\left(e^{T\theta}-T\theta-1\right)$$
$$-\frac{(h_2-h_1)}{T\theta^2}\left\{K\left(e^{T_e\theta}-T_a\theta-1\right)+w\theta^2(T-T_a)\right\}$$
$$-\frac{1}{2T}KpI_e(T-B)^2 + \frac{1}{2T}(cQ-w^*)(B-M)I_p \quad (3.10)$$

where,

$$B = M + \frac{2(Qc-w^*)}{2pK-(Qc-w^*)I_p},$$
$$W^* = KMp + \frac{1}{2}KpI_e\left(M^2-N^2\right),$$
$$Q = \frac{K}{\theta}\left(e^{T\theta}-1\right) \Rightarrow \frac{dQ}{dT}=Ke^{T\theta}$$

$$\frac{dB}{dT} = \frac{4K^2 cpe^{T\theta}}{\left\{2Kp-(cQ-w^*)I_p\right\}^2} > 0 \quad (3.11)$$

$$TC_5'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+c\theta)\left(Te^{T\theta}-e^{T\theta}+1\right)\right.$$
$$\left.-K(h_2-h_1)\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)\right]$$
$$+\frac{1}{2T^2}KpI_e(T-B)^2 - \frac{1}{2T}KpI_e 2(T-B)$$
$$\left(1-\frac{dB}{dT}\right) - \frac{1}{2T^2}(cQ-w^*)(B-M)I_p$$
$$+\frac{1}{2T}c\frac{dQ}{dT}(B-M)I_p + \frac{1}{2T}(cQ-w^*)\frac{dB}{dT}I_p \quad (3.12)$$

$$TC_5''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left\{\theta^2 T^2 e^{\tau\theta}-2T\theta e^{T\theta}+2e^{T\theta}\right.$$
$$\left.-2\right\}+\frac{2K(h_2-h_1)}{T^3\theta^2}\left(T_a\theta e^{T_e\theta}-e^{T_e\theta}+1\right)$$
$$+\frac{1}{T^3}I_p(B-M)\left(cQ-w^*-TKce^{\theta T}\right)$$
$$+\frac{1}{T^3}KpI_e\left(T^2-B^2\right)+\frac{1}{2T^2}I_p(B-M)\left(c\frac{dQ}{dT}\right.$$
$$\left.-T\theta Dce^{T\theta}-Dce^{T\theta}\right)-\frac{1}{2T^2}KpI_e\left(2T-2B\frac{dB}{dT}\right)$$
$$+\frac{1}{\left\{T\left\{2Kp-(cQ-w^*)I_p\right\}^2\right\}^2}$$

$$\left[\begin{array}{l} \left\{2K^2 cp\theta e^{T\theta}\left\{(cQ-w^\circ)I_p+2KpI_e\right\}\right. \\ \left.+2K^2 c^2 pe^{T\theta}\frac{dQ}{dT}I_p\right\}T\left\{2Kp-(cQ-W^*)I_p\right\}^2 \\ -\left\{\left\{2Kp-(cQ-w^*)I_p\right\}^2+T\left\{-c\frac{dQ}{dT}I_p\right\}^2\right\} \\ \left\{2K^2 cpe^{T\theta}\left\{(cQ-w^*)I_p+2KpI_e\right\}\right\} \end{array}\right] \quad (3.13)$$

$$TC_6'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+c\theta)\left(T\theta e^{T\theta}-e^{T\theta}+1\right)\right]$$

$$-\frac{1}{2T^2}\left[-3KpT^3I_e^2+2KpT^2MI_e^2-KpT^2I_e\right.$$

$$\left.+KpN^2I_e-KpT^2M^2I_e^2\right] \qquad (3.14)$$

$$TC_6''(T) = \frac{2a}{T^3} + \frac{K(h_1+\theta c)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta}-2T\theta e^{T\theta}\right.$$

$$\left.+2e^{T\theta}-2\right\}+\frac{KpI_e}{2T^3}\left(3T^3+N^2\right) \qquad (3.15)$$

$$TC_7'(T) = \frac{1}{(T\theta^2)^2}\left[-a\theta^2+K(h_1+\theta c)\left(T\theta^2 e^{T\theta}-e^{T\theta}\right.\right.$$

$$\left.\left.+1\right)\right]-\frac{I_p}{2T^2}\left[(cQ-B)(B-M)-TcQ'(B-M)\right.$$

$$\left.-(cQ-w^*)TB'\right]-\frac{DpI_e}{2T^2}\left(B-TB'\right)(T-B)$$

$$(3.16)$$

$$TC_7''(T) = \frac{2a}{T^3}+\frac{K(h_1+\theta c)}{\theta^2 T^3}\left\{T^2\theta e^{T\theta}-2T\theta e^{T\theta}+2e^{T\theta}-2\right\}$$

$$+\frac{I_p}{T^3}\left[\begin{array}{c}(B-M)(cQ-B)\\-TcQ'(B-M)-(cQ-w^*)B'T\end{array}\right]$$

$$-\frac{I_p}{2T^2}\left[\begin{array}{c}(Q'c-B')(B-M)+(Qc-B)B'\\-Q'c(B-M)-cT(B-M)Q''\\-2cQ'B'T-(Qc-w^*)B'\\-(Qc-w^*)TB'\end{array}\right]$$

$$+\frac{KpI_e}{T^3}\left(B-B'T\right)(T-B)-\frac{KpI_e}{2T^2}\left[(B-B'T)\right.$$

$$\left.(1-B')+(B'-TB''-B')(T-B)\right] \qquad (3.17)$$

$$TC_8'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2+K(h_1+\theta c)\left(T\theta e^{T\theta}\right.\right.$$

$$\left.\left.-e^{T\theta}+1\right)\right]+\frac{I_e}{T}cQ'(T-M)-\frac{MI_e}{T^2}(w^*-cQ)$$

$$+\frac{KpI_e}{2T^2}\left(T^2-M^2\right)(B-TB') \qquad (3.18)$$

$$TC_8''(T) = \frac{2a}{T^3}+\frac{K(h_1+\theta c)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta}-2T\theta e^{T\theta}\right.$$

$$\left.+2e^{T\theta}-2\right\}+\frac{2MI_e(w^*-Qc)}{T^3}+\frac{KpI_e M^2}{T^3}$$

$$+\frac{I_e}{T}cQ''(T-M)+2cMQ' \qquad (3.19)$$

4. Convexity of the cost function

Theorem 4.1. *Prove that $TC_1(T)$ is convex function when $T>0$.*

Firstly we have to prove the following lemma 4.2, after that prove this theorem.

Lemma 4.2. $(T\theta)^2-2T\theta+2-2e^{-T\theta}>0$, *when $T\theta>0$*

Proof. Let

$$g(y) = y^2-2y+2-2e^{-y} \qquad (4.1)$$

for $y\geq 0$

$$g'(y) = 2y-2+2e^{-y}$$
$$g''(y) = 2-2e^{-y}>0$$

$\because e^{-y}<1$. Hence $g'(y)$ is an increasing function of y. $g'(0)=0$.

Since $g'(y)>g'(0)=0$.
This implies that $g'(y)>0$.
Hence $g'(y)$ is an increasing function of $y>0$. Substitute $y=T\theta$ in equation (4.1), then we get

$$g(T\theta)=(T\theta)^2-2T\theta+2-2e^{-T\theta}>0 \qquad (4.2)$$

if $T\theta>0$. $\qquad\square$

Proof of Theorem 4.1: From the equation (2.1), (3.1) and (3.2)

$$TC_1(T) = \frac{a}{T}+\frac{K(h_1+c\theta)}{T\theta^2}\left(e^{T\theta}-T\theta-1\right)-pKI_e(M-N)$$

$$TC_1'(T) = -\frac{a}{T^2}-\frac{K(h_1+c\theta)}{T^2\theta^2}\left(e^{T\theta}-T\theta-1\right)$$

$$+\frac{D(h_1+c\theta)}{T\theta^2}\left(\theta e^{T\theta}-1\right)$$

$$=\frac{1}{(T\theta)^2}\left[-\theta^2 A+K(h_1+c\theta)\left\{T\theta e^{T\theta}-e^{T\theta}+1\right\}\right]$$

$$TC_1''(T) = \frac{2a}{T^3}+\frac{K(h+c\theta)}{T^3\theta^2}e^{T\theta}\left\{(T\theta)^2-2T\theta+2-2e^{-T\theta}\right\}$$

By Lemma 4.2, $TC_1'(T)>0$ When $T>0$. This implies that $TC_1(T)$ is convex function on $T>0$.

Theorem 4.3. *Prove that $TC_2(T)$ is convex function of T when $T>0$.*

Firstly we have to prove the following lemma 4.4, after that prove this theorem.

Lemma 4.4. $\left(T_a\theta-1+e^{-T_e\theta}\right)>0$ *for $T_a\theta>0$.*

Proof. Let

$$h(y)=y-1+e^{-y} \text{ for } y\geq 0 \qquad (4.3)$$

$$h'(y) = 1-e^{-y}$$
$$h''(y) = e^{-y}>0$$

Hence $h'(y)$ is an increasing function of y. Substitute $y=T_a\theta$ in equation (4.3), then we get $h(T_a\theta)=T_a\theta-1+2e^{-T_e\theta}>0$. If $T_a\theta>0$. $\qquad\square$

Proof of Theorem 4.3: From the equation (2.2), (3.3) and (3.4) We know that,

$$TC_2(T) = \frac{a}{T} + \frac{K(h_2+c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2-h_1)}{\theta^2 T}$$
$$\left\{K\left(e^{\tau_s\theta} - T\theta_a - 1\right) + w\theta^2(T-T_a)\right\}$$
$$- pKI_e(M-N)$$

$$TC_2'(T) = \frac{1}{(T\theta)^2}\left[-\theta^2 A + K(h_2+c\theta)\left(T\theta e^{T\theta} - e^{T\theta} + 1\right)\right.$$
$$\left. - K(h_2-h_1)\left(T_a\theta e^{T-\theta} - e^{T_E^{T\theta}} + 1\right)\right]$$

$$TC_2''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta}\right.$$
$$\left. - 2\right\} + \frac{2K(h_2-h_1)}{T^3\theta^2}\left(T_a\theta e^{\tau_e\theta} - e^{\tau_s\theta} + 1\right)$$

$$TC_2''(T) = TC_1''(T) + \frac{2K(h_2-h_1)}{T^3\theta^2}\left(T_a\theta e^{\tau_s\theta} - e^{\tau_e\theta} + 1\right)$$

From Lemma 4.4 and $TC_1(T)$ is convex so that $TC_1''(T) > 0$. This implies that $TC_2''(T) > 0$ When $T > 0$.

Theorem 4.5. *Prove that $TC_3(T)$ is convex on $T > 0$.*

Proof. From the equation (2.3), (3.5) and (3.6)

$$TC_3(T) = \frac{a}{T} + \frac{K(h_2+c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2-h_1)}{\theta^2 T}$$
$$\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2(T-T_a)\right\}$$
$$- \frac{1}{2T}\left\{KpI_e\left(T^2-N^2\right)\right.$$
$$\left. + \left(2KTp + Kp\left(T^2-N^2\right)I_e\right)I_e(M-T)\right\}$$

$$TC_3'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+\theta c)\left(T\theta e^{T\theta} - e^{T\theta} + 1\right)\right.$$
$$\left. - K(h_2-h_1)\left(T_a\theta e^{T_e\theta} - e^{T_e\theta} + 1\right)\right]$$
$$- \frac{KpI_e}{2T^2}\left\{(N^2-T^2) + (1+TI_e) + (N^2+T^2)I_e\right\}$$

$$TC_3''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left\{T^2\theta^2 e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta} - 2\right\}$$
$$+ \frac{2K(h_2-h_1)}{\theta^2 T^3}\left(T_a\theta e^{T_e\theta} - e^{T_e\theta} + 1\right)$$
$$+ \frac{KpI_e}{2T^3}\left\{N^2+T^2+I_e\left(N^2+2T^3-T^2\right)\right\}$$

$$TC_3''(T) = TC_2''(T) + \frac{KpI_e}{2T^3}\left\{N^2+T^2+I_e\left(N^2+2T^3-T^2\right)\right\}$$
$$\tag{4.4}$$

□

Lemma 4.6. *Prove that $N^2+T^2+I_e\left(N^2+2T^3-T^2\right) > 0$.*

Proof. Let

$$h(T) = N^2+T^2+I_e\left(N^2+2T^3-T^2\right) \tag{4.5}$$

$$h'(T) = 2T + 6T^2 I_e - 2I_e T$$
$$h'(T) = 2T(1-I_e) + 6T^2 I_e$$

$h'(T) > 0$ $\because 0 < I_e < 1$. Hence, $h(T)$ is increasing function of T and $f(0) = (1+I_e)N^2 > 0$. Hence, $f(T) > 0$, when $T > 0$. Now continue from (4.4) and using (4.5)

$$TC_3''(T) = TC_2''(T) + \frac{KpI_e}{2T^3}h(T)$$

From Lemma 4.6, $h(T) > 0$ and $TC_2(T)$ is convex so that $TC_3''(T) > 0$. Hence, $TC_3(T)$ is a convex function when $T > 0$. □

Theorem 4.7. *Prove that $TC_4(T)$ is convex function when $T > 0$.*

Proof. From Equation (2.4), (3.7) and (3.8), we have

$$TC_4(T) = \frac{a}{T} + \frac{K(h_2+c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2-h_1)}{T\theta^2}$$
$$\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2(T-T_a)\right\}$$
$$- \frac{1}{T}I_e(w^*-cQ)(T-M) - \frac{1}{2T}KpI_e(T-M)^2$$

$$TC_4'(T) = \frac{1}{(T\theta)^2}\left[-\theta^2 a + K(h_2+c\theta)\left(\theta Te^{T\theta} - e^{T\theta} + 1\right)\right.$$
$$\left. - K(h_2-h_1)\left(T_a\theta e^{T_e\theta} - e^{T_e\theta} + 1\right)\right]$$
$$+ \frac{1}{T^2}\left[I_e(w^*-cQ)(T-M) + \frac{KpI_e}{2}(T-M)^2\right]$$
$$- \frac{1}{T}\left[-I_ec\frac{dQ}{dT}(T-M) + I_e(w^*-cQ) + KpI_e(T-M)\right]$$

$$TC_4'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2+c\theta)\left(T\theta e^{T\theta} - e^{T\theta} + 1\right)\right.$$
$$\left. - K(h_2-h_1)\left(T_a\theta e^{T_e\theta} - e^{T_e\theta} + 1\right)\right]$$
$$+ \frac{1}{T^2}\left[I_e(w^*-cQ)(T-M) + \frac{KpI_e}{2}(T-M)^2\right]$$
$$- \frac{1}{T}\left[-cKe^{T\theta}(T-M)I_e + (w^*-cQ)I_e + Kp(T-M)I_e\right]$$

$$TC_4''(T) = \frac{2a}{T^3} + \frac{K(h_2+c\theta)}{T^3\theta^2}\left[(T\theta)^2 e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta} - 2\right]$$
$$+ \frac{2K(h_2-h_1)}{T^3\theta^2}\left[T_a\theta e^{T_e\theta} - e^{T_e\theta} + 1\right]$$
$$+ \frac{1}{T^3}\left[\begin{array}{c}2MI_e(w^*-cQ) + (T-M)\\ \{cKI_ee^{T\theta}(T^2\theta-2) + KpTI_e\}\\ +2cKT^2I_ee^{T\theta} + KpT^2 I_e\end{array}\right]$$

$$TC_4'(T) = TC_2'(T) + \frac{1}{T^2}\left[I_e(w^*-cQ)(T-M) + \frac{KpI_e}{2}(T-M)^2\right]$$
$$- \frac{1}{T}\left[-cKe^{T\theta}(T-M)I_e + (w^*-cQ)I_e + Kp(T-M)I_e\right]$$

$$TC_4''(T) = TC_2''(T) + \frac{1}{T^3}$$
$$\left[\begin{array}{c}2MI_e(w^*-Qc) + (T-M)\\ \{cKI_ee^{T\theta}(T^2\theta-2) + KpTI_e\}\\ +2cKT^2I_ee^{T\theta} + KpT^2 I_e\end{array}\right]$$

$$TC_4''(T) = TC_2''(T) + \frac{1}{T^3}$$

$$\begin{bmatrix} 2MI_e\left(w^* - cQ\right) + KpTI_e(T - M) \\ +2cKI_eT\left(\theta M + T + \theta T^2 - 1\right) \\ +KpT^2I_e \end{bmatrix} \qquad (4.6)$$

$TC_2(T)$ is a convex function. Therefore $TC_2''(T) > 0$. This implies that $TC_4''(T) > 0$, if $M\theta + T + T^2\theta - 1 > 0$. Hence, $TC_4(T)$ is convex function when $T > 0$ and $M\theta + T + T^2\theta - 1 > 0$. $\qquad \square$

Theorem 4.8. *Prove that $TC_5(T)$ is convex on $T > 0$.*

Proof. From the equation (2.5), (3.9) and (3.10), then

$$TC_5(T) = \frac{a}{T} + \frac{K\left(h_2 + c\theta\right)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right) - \frac{(h_2 - h_1)}{T\theta^2}$$
$$\left\{K\left(e^{T_e\theta} - T_a\theta - 1\right) + w\theta^2\left(T - T_a\right)\right\}$$
$$- \frac{1}{2T}KpI_e(T - B)^2 + \frac{1}{2T}\left(cQ - w^*\right)(B - M)I_p$$

$$\frac{dB}{dT} = \frac{4K^2cpe^{T\theta}}{\left\{2Kp - (cQ - w^*)I_p\right\}^2} > 0 \qquad (4.7)$$

$$TC_5(T) = TC_2(T) + pKI_e(M - N) - \frac{1}{2T}KpI_e(T - B)^2$$
$$+ \frac{1}{2T}\left(Qc - w^*\right)(B - M)I_p$$
$$TC_5'(T) = TC_2'(T) + \frac{1}{2T^2}KpI_e(T - B)^2$$
$$- \frac{1}{2T}KpI_e2(T - B)\left(1 - \frac{dB}{dT}\right)$$
$$- \frac{1}{2T^2}\left(cQ - w^*\right)(B - M)I_p$$
$$+ \frac{1}{2T}c\frac{dQ}{dT}(B - M)I_p + \frac{1}{2T}\left(cQ - w^*\right)\frac{dB}{dT}I_p$$

$$TC_5''(T) = TC_2''(T) + \frac{1}{T^3}I_p(B - M)\left(cQ - w^* - TKce^{\theta T}\right)$$
$$+ \frac{1}{T^3}KpI_e\left(T^2 - B^2\right)$$
$$+ \frac{1}{2T^2}I_p(B - M)\left(c\frac{dQ}{dT} - T\theta Dce^{T\theta} - Dce^{T\theta}\right)$$
$$- \frac{1}{2T^2}KpI_e\left(2T - 2B\frac{dB}{dT}\right)$$
$$+ \frac{1}{\left\{T\left\{2Kp - (cQ - w^*)I_p\right\}^2\right\}^2}$$

$$\begin{bmatrix} \dfrac{\left\{2K^2cp\theta e^{T\theta}\left\{(cQ - w^*)I_p + 2KpI_e\right\} + 2K^2c^2pe^{T\theta}\frac{dQ}{dT}I_p\right\}}{T\left\{2Kp - (cQ - W^*)I_p\right\}^2} \\ - \dfrac{\left\{\left\{2Kp - (cQ - w^*)I_p\right\}^2 + T\left\{-c\frac{dQ}{dT}I_p\right\}^2\right\}}{\left\{2K^2cpe^{\tau\theta}\left\{(cQ - w^*)I_p + 2KpI_e\right\}\right\}} \end{bmatrix}$$

$$TC_5(T) = TC_2(T) + pKI_e(M - N) - \frac{1}{2T}KpI_e(T - B)^2$$
$$+ \frac{1}{2T}\left(Qc - W^*\right)(B - M)I_p$$

where,

$$B = M + \frac{2\left(Qc - w^*\right)}{2Kp - (Qc - w^*)I_p},$$

$$w^* = DMp + \frac{1}{2}DpI_e\left(M^2 - N^2\right)$$

$$Q = \frac{K}{\theta}\left(e^{T\theta} - 1\right) \Rightarrow \frac{dQ}{dT} = De^{T\theta}$$

$$\frac{dB}{dT} = \frac{\left\{2Kp - (Qc - w^*)I_p\right\}2c\frac{dQ}{dT} - 2(Qc - w^*)\left\{-c\frac{dQ}{dT}I_p\right\}}{\left\{2Kp - (Qc - w^*)I_p\right\}^2}$$

$$\frac{dB}{dT} = \frac{4K^2cpe^{\theta T}}{\left\{2Dp - (Qc - w^*)I_p\right\}^2} > 0$$

$$\frac{d^2B}{dT^2} = \frac{4K^2\theta cpe^{\theta T}}{\left\{2Kp - (Qc - w^*)I_p\right\}^3}$$
$$\left[2Kp\theta - (Qc - w^*)\theta I_p + 2cDe^{\theta T}I_p\right] > 0$$

$$TC_5'(T) = TC_2'(T) + \frac{1}{2T^2}KpI_e(T - B)^2$$
$$- \frac{1}{2T}KpI_e2(T - B)\left(1 - \frac{dB}{dT}\right)$$
$$- \frac{1}{2T^2}\left(Qc - w^*\right)(B - M)I_p$$
$$+ \frac{1}{2T}c\frac{dQ}{dT}(B - M)I_p + \frac{1}{2T}\left(Qc - w^*\right)\frac{dB}{dT}I_p$$
$$TC_5'(T) = TC_2'(T) + \frac{1}{2T^2}I_p(B - M)\left((Qc - w^*) - TDce^{\theta T}\right)$$
$$- \frac{1}{2T^2}KpI_e\left(T^2 - B^2\right)$$
$$+ \frac{2K^2cpe^{\theta T}\left\{(Qc - w^*)I_p + 2KpI_e\right\}}{T\left\{2Kp - (Qc - w^*)I_p\right\}^2}$$

$$TC_5''(T) = TC_2''(T) + \frac{1}{T^3}\left[I_p(B - M)\left(cQ - w^* - TKce^{T\theta}\right)\right.$$
$$\left. + KpI_e\left(T^2 - B^2\right)\right]$$
$$+ \frac{1}{2T^2}\left[I_p(B - M)\left(c\frac{dQ}{dT} - T\theta Kce^{\theta\theta} - Kce^{T\theta}\right.\right.$$
$$\left.\left. - 2K_pI_eT + 2KpI_eBB'\right)\right]$$

$$+ \frac{1}{\left\{T\left\{2Kp - (Qc - w^*)I_p\right\}^2\right\}^2}$$

$$\begin{bmatrix} \left\{2K^2cp\theta e^{T\theta}\left\{(Qc - w^*)I_p + 2KpI_e\right\}\right. \\ \left. + 2K^2c^2pe^{T\theta}\frac{dQ}{dT}I_p\right\}T\left\{2Kp - (Qc - w^*)I_p\right\}^2 \\ \left\{\left\{2Kp - (Qc - w^*)I_p\right\}^2 + T\left\{-c\frac{dQ}{dT}I_p\right\}\right\} \\ \left\{2K^2cpe^{T\theta}\left\{(Qc - w^*)I_p + 2KpI_e\right\}\right\} \end{bmatrix} \quad (4.8)$$

This implies that $TC_5''(T) > 0$. If $\left(cQ - w^* - TKce^{T\theta}\right) > 0$, and $c\frac{dQ}{dT} - T\theta Kce^{T\theta} - Kce^{T\theta} - 2KpI_\varepsilon T + 2KpI_\varepsilon BB' > 0$ Using all above conditions, this shows that $TC_5(T)$ is a convex function on T. $\qquad\square$

Theorem 4.9. *Prove that $TC_6(T_j)$ is convex on $T_j > 0$.*

Proof. From the equation (2.6), (3.11) and (3.12), then

$$TC_6(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{T\theta^2}\left(e^{T\theta} - T\theta - 1\right)$$
$$- \frac{1}{2T}\left\{KpI_e\left(T^2 - N^2\right) + \left(2KTp + Kp\left(T^2 - N^2\right)I_e\right)I_e(M - T)\right\}$$

$$TC_6'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_2 + c\theta)\left(T\theta e^{T\theta} - e^{T\theta} + 1\right)\right]$$
$$- \frac{1}{2T^2}\left[-3KpT^3I_e^2 + 2KpT^2MI_e^2 - KpT^2I_e + KpN^2I_e - KpT^2M^2I_e^2\right]$$

$$TC_6''(T) = \frac{2a}{T^3} + \frac{K(h_1 + \theta c)}{T^3\theta^2}$$
$$\left\{T^2\theta^2 e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta} - 2\right\}$$
$$+ \frac{KpI_e}{2T^3}\left(3T^3 + N^2\right)$$
$$TC_6''(T) = TC_1''(T) + \frac{DpI_e}{2T^3}\left(3T^3 + N^2\right)$$

By theorem 4.1, $TC_6''(T) > 0$ since $TC_1''(T) > 0$. Hence, $TC_6(T)$ is convex on $T > 0$. $\qquad\square$

Theorem 4.10. *Prove that $TC_7(T)$ is convex on $T > 0$.*

Proof.

$$TC_7(T) = \frac{a}{T} + \frac{K(h_1 + c\theta)}{\theta^2 T}\left(e^{\theta T} - \theta T - 1\right)$$
$$- \frac{1}{2T}KpI_e(T - B)^2 + \frac{1}{2T}(cQ - w^*)(B - M)I_p$$

$$TC_7'(T) = \frac{1}{(T\theta^2)^2}\left[-a\theta^2 + K(h_1 + \theta c)\left(T\theta^2 e^{T\theta} - e^{T\theta} + 1\right)\right]$$
$$- \frac{I_p}{2T^2}\left[(cQ - B)(B - M) - TcQ'(B - M)\right.$$
$$\left. - (cQ - w^*)TB'\right] - \frac{DpI_e}{2T^2}\left(B - TB'\right)(T - B)$$

$$TC_7''(T) = \frac{2a}{T^3} + \frac{K(h_1 + \theta c)}{\theta^2 T^3}\left\{T^2\theta e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta} - 2\right\}$$
$$+ \frac{I_p}{T^3}\begin{bmatrix}(B - M)(cQ - B) \\ -TcQ'(B - M) - (cQ - w^*)B'T\end{bmatrix}$$
$$- \frac{I_p}{2T^2}\begin{bmatrix}(Q'c - B')(B - M) + (Qc - B)B' \\ -Q'c(B - M) - cT(B - M)Q'' \\ -2cQ'B'T - (Qc - w^*)B' \\ -(Qc - w^*)TB'\end{bmatrix}$$
$$+ \frac{KpI_e}{T^3}(B - B'T)(T - B) - \frac{KpI_e}{2T^2}$$
$$\left[(B - B'T)(1 - B') + (B' - TB'' - B')(T - B)\right]$$

$$TC_7'(T) = TC_1'(T) - \frac{I_p}{2T^2}\left[(cQ - B)(B - M) - TcQ'(B - M)\right.$$
$$\left. - (cQ - W^*)B'T\right] - \frac{KpI_e}{2T^2}(T - B)(B - B'T)$$

$$TC_7''(T) = TC_1''(T) + \frac{I_p}{T^3}\begin{bmatrix}(B - M)(cQ - B) \\ -TcQ'(B - M) - (cQ - w^*)B'T\end{bmatrix}$$
$$+ \frac{1}{2T^2}\begin{bmatrix}I_p\begin{Bmatrix}+Q'c(B - M) + cT(B - M)Q'' \\ +2cQ'B'T + (Qc - w^*)B' \\ +(Qc - w^*)TB' - (Q'c - B') \\ (B - M) - (Qc - B)B'\end{Bmatrix} \\ -KpI_e\{(B - B'T)(1 - B') \\ +(B' - TB'' - B')(T - B)\}\end{bmatrix}$$
$$+ \frac{KpI_e}{T^3}(B - B'T)(T - B)$$

$$TC_7''(T) = TC_1''(T) + \frac{I_p}{T^3}X + \frac{1}{2T^2}Y + \frac{KpI_\varepsilon}{T^3}Z$$

By theorem 4.1, $TC_7''(T) > 0$ if $X > 0, Y > 0$ and $Z > 0$. Where,

$$X = (B - M)(cQ - B) - TcQ'(B - M) - (cQ - w^*)B'T$$

$$Y = I_p\begin{Bmatrix}+Q'c(B - M) + cT(B - M)Q'' + 2cQ'B'T \\ +(Qc - w^*)B' + (Qc - w^*)TB' \\ -(Q'c - B')(B - M) - (Qc - B)B'\end{Bmatrix}$$
$$Z = (B - B'T)(T - B) - KpI_\varepsilon\left\{(B - B'T)(1 - B') \right.$$
$$\left. + (B' - TB'' - B')(T - B)\right\}$$

Hence, $TC_7(T)$ is a convex function when $T > 0$. $\qquad\square$

Table 1: Values of parameter for different examples

Example	$c(\$)$	$p(\$)$	$a(\$)$	$h_l(\$)$	$h_2(\$)$	θ	K	w	M	N	I_2	I_p
									year		per $	year
$Ta \leq N < M$	5	10	160	1	3	0.1	250	50	0.3	0.2	0.12	0.15
$N < Ta \leq M$	5	10	160	1	3	0.1	250	50	0.3	0.15	0.12	0.15
$N < M < T_a$	5	10	160	1	3	0.1	250	50	0.15	0.1	0.12	0.15

Table 2: Solution of examples

Cases	Sub cases	W^*	B	T_a	Q	T	$TC(T)$
Ta≤N < M	$0 < T < T_a$	–	–	–	–	–	Infeasible
	$T_a < T \leq N$	–	–	–	–	–	Infeasible
	$N < T \leq M$	–	–	–	–	–	Infeasible
	$M < T \& Qc \leq w^*$	–	–	–	–	–	Infeasible
	$M < T \& Qc > w^*$	751.31	0.459	0.198	155	0.60	438.32
$N < Ta \leq M$	$0 < T \leq N$	–	–	–	–	–	Infeasible
	$N < T \leq T_a$	–	–	–	–	–	Infeasible
	$T_a < T \leq M$	–	–	–	–	–	Infeasible
	$M < T \& Qc \leq w^*$	–	–	–	–	–	Infeasible
	$M < T \& Qc > w^*$	651.33	0.459	0.198	154	0.6	438.32
$N < M < T_a$	$0 < T \leq N$	–	–	–	–	–	Infeasible
	$N < T \leq M$	–	–	–	–	–	Infeasible
	$M < T \leq T_a \& Qc \leq w^*$	–	–	–	–	–	Infeasible
	$M < T \leq T_a \& Qc > w^*$	–	–	–	–	–	Infeasible
	$M < T_s \leq T \& Qc \leq w^*$	–	–	–	–	–	Infeasible
	$M < T_a \leq T \& Q > w^*$	375.33	0.38	0.198	152	0.59	443.23

Table 3: Sensitivity analysis on parameters.

Parameters	% Change of parameters	% change				
		B	W^*	Q	T	$TC(T)$
A	−20%	−6.71	0	−9.86	−9.59	−12.78
	−10%	−3.27	0	−4.80	−4.66	−6.22
	+10%	+3.12	0	+4.57	+4.43	+5.95
	+20%	+6.12	0	+8.95	+8.67	+11.66
h	−20%	+0.39	0	+0.58	+0.56	−1.53
	−10%	+0.2	0	+0.29	+0.28	−0.77
	+10%	−0.2	0	−0.29	−0.28	+0.76
	+20%	−0.4	0	−0.58	−0.56	+1.53
k	−20%	+5.75	0	+8.41	+8.15	−5.14
	−10%	+2.68	0	+3.92	+3.80	−2.4
	+10%	−2.36	0	−3.46	−3.36	+2.30
	+20%	−4.45	0	−6.54	−6.36	+4.45
c	−20%	−10.58	0	+5.55	+5.38	−3.61
	−10%	−5.17	0	+2.67	+2.59	−1.78
	+10%	+4.96	0	−2.49	−2.42	+1.72
	+20%	+9.72	0	−4.83	−4.70	+3.39
p	−20%	14.02	−19.97	−3.61	−3.51	+1.93
	−10%	6.19	−9.98	−1.83	−1.78	+0.97
	+10%	−5.01	+9.98	+1.90	+1.84	−0.98
	+20%	−9.13	+19.13	+3.88	+3.76	−1.99
M	−20%	−6.80	−20.03	−0.69	−0.67	+0.32
	−10%	−3.41	−10.02	−0.35	−0.34	+0.14
	+10%	+3.44	+10.02	+0.38	+0.37	−0.10
	+20%	+6.91	+20.04	+0.79	+0.77	−0.15

Theorem 4.11. *Prove that $TC_8(T)$ is convex on $T > 0$.*

Proof. From the equation (8), (23) and (24), we have

$$TC_8(T) = \frac{a}{T} + \frac{K(h+c\theta)}{\theta^2 T}\left(e^{T\theta} - T\theta - 1\right)$$

$$-\frac{1}{T}(w^* - cQ)(T-M)I_e + \frac{1}{2T}KpI_e(T-M)^2$$

$$TC_8'(T) = \frac{1}{(T\theta)^2}\left[-a\theta^2 + K(h_1 + \theta c)\left(T\theta e^{T\theta} - e^{T\theta} + 1\right)\right]$$

$$+ \frac{I_e}{T} cQ'(T-M) - \frac{MI_e}{T^2}(w^* - cQ)$$
$$+ \frac{KpI_e}{2T^2}(T^2 - M^2)(B - TB')$$

$$TC_8''(T) = \frac{2a}{T^3} + \frac{K(h_1 + \theta c)}{T^3 \theta^2}\left\{ T^2\theta^2 e^{T\theta} - 2T\theta e^{T\theta} + 2e^{T\theta} - 2 \right\}$$
$$+ \frac{2MI_e(w^* - Qc)}{T^3} + \frac{KpI_e M^2}{T^3}$$
$$+ \frac{I_e}{T}cQ''(T-M) + 2cMQ'$$

$$TC_8''(T) = TC_1''(T) + \frac{2MI_e(w^* - Qc)}{T^3} + \frac{KpI_e M^2}{T^3}$$
$$+ \frac{I_e}{T}cQ''(T-M) + 2cMQ'$$

Since $TC_1''(T) > 0$ by theorem 4.1 and $W^* \geq cQ, T > M$ in that case. This implies that $TC_8''(T) > 0$. Hence, $TC_8(T)$ is convex on $T > 0$ □

5. Numerical Examples

This EOQ model has been demonstrated on three theoretical examples which are shown by in Table1. These are solved by using Microsoft Excel Solver to obtain the EOQ i.e. Q, replenishment time T, Amount at Mi.e. W^*, Settlement point B, the time during inventory w (OW capacity) is exhausted i.e. T_a, and total cost $TC(T)$.

5.1 Sensitivity Analysis

To affirmation of present model, sensitivity analyses have been performed on different parameters, ie a, h1, h2, c, p, and M on the optimal policies. These studies have been performed on shifting – 20 % to 20 % for single variable and rest variables are as same. The results of This Table3 are Shows that present model is satisfied on realistic circumstances.

6. Conclusion

In present model, a capacity constraint EOQ model for deteriorating items with two level trade credit under financial environment has been developed. The present paper generalized the two level trade credit problems under financial environment. Further, the validity of this model shows by three hypothetical numerical examples and sensitivity analysis of different parameters.

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