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Theoretical fixed point theorem on S-metric space under binary relation via implicit contractive condition with an application

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Abstract
In this paper, we extend and generalises the results by Ahmadullah et al. to self mappings on the S-metric space under a binary relation via implicit contractive condition with an application to an integral equation. We also provided an illustrative example.

Keywords
S-metric space, binary relation, self-mapping, fixed point, implicit relation, integral equation.

AMS Subject Classification
47H10, 54H25.

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1. Introduction

In 1992, Bapure Dhage [5] in his PhD thesis introduced a new class of generalised metric space called the D-metric space. In this work, he defined topological properties, completeness and compactness for D-metric spaces.

The study of fixed point theorems on S-metric space was initiated by Sedghi et al. [17]. They gave an interesting generalization of D-metric space to S-metric space by formulating its properties as follows:

Definition 1.1. [17] *Let X be a non empty set. A S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for all $x,y,z,a \in X$.*

(S_1) $S(x,y,z) \geq 0$;

(S_2) $S(x,y,z) = 0$ *if and only if* $x = y = z$; *and*

(S_3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called the S-metric space.

Some of the examples which satisfies the above characteristics are:

(1) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then,

$$S(x,y,z) = \|y+z-2x\| + \|y-z\|,$$

is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then,

$$S(x,y,z) = \|x-z\| + \|y-z\|,$$

is an S-metric on X.

(3) Let X be a non empty set, d is ordinary metric on X, then

$$S(x,y,z) = d(x,z) + d(y,z),$$

is an S-metric on X.

Sedghi *et al.* [17] proved that the D-metric is the S-metric, but in general the converse is not true.

We give some vivid illustrative examples on S-metric spaces as follows:

Example 1.2. [17] *Let $X = \mathbb{R}^2$, d is an ordinary metric on X, therefore,*

$$S(x,y,z) = d(x,y) + d(x,z) + d(y,z),$$

is a S-metric on X. If we connect the points x,y,z by a line we have a triangle and if we choose a point a mediating this triangle then the inequality

$$S(x,y,z) = S(x,x,a) + S(y,y,a) + S(z,z,a),$$

holds.

Example 1.3. *[17] Let $X = \mathbb{R}$, then*

$$S(x,y,z) = \|x-z\| + \|y-z\|,$$

is a metric space on X. Define a self map F on X by: $Fx = \frac{1}{2}\sin x$. We have

$$\begin{aligned} S(Fx,Fx,Fy) &= |\frac{1}{2}(sinx - siny)| + |\frac{1}{2}(sinx - siny)|, \\ &\leq |\frac{1}{2}(x-y)| + |\frac{1}{2}(x-y)| \\ &\leq \frac{1}{2}(|(x-y)| + |(x-y)|), \\ &= \frac{1}{2}S(x,x,y), \end{aligned}$$

for every $x,y \in X$.

Also, [17] proved the following lemma to satisfy S-metric

Lemma 1.4. [17] *In an S-metric space, we have*

$$S(x,x,y) = S(y,y,x).$$

Lemma 1.5. [17] *Let (X,S) be an S-metric space. If $\lim_{n\to\infty} x_n \to x$ and $\lim_{n\to\infty} y_n \to y$, then*

$$\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y).$$

Lemma 1.6. [18] *Let (X,S) be a S-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} S(x_n,x_n,y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.*

Definition 1.7. [17] *Let (X,S) be the S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows:*

(B_1) *$B_S(x,r) = \{y \in X : S(y,y,x) < r\}$;*

(B_2) *$B_S[x,r] = \{y \in X : S(y,y,x) \leq r\}$.*

The topology induced by the S-metric is the topology generated by all open balls' base in X.

Definition 1.8. [17] *Let (X,S) be an S-metric space.*

(i) *A sequence $\{x_n\} \in X$ converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$ and we denote this by $\lim_{n\to\infty} x_n \to x$.*

(ii) *A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq 0$.*

(iii) *The S-metric space (X,S) is complete if every Cauchy sequence is a convergent sequence.*

Since then, several researchers have been working in this direction to generalise the results in different spaces using the S-metric space. For more details, one can see [6, 9, 16] and the references therein.

2. Preliminaries

2.1 Implicit relation

In 2018, Popa and Patriciu [14] gave the concept of implicit functions in S-metric space which includes most of the existing literature's well-known contractions besides several new ones. Popa and Patriciu [14] proved a generalised fixed point theorem for two pairs of compatible mappings in S - metric spaces. They considered the family $\mathscr{F}_{CS}$ be the set of all real continuous functions.

$F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is non-increasing in variable t_5 and t_6;

(F_2) there exists $h \in [0,1)$ such that for all $u,v \geq 0$,

(F_{2a}) $F(u,v,v,u,0,2u+v) \leq 0$, or

(F_{2b}) $F(u,v,u,v,2u+v,0) \leq 0$, implies $u \leq hv$,

(F_3) $F(t,t,0,0,t,t) \leq 0, \forall\, t > 0$.

Popa and Patriciu [14] used it to unify and extend various findings in the literature. Fore more details, one can see paper by Imdad *et al.* [8].

These are some examples of implicit functions which satisfies the above implicit relations.

Example 2.1 The function of $F \in \mathscr{F}_{CS}$ satisfies the properties (F_1) - (F_3) (see, Popa and Patriciu [14]).

(1) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}$, where $k \in \left[0, \frac{1}{3}\right)$.

(2) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geq 0$ and $a+b+c+3d+3e \leq 1$.

(3) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\max\{t_3,t_4,t_5,t_6\}$, where $a,b \geq 0$ and $a+3b < 1$.

(4) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - bt_3 - ct_4 - d\max\{t_5,t_6\}$, where $a,b,c,d \geq 0$ and $a+b+c+3d < 1$.

(5) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - d\max\{t_3,t_4\} - bt_5 - ct_6$, where $a,b,c,d \geq 0$ and $a+d+3(b+c) < 1$.

(6) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a(t_5+t_6) - bt_2 - c\max\{t_3,t_4\}$, where $a,b,c \geq 0$ and $3a+b+c < 1$.

(7) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a(t_3+t_4) - bt_2 - c\max\{t_5,t_6\}$, where $a,b,c \geq 0$ and $2a+b+3c<1$.

(8) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\{t_4+t_5, t_3+t_6\} - bt_2$, where $a,b \geq 0$ and $4a+b<1$.

(9) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - t_1(at_2+bt_3+ct_4) - dt_5t_6$, where $a,b,c,d \geq 0, a+b+c<1$ and $a+d<1$.

(10) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_1t_2 - bt_3t_4 - ct_5t_6$, where $a,b,c \geq 0, a+b<1$ and $a+c<1$.

(11) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{3}\right\}$, where $k \in [0,1)$.

2.2 Relation theoretic in S-metric space

In this section, we recall some definitions of binary relations relevant to relation-theoretical variants and some metrical concepts such as completeness and continuity. They will be useful in developing our main results.

Definition 2.1. [11] *A binary relation on X is a non-empty subset $\mathscr{R}$ of $X \times X$. $\mathscr{R}$ is called transitive binary relation if $(x,z) \in \mathscr{R}$ for all $x,y,z \in X$ such that $(x,y) \in \mathscr{R}$ and $(y,z) \in \mathscr{R}$. If $(x,y) \in \mathscr{R}$, we also denote it by $x\mathscr{R}y$, and we say "x is related to y".*

Definition 2.2. [10] *Let X be a non-empty set and $\mathscr{R}$ be a binary relation on X. Let l be a natural number, a path length in $\mathscr{R}$ from x to y is a finite sequence $\{z_0,z_1,z_2,...,z_l\} \in X$ beginning with x ending with y, such that*

$$z_0\mathscr{R}z_1, z_1\mathscr{R}z_2, z_2\mathscr{R}z_3, z_3\mathscr{R}z_4, \dots z_{n-1}\mathscr{R}z_l,$$

which satisfies the following conditions: $z_0 = x, z_l = y$ and $[z_i,z_i,z_{i+1}] \in \mathscr{R}$ for each $i \in \{0,1,2,3,...l-1\}$, then this finite sequence is called a path of length l (where $l \in \mathbb{N}$) joining x to y in $\mathscr{R}$. We denote the family of all paths in $\mathscr{R}$ from x to y by $\gamma(x,y,\mathscr{R})$.

Note that, a path of length l involves $(l+1)$ elements of X that need not be distinct in general.

Definition 2.3. [19] *A metric space (X,d) endowed with a binary relation $\mathscr{R}$ is $\mathscr{R}$-non decreasing regular if for any sequence $\{x_n\} \in X$,*

$$(x_n, x_{x_{n+1}}) \in \mathscr{R}, \forall n \in \mathbb{N},$$
$$x_n \to x^* \in X,$$
$$\Rightarrow (x_n, x^*) \in \mathscr{R}, \forall n \in \mathbb{N}.$$

We denote by $X(F,\mathscr{R})$ the set of all points $x \in X$ satisfying $(x,Fx) \in \mathscr{R}$, where $\mathscr{R}$ be a binary relation on a non-empty set X and $F : X \to X$ a self mapping.

Motivated by definition given by Roldan and Lopez [19], *we extend it to S-metric space notion.*

Definition 2.4. *A S-metric space (X,S) endowed with a binary relation $\mathscr{R}$ is $\mathscr{R}$-non decreasing if for any sequence $\{x_n\} \in X$,*

(i) $(x_n, x_n, x_{x_{n+1}}) \in \mathscr{R}, \forall n \in \mathbb{N}$,

(ii) $x_n \to x \in X$,

(iii) $\Rightarrow (x_n, x_n, x) \in \mathscr{R}, \forall n \in \mathbb{N}$.

We denote by $X(T,\mathscr{R})$ the set of all points $x \in X$ satisfying $(x,x,Tx) \in \mathscr{R}$, where $\mathscr{R}$ be a binary relation on a non-empty set X and $T : X \to X$ a self mapping.

Definition 2.5. *Let (X,S) be the S- metric space. A binary relation $\mathscr{R}$ defined on X is called S-self closed if whenever $\{x_n\}$ is a $\mathscr{R}$-preserving sequence and $x_n \xrightarrow{S} x$, then there is a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x_{n_k}, x] \in \mathscr{R}$ for all $k \in \mathbb{N}_0$.*

Definition 2.6. *Let $(X,S,\mathscr{R})$ be the S- metric space equipped with a binary relation $\mathscr{R}$ defined on X. Then a subset D of X is called $\mathscr{R}$-directed if for every pair of points $x,y \in D$, there is z in X such that $(x,x,z) \in \mathscr{R}$ and $(y,y,z) \in \mathscr{R}$.*

Several scholars worked along this line, proved and generalized a binary relation notion in different spaces. One can see [1, 3, 4, 7, 12, 15] and the references therein.

Furthermore, Ahmadullah et al. [2] gave an interesting result on metric spaces equipped with binary relation as follows:

Theorem 2.7. [2] *Let $(X,d,\mathscr{R})$ be a metric space equipped with a binary relation $\mathscr{R}$ defined on X and T a self-mapping on X. Assume that the following conditions holds:*

(a) *(X,d) is $\mathscr{R}$-complete,*

(b) *$X(T;\mathscr{R})$ is non-empty,*

(c) *$\mathscr{R}$ is T-closed,*

(d) *either T is $\mathscr{R}$ continuous or $\mathscr{R}$ is d-self-closed,*

(e) *There exists an implicit function $F \in \mathscr{F}$ with*

$$F(d(Tx,Ty), d(x,y), S(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \leq 0,$$

for all $x,y \in X$ such that $x,y \in \mathscr{R}$.

Then T has a fixed point.

Theorem 2.8. [2] *In addition to the hypothesis $(a)-(e)$ of Theorem 2.7, suppose that the following condition hold:*

(f) *γ_T $(x,y,\mathscr{R}^s)$ is non-empty for each $x,y \in X$, wherein F also enjoys (F_3). Then T has a unique fixed point.*

The purpose of this paper is to study the existence of a fixed point employing an arbitrary binary relation under self-mappings in S-metric spaces via implicit contraction condition. Our results improve and extend

Ahmadullah et al. result in [2] from metric space to S-metric space notion.

3. Main Results

In this section, we will extend the results due to Ahmadullah et al. [2]. The extended theorem is as follows:

Theorem 3.1. *Let (X,S) be the S- metric space equipped with a binary relation $\mathscr{R}$ on X and T a self-mapping on X. Assume that the following condition holds:*

(a) *(X,S) is $\mathscr{R}$-complete,*

(b) *$X(T,\mathscr{R})$ is non-empty,*

(c) *$\mathscr{R}$ is T-closed,*

(d) *either T is $\mathscr{R}$ continuous or $\mathscr{R}$ is S-self-closed,*

(e) *there exists an implicit function $F \in \mathscr{F}_{CS}$ with*

$$F(S(Tx,Tx,Ty),S(x,x,y),S(Tx,Tx,x),\\ S(Ty,Ty,y),S(Tx,Tx,y),S(Ty,Ty,x)) \leq 0, \quad (3.1)$$

for all $x,y \in X$ such that $x,y \in \mathscr{R}$.

(f) *$\gamma_T\ (x,x,y,\mathscr{R}^s)$ is non-empty for each $x,y \in X$, where in F also satisfies (F_3). Then T has a fixed point.*

Proof. Let $x_0 \in X(T,\mathscr{R})$ be non empty set, as from (a), we construct a Picard sequence $\{x_n\}$ such that $x_n = T^n x_0, \forall n \in \mathbb{N}_0$. Since $(x_0,x_0,Tx_0) \in \mathscr{R}$ and $\mathscr{R}$ is T-closed using (c) we have

$$S(Tx_0,Tx_0,T^2x_0),S(T^2x_0,T^2x_0,T^3x_0),\ldots\\ S(T^nx_0,T^nx_0,T^{n+1}x_0),\cdots \in \mathscr{R}.$$

From Definition 2.4, we have

$$(x_n,x_n,x_{n+1}) \in \mathscr{R}, \forall n \in \mathbb{N}_0.$$

Such that the sequence $\{x_n\}$ is $\mathscr{R}$-preserving. By (3.1) for $x = x_n$ and $y = x_{n+1}$ we have

$$F(S(Tx_n,Tx_n,Tx_{n+1}),S(x_n,x_n,x_{n+1}),S(Tx_n,Tx_n,x_n),\\ S(Tx_{n+1},Tx_{n+1},x_{n+1}),S(Tx_n,Tx_n,x_{n+1}),\\ S(Tx_{n+1},Tx_{n+1},x_n)) \leq 0.$$

Equivalently to

$$F(S(x_{n+1},x_{n+1},x_{n+2}),S(x_n,x_n,x_{n+1}),S(x_{n+1},x_{n+1},x_n),\\ S(x_{n+2},x_{n+2},x_{n+1}),S(x_{n+1},x_{n+1},x_{n+1}),S(x_{n+2},x_{n+2},x_n)) \leq 0.$$

Implies that

$$F(S(x_{n+1},x_{n+1},x_{n+2}),S(x_n,x_n,x_{n+1}),S(x_{n+1},x_{n+1},x_n),\\ S(x_{n+2},x_{n+2},x_{n+1}),0,S(x_{n+2},x_{n+2},x_n)) \leq 0. (3.2)$$

By Lemma 1.4,

$$S(x_{n+1},x_{n+1},x_{n+2}) = S(x_{n+2},x_{n+2},x_{n+1})\\ S(x_n,x_n,x_{n+1}) = S(x_{n+1},x_{n+1},x_n).$$

By (S_3) we have

$$S(x_{n+2},x_{n+2},x_n) \leq 2S(x_{n+2},x_{n+2},x_{n+1}) + S(x_n,x_n,x_{n+1}).$$

Then, by (3.2) we obtain

$$F(S(x_{n+1},x_{n+1},x_{n+2}),S(x_n,x_n,x_{n+1}),S(x_{n+1},x_{n+1},x_n),\\ S(x_{n+2},x_{n+2},x_{n+1}),0,2S(x_{n+2},x_{n+2},x_{n+1})\\ +S(x_n,x_n,x_{n+1})) \leq 0. (3.3)$$

Letting

$$u = S(x_{n+1},x_{n+1},x_{n+2}), \quad (3.4)$$

and

$$v = S(x_n,x_n,x_{n+1}), \quad (3.5)$$

Applying (3.4) and (3.5) in (3.3), we get

$$F(u,v,v,u,0,2u+v) \leq 0.$$

By (F_1), F is non-decreasing in the sixth variable.

$$F(u,v,v,u,0,2u+v) \leq 0.$$

Implying thereby using (F_1) there exists some $h \in [0,1)$, such that $u \leq hv$, which amounts to say

$$S(x_{n+1},x_{n+1},x_{n+2}) \leq hS(x_n,x_n,x_{n+1}),$$

By induction, it gives rise to

$$S(x_{n+1},x_{n+1},x_{n+2}) \leq h^{n+1}S(x_0,x_0,x_1), \ \forall n \in \mathbb{N}_0. \ (3.6)$$

Thus for all $n < m$, by S_3, Lemma 1.4 and (3.6), we have

$$\begin{aligned} S(x_n,x_n,x_m) &\leq 2S(x_n,x_n,x_{n+1}) + S(x_m,x_m,x_{n+1}).\\ &= 2S(x_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},x_m),\ldots\\ &\leq 2[h^n + h^{n+1} + h^{n+2}\cdots + h^{m-1}]S(x_0,x_0,x_1)\\ &\leq 2h^nS(x_0,x_0,x_1)[1+h+h^2+\ldots+h^{m-n-1}]\\ &\leq 2h^nS(x_0,x_0,x_1)[1+h+h^2+\ldots+h^{m-n-1}]\\ &\leq 2h^n\sum_{i=0}^{m-n-1}h^iS(x_0,x_0,x_1)\\ &\leq \frac{2h^n}{1-h}S(x_0,x_0,x_1). \end{aligned} \quad (3.7)$$

If $x = x_n$ and $y = x_{n-1}$, by (3.1) we obtain

$$F(S(Tx_n,Tx_n,Tx_{n-1}),S(x_n,x_n,x_{n-1}),S(Tx_n,Tx_n,x_n),\\ S(Tx_{n-1},Tx_{n-1},x_{n-1}),S(Tx_n,Tx_n,x_{n-1}),S(Tx_{n-1},Tx_{n-1},x_n)) \leq 0.$$

Equivalently to

$$F(S(x_{n+1},x_{n+1},x_n),S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n),\\ S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_{n-1}),S(x_n,x_n,x_n)) \leq 0.$$

Implies that

$$F(S(x_{n+1},x_{n+1},x_n),S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n),\\ S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_{n-1}),0) \leq 0. (3.8)$$

By Lemma 1.4,

$$\begin{aligned} S(x_{n+1},x_{n+1},x_{n-1}) &= S(x_{n-1},x_{n-1},x_{n+1}) \\ S(x_n,x_n,x_{n-1}) &= S(x_{n-1},x_{n-1},x_n). \end{aligned}$$

By (S_3) and Lemma 1.4 we have

$$\begin{aligned} S(x_{n+1},x_{n+1},x_{n-1}) &\leq 2S(x_{n+1},x_{n+1},x_n)+S(x_{n-1},x_{n-1},x_n) \\ S(x_{n+1},x_{n+1},x_{n-1}) &\leq 2S(x_{n+1},x_{n+1},x_n)+S(x_n,x_n,x_{n-1}). \end{aligned}$$

Then, by (3.8) we obtain

$$F(S(x_{n+1},x_{n+1},x_n),S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n), 2S(x_{n+1},x_{n+1},x_n)+S(x_n,x_n,x_{n-1}),0)\leq 0. \quad (3.9)$$

Letting

$$u = S(x_{n+1},x_{n+1},x_n), \quad (3.10)$$

and

$$v = S(x_n,x_n,x_{n-1}), \quad (3.11)$$

Applying (3.10) and (3.11) in (3.9), we get

$$F(u,v,v,u,2u+v,0)\leq 0.$$

But from (F_1), we have, function F non-decreasing in the sixth variable. Thus by applying (F_{2b}), there exists some $h\in[0,1)$, such that $u\leq hv$, i. e.

$$S(x_{n+1},x_{n+1},x_n) \leq hS(x_n,x_n,x_{n-1}),$$

which inductively gives rise to

$$S(x_{n+1},x_{n+1},x_n) \leq h^{n+1}S(x_0,x_0,x_1), \quad \forall n\in\mathbb{N}_0. \quad (3.12)$$

We notice that (3.12) is identical to (3.6). So we follow a similar proof for the condition (F_{2b}) and conclude that a sequence $\{x_n\}$ is Cauchy sequence in X. Hence, $\{x_n\}$ is an $\mathscr{R}$-preserving Cauchy sequence in X.

From (d) assume that T-is $\mathscr{R}$-continuous. Since X is complete there exists $x\in X$ with

$$\lim_{n\to\infty} Tx_n = x.$$

Since T is continuous, we have

$$x=\lim_{n\to\infty} Tx_{n+1}=\lim_{n\to\infty} T(Tx_n)=Tx.$$

Therefore, x is a fixed point of T.

Letting $m\to\infty$ in (3.7), we have

$$S(Tx_n,Tx_n,x) \leq \frac{2h^n}{1-h}S(x,x,x_0).$$

Next, suppose that $\mathscr{R}$ is S-self closed and $x_n\xrightarrow{S} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k},x_{n_k},x]\in\mathscr{R}$, $\forall k\in\mathbb{N}_0$. This implies that either $(x_{n_k},x_{n_k},x)\in\mathscr{R}$, $\forall k\in\mathbb{N}_0$ or $(x,x,x_{n_k})\in\mathscr{R}$, $\forall k\in\mathbb{N}_0$.

Applying condition (e) to $(x_{n_k},x_{n_k},x)\in\mathscr{R}$, $\forall k\in\mathbb{N}_0$, for $x=x_{n_k}$, $y=x$ and by Lemma 1.8, we have

$$F(S(Tx_{n_k},Tx_{n_k},Tx),S(x_{n_k},x_{n_k},x),S(Tx_{n_k},Tx_{n_k},x_{n_k}), S(Tx,Tx,x),S(Tx_{n_k},Tx_{n_k},x),S(Tx,Tx,x_{n_k}))\leq 0.$$

or

$$F(S(x_{n_{k+1}},x_{n_{k+1}},Tx),S(x_{n_k},x_{n_k},x),S(x_{n_{k+1}},x_{n_{k+1}},x_{n_k}), S(Tx,Tx,x),S(x_{n_{k+1}},x_{n_{k+1}},x),S(Tx,Tx,x_{n_k}))\leq 0.$$

Letting $n\to\infty$ by Lemma 1.4 and using $x_{n_k}\xrightarrow{S} x$ and continuity of F and S, we obtain

$$\begin{aligned} &F(S(x,x,Tx),S(x,x,x),S(x,x,x),S(Tx,Tx,x), S(x,x,x),S(Tx,Tx,x))\leq 0. \\ &F(S(x,x,Tx),0,0,S(Tx,Tx,x),0,S(Tx,Tx,x))\leq 0, \end{aligned}$$

a contradiction to (F_3). Hence, we obtain $S(x,x,Tx)=0$, so that $Tx=x$. x is a fixed point of T.

Again, if $(x,x,x_{n_k})\in\mathscr{R}$, $\forall k\in\mathbb{N}_0$. Then owing to (F_1), we obtain $S(x,x,Tx)=0$, so that $Tx=x$. Hence x is a fixed point of T.

By assumption (f), there exists a path say $\{z_0,z_1,z_2,...,z_l\}$ of some finite length l in $\mathscr{R}^s$ from x to y so that $z_0=x, z_l=y, [z_n^i,z_n^i,z_{n+1}^i]\in\mathscr{R}$ for each $i(0\leq i\leq l-1)$. As $\mathscr{R}$ is T-closed, we have $[T^nz_n^i,T^nz_n^i,T^nz_{n+1}^i]\in\mathscr{R}$ for each $i(0\leq i\leq l-1)$ and each $n\in\mathbb{N}_0$. Let

$$Tx=x \textit{ and } Ty=y. \quad (3.13)$$

We show that x= y. By (f), there exists a path $(\{z_0,z_1,z_2,...,z_l\})$ of finite length l in $\mathscr{R}^s$ from x to y with

$$z_0=x \textit{ and } z_l=y,\ [z_i,z_i,Tz_{i+1}]\in\mathscr{R}, \textit{for each } i\in\{0,1,2,\ldots l-1\}, \quad (3.14)$$

and

$$[z_i,z_i,Tz_i]\in\mathscr{R},\ \textit{for each } i\in\{1,2,\ldots l-1\}. \quad (3.15)$$

We construct two sequences

$$z_n^0=x \textit{ and } z_n^l=y. \quad (3.16)$$

By using (3.13), we get

$$Tz_n^0=Tx=x\ \forall n\in\mathbb{N}_0,\ \textit{and } Tz_n^l=Ty=y\ \forall n\in\mathbb{N}_0.$$

Setting, p

$$z_0^i=z_i \textit{ for } i\in\{0,1,2,3,...,l-1\}, \quad (3.17)$$

we construct a sequence $\{z_n^i\}$, such that $T^iz_n=T^iz_{n+1}$ corresponding to each z_i. Since $[z_0^i,z_0^i,z_l^i]\in\mathscr{R}$ and $\mathscr{R}$ is T-closed, on using (3.6), we get

$$\lim_{n\to\infty} S(z_n^i,z_n^i,z_{n+1}^i) = 0,\ \forall\, i\in\{0,1,2,3,...,l-1\}.$$

By Using $[z_0^i, z_0^i, z_0^{i+1}] \in \mathscr{R}$ due to (3.14) and (3.15) and $\mathscr{R}$ is T closed, we obtain

$$[Tz_0^i, Tz_0^i, Tz_0^{i+1}] \in \mathscr{R}, \text{for each } i \in \{0,1,2,\dots l-1\} \text{ and for all } n \in \mathbb{N}_0,$$

$$\implies [z_0^i, z_0^i, z_0^{i+1}] \in \mathscr{R}, \text{for each } i \in \{0,1,2,\dots l-1\} \text{ and for all } n \in \mathbb{N}_0.$$

Define $S_n^i = S(z_n^i, z_n^i, z_n^{i+1}), for\ all\ n \in \mathbb{N}_0\ and\ for\ each\ i \in \{0,1,2,\dots l-1\}$. Equivalently to

$$\lim_{n\to\infty} S_n^i = 0.$$

By Lemma 1.6, assume that $\lim_{n\to\infty} S_n^i = t > 0$. Since $[z_n^i, z_n^i, z_n^{i+1}] \in \mathscr{R}$, either $[z_n^i, z_n^i, z_n^{i+1}] \in \mathscr{R}$ or $[z_n^{i+1}, z_n^{i+1}, z_n^i] \in \mathscr{R}$, now on applying condition (e) to it, we obtain

$$F(S(Tz_n^{i+1}, Tz_n^{i+1}, Tz_n^i), S(z_n^{i+1}, z_n^{i+1}, z_n^i), S(Tz_n^{i+1}, Tz_n^{i+1}, z_n^{i+1}), S(Tz_n^i, Tz_n^i, z_n^i), S(Tz_n^{i+1}, Tz_n^{i+1}, z_n^i), S(Tz_n^i, Tz_n^i, z_n^{i+1})) \le 0.$$

or

$$F(S(z_{n+1}^{i+1}, z_{n+1}^{i+1}, z_{n+1}^i), S(z_n^{i+1}, z_n^{i+1}, z_n^i), S(z_{n+1}^{i+1}, z_{n+1}^{i+1}, z_n^{i+1}), S(z_{n+1}^i, z_{n+1}^i, z_n^i), S(z_{n+1}^{i+1}, z_{n+1}^{i+1}, z_n^i), S(z_{n+1}^i, z_{n+1}^i, z_n^{i+1})) \le 0.$$

Taking $n \to \infty$ and using $\lim_{n\to\infty} S_n^i = t$ in the above inequality, we get

$$F(t,t,0,0,t,t) \le 0,$$

which is a contradiction by (F_3) and hence

$$\lim_{n\to\infty} S_n^i = t = 0.$$

Similarly, if $(z_n^i, z_n^i, z_n^{i+1}) \in \mathscr{R}$, then, from above

$$\lim_{n\to\infty} S_n^i = t = 0.$$

Therefore,

$$\lim_{n\to\infty} S_n^i = \lim_{n\to\infty} S(z_n^i, z_n^i, z_n^{i+1}) = 0, \text{ for each } i \in \{0,1,2,\dots l-1\}. \tag{3.18}$$

Using (S_3) and Lemma 1.4, we obtain

$$\begin{aligned} S(x,x,y) = S(z_n^0, z_n^0, z_n^l) &\le \sum_{i=0}^{l-1} S(z_n^i, z_n^i, z_n^{i+1}) \\ &= \sum_{i=0}^{l-1} S_n^i, \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

From Theorem 3.1, we can deduce a number of corollaries which appeared as given below:

Corollary 3.2. *The results of Theorem 3.1 remain true for all $x, y \in X$ with $(x,x,y) \in \mathsf{R}$, the implicit relation (e) is replaced by one of the following:*

(i)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & k\max\{S(x,x,y), S(Tx,Tx,x), S(Ty,Ty,y), \\ & S(Tx,Tx,y), S(Ty,Ty,x)\}, \end{aligned} \tag{3.19}$$

where $k \in \left[0, \frac{1}{3}\right)$.

(ii)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & aS(x,x,y) + bS(Tx,Tx,x) + cS(Ty,Ty,y) \\ & + dS(Tx,Tx,y) + eS(Ty,Ty,x), \end{aligned} \tag{3.20}$$

where $a,b,c,d,e \ge 0$ and $a+b+c+3d+3e \le 1$.

(iii)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & k\max\{S(x,x,y), S(Tx,Tx,x), S(Ty,Ty,y), \\ & S(Tx,Tx,y), S(Ty,Ty,x)\}, \end{aligned} \tag{3.21}$$

where $k \in \left[0, \frac{1}{3}\right)$.

(iii)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & aS(x,x,y) + b\max\{S(Tx,Tx,x), S(Ty,Ty,y), \\ & S(Tx,Tx,y), S(Ty,Ty,x)\}, \end{aligned} \tag{3.22}$$

where $a,b \ge 0$ and $a+3b < 1$.

(iv)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & aS(x,x,y) + bS(Tx,Tx,x) + \\ & cS(Ty,Ty,y) + \\ & d\max\{S(Tx,Tx,y), S(Ty,Ty,x)\}, \end{aligned} \tag{3.23}$$

where $a,b,c,d \ge 0$ and $a+b+c+3d < 1$.

(v)

$$\begin{aligned} &S(Tx,Tx,Ty) \\ \le\ & aS(x,x,y) + d\max\{S(Tx,Tx,x), \\ & S(Ty,Ty,y)\} + bS(Tx,Tx,y) + \\ & cS(Ty,Ty,x), \end{aligned} \tag{3.24}$$

where $a,b,c,d \ge 0$ and $a+d+3(b+c) < 1$.

(vi)

$$S(Tx,Tx,Ty) \leq a(S(Tx,Tx,y)+S(Ty,Ty,x))+ bS(x,x,y)+ c\max\{S(Tx,Tx,x),S(Ty,Ty,y)\}, \quad (3.25)$$

where $a,b,c \geq 0$ and $3a+b+c<1$.

(vii)

$$S(Tx,Tx,Ty) \leq a(S(Tx,Tx,x)+S(Ty,Ty,y)) +bS(x,x,y)+c\max\{S(Tx,Tx,y), S(Ty,Ty,x)\}, \quad (3.26)$$

where $a,b,c \geq 0$ and $2a+b+3c<1$.

(viii)

$$S(Tx,Tx,Ty) \leq a\max\{S(Ty,Ty,y)+S(Tx,Tx,y), S(Tx,Tx,x)+S(Ty,Ty,x)\} +bS(x,x,y), \quad (3.27)$$

where $a,b \geq 0$ and $4a+b<1$.

(ix)

$$S(Tx,Tx,Ty)^2 \leq S(Tx,Tx,Ty)\{aS(x,x,y)+ bS(Tx,Tx,x)+cS(Ty,Ty,y)\} -dS(Tx,Tx,y)S(Ty,Ty,x), \quad (3.28)$$

where $a,b,c,d \geq 0, a+b+c<1$ and $a+d<1$.

(x)

$$S(Tx,Tx,Ty)^2 \leq aS(Tx,Tx,Ty)S(x,x,y)+ bS(Tx,Tx,x)S(Ty,Ty,y)+ cS(Tx,Tx,y)S(Ty,Ty,x), \quad (3.29)$$

where $a,b,c \geq 0, a+b<1$ and $a+c<1$.

(xi)

$$S(Tx,Tx,Ty) \leq k\max\{S(x,x,y),S(Tx,Tx,x),S(Ty,Ty,y), \frac{S(Tx,Tx,y)+S(Ty,Ty,x)}{3}\}, \quad (3.30)$$

where $k \in [0,1)$.

Example 3.3. *Let $X=[1,2]$. Define the usual S-metric as $S(x,y,z)=\|x-z\|+\|y-z\|$ for all $x,y,z \in X$. Then (X,S) is a complete S- metric space. Let T be a self mapping defined on X as:*

$$Tx = \begin{cases} 0, & for \in [0,1], \\ 1, & for\, x \in (1,2]. \end{cases}$$

Now, $\mathscr{R}$ can be a set of binary relation.

$$\mathscr{R} = \{(0,0)(0,1),(0,2),(1,1),(1,2),(2,2)\},$$

on X. Obviously, $\mathscr{R}$ is T-closed but T is not continuous. We choose $\mathscr{R}$ - preserving sequence $\{x_n\}$ with $x_n \xrightarrow{S} x$ such that $(x_n,x_n,x_{n+1}) \in \mathscr{R}$, for all $n \in \mathbb{N}_0$.

Here, one may notice that $(x_n,x_n,x_{n+1}) \in \mathscr{R}$, for all $n \in \mathbb{N}_0$ and there exists an integer $N \in \mathbb{N}_0$ such that $x_n = x \in \{0,1,2\}$ for $n \leq \mathbb{N}$. So, we can take subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k}=x$ for all $k \in \mathbb{N}_0$. For which it amounts saying that $[x_{n_k},x_{n_k},x] \in \mathscr{R}$, for all $k \in \mathbb{N}_0$. Therefore $\mathscr{R}$ is S-closed.

Define a continuous function $F:\mathbb{R}^6_+ \to \mathbb{R}$ by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{99}{100}t_5 - \frac{9}{10}t_6.$$

i.e.,

$$S(Tx,Tx,Ty) \leq \frac{99}{100}S(Tx,Tx,y)+\frac{9}{10}S(Ty,Ty,x).$$

For

$(x,y) \in \{(0,0)(0,1),(0,2),(1,1),(1,2),(2,2)\}, \forall x,y \in \mathscr{R}$.

$$S(Tx,Tx,Ty) = 0,$$

hence obvious.

For $(x,y) \in (0,2)$

$$\begin{aligned} S(Tx,Tx,Ty) &= S(T0,T0,T2)=2. \\ S(Tx,Tx,y) &= S(T0,T0,2)=2. \\ S(Ty,Ty,x) &= S(T2,T2,0)=2. \end{aligned}$$

$$\begin{aligned} S(T0,T0,T2) &\leq \frac{99}{100}S(T0,T0,2)+\frac{9}{10}S(T2,T2,0). \\ 2 &\leq \frac{99}{100}\times 2+\frac{9}{10}\times 2. \\ 2 &\leq \frac{198}{100}+\frac{18}{10}. \\ 2 &\leq \frac{378}{100}. \end{aligned}$$

For $(x,y) \in (1,2)$

$$\begin{aligned} S(Tx,Tx,Ty) &= S(T1,T1,T2)=2. \\ S(Tx,Tx,y) &= S(T1,T1,2)=4. \\ S(Ty,Ty,x) &= S(T2,T2,1)=0. \end{aligned}$$

$$\begin{aligned} S(T1,T1,T2) &\leq \frac{99}{100}S(T1,T1,2)+\frac{9}{10}S(T2,T2,1).\\ 2 &\leq \frac{99}{100}\times 4+\frac{9}{10}\times 0.\\ 2 &\leq \frac{396}{100}+0.\\ 2 &\leq \frac{396}{100}. \end{aligned}$$

Which shows that all assertion of Theorem 3.1 are satisfied. Hence $x=0$ is a fixed point of T. □

4. An application

In this section, we provide an application of Theorem 3.1 in the form of the existence of a solution for an integral equation.

In 2018, Özgür and Taş [13] gave a generalization of completeness of S_∞-space by using the metric S_∞ defined as S_∞-metric generated by d_∞.

Let $F=\mathbb{R}$ or $F=\mathbb{C}$ and

$$C[a,b]=\{f:[a,b]\to F\}$$

where F is a continuous function.

The function

$$S_\infty : C[a,b]\times C[a,b]\times C[a,b]\to[0,\infty)$$

defined as

$$\begin{aligned} S_\infty(f,g,h) &= d_\infty(f,h)+d_\infty(g,h)\\ &= \sup_{x\in[a,b]}|f(x)-h(x)|+\sup_{x\in[a,b]}|g(x)-h(x)|, \end{aligned}$$

for all $f,g,h\in[a,b]$ is the S-metric on $[a,b]$ and $(C[a,b],S_\infty)$ is the S-metric space.

Definition 4.1. [13] *Assume that* $r=\min\{a,\frac{b}{K}\}$ *and*

$$X=\{y\in C[x_0-r,x_0+r]:S(y(x),y(x),y_0)\leq b \quad for\ x\in[x_0-r,x_0+r]\}$$

where S is the S-metric spaces defined in Definition 1.7. Hence, (X,S_∞) is a complete S-metric space.

Definition 4.2. [13] *The function f is bounded if there exists*

$$K = \sup\{|f(x,y)|:(x,y)\in I\}.$$

We notice that $(x,x,y)\in I$ for $S(x,x,x_0)\leq r$ and $y\in X$.

Proposition 4.3. [13] $(C[a,b],S_\infty)$ *is a complete S-metric space.*

Consider the following integral equation

$$\begin{cases} y'(x)=f(x,y(x)),\\ y(x_0)=y_0. \end{cases} \tag{4.1}$$

The solution of this problem can be written in the following form

$$y(x)=y(x_0)+\int_{x_0}^{x}f(t,y(t))dt,\forall\, t\in[a,b], \tag{4.2}$$

where $y(x)$ is an unknown function on $I=[a,b]$, $y(x_0)$ known continuous function on I and F a kernel defined on $D=\{(x,y):x\in I\}$ for all $(x,y_1),(x,y_2)\in D$.

Definition 4.4. *A lower solution for (4.2) is a function* $\alpha\in C([a,b],\mathbb{R})$

$$\alpha(x)\leq y(x_0)+\int_{x_0}^{x}f(t,y_1(t))dt,\forall\, t\in[a,b].$$

Definition 4.5. *An upper solution for (4.2) is a function* $\beta\in C([a,b],\mathbb{R})$

$$\beta(x)\leq y(x_0)+\int_{x_0}^{x}f(t,y_2(t))dt,\forall\, t\in[a,b].$$

Theorem 4.6. *Assume that* $f:[a,b]\times[a,b]\times[a,b]\to\mathbb{R}$ *is continuous for all* $x,y\in C([a,b],\mathbb{R})$*and there exists a real number* $M>0$ *such that*

$$0\leq f(t,y(t))\leq M,$$

for all $t\in[a,b]$ *and* $x,y\in\mathbb{R}$. *Then the boundary valued problem (4.1) has a lower solution which ensures the existence of a unique solution of (4.2).*

Proof: We start by formulating (4.1) as a fixed point equation

$$Ty=y.$$

$Ty\in X$ and $T:X\longrightarrow X$ defined by

$$Ty(x)=y(x_0)+\int_{x_0}^{x}f(t,y(t))dt,\ t\in[a,b], \tag{4.3}$$

and an S- binary relation

$$\mathscr{R}=\{(x,x,y)\in C([a,b],\mathbb{R})|x(t)\leq y(t),\forall\, t\in[a,b]\}.$$

(i) Assume that $X=C([a,b]^3,\mathbb{R})$ is the space of all continuous functions and define a S-metric space on X endowed with S_∞- metric.

$$S_\infty(x,x,y) = \sup_{t\in[a,b]}|x(t)-y(t)|+\sup_{t\in[a,b]}|x(t)-y(t)|,$$

for all $x,y\in X$ is a complete S-metric space and hence $C((I,\mathbb{R}),S_\infty)$ is $\mathscr{R}$-complete.

(ii) Choosing an $\mathscr{R}$-preserving sequence $\{y_n\}$ such that $y_n\stackrel{S}{\to}y$, we get for all $t\in I$,

$$y_0(t)\leq y_1(t)\leq y_2(t)\leq\cdots\leq y_n(t)\leq y_{n+1}(t)\ldots$$

and it converges to $y(t)$ implying that $y_n(t)\leq y(t)$ for all $t\in I,n\in\mathbb{N}_0$, which is equivalent to $[y_n,y_n,y]\in\mathscr{R}$, for all $n\in\mathbb{N}_0$. Hence, $\mathscr{R}$ is S- self-closed.

(iii) We prove that a mapping T defined in (4.3) is a contraction for two continuous functions y_1 and y_2 on $C([a,b],\mathbb{R})$. For any $(y_1,y_1,y_2)\in\mathscr{R}$, $y_1(t)\le y_2(t)$ for all $t\in[a,b]$, $\lambda>0$ *and* $f(t,y(t))\ge 0$, we obtain

$$\begin{aligned} Ty_1(t) &= y(x_0)+\int_{x_0}^{x} f(t,y_0(t))dt \\ &\le y(x_0)+\int_{x_0}^{x} f(t,y_1(t))dt \\ &\le Ty_2(t), \end{aligned}$$

which shows that $(Ty_1,Ty_1,Ty_2)\in\mathscr{R}$, thus $\mathscr{R}$-is T-closed.

(iv) For all $(y_1,y_1,y_2)\in\mathscr{R}$, we have

$$\begin{aligned} S(Ty_1(x),Ty_1(x),Ty_2(x)) &= 2|Ty_1(x)-Ty_2(x)| \\ &\le \int_{x_0}^{x} |f(t,y_1(t))-f(t,y_2(t))|dt \\ &\le 2M\int_{x_0}^{x} |y_1(t)-y_2(t)|dt \\ &\le 2M\,|y_1(t)-y_2(t)|\Big(\int_{x_0}^{x} dt\Big) \\ &\le MS_\infty(y_1,y_1,y_2)\Big(\int_{x_0}^{x} dt\Big) \\ &\le M(x-x_0)S_\infty(y_1,y_1,y_2). \\ &\le MS_\infty(y_1,y_1,y_2). \end{aligned}$$

where $M>0$ and $x>x_0$. This shows that T satisfies assertion (e) of Theorem 3.1.

(v) Next, a lower solution for (4.2), by Definition 4.4 is a function $\alpha(x)$, let $\alpha(x)=y_1(x)\in C([a,b],\mathbb{R})$ for all $t\in I$,

$$\begin{aligned} y_1(x) &\le y(x_0)+\int_{x_0}^{x} f(t,y_1(t))dt, \forall\, t\in[a,b], \\ &= Ty_1(x), \end{aligned}$$

which shows that the function y_1 satisfies if and only if a lower solution is bounded.

Using the definition of r and Definition 4.2, if $S(x,x,x_0)\le r$ then we have

$$\begin{aligned} S(Ty_1(x),Ty_1(x),y_0) &= 2|Ty_1(x)-y_0| \\ &\le 2|\int_{x_0}^{x} f(t,y_1(t))|dt \\ &\le 2\int_{x_0}^{x}|f(t,y_1(t))|dt \\ &\le 2\,|f(t,y_1(t))|\Big(\int_{x_0}^{x} dt\Big) \\ &\le 2K|x-x_0| = KS(x,x,x_0). \\ &= Kr\le b, \end{aligned}$$

which shows that a lower solution exists and is bounded. Implies that $(y_1,y_1,Ty_1)\in\mathscr{R}$ therefore $X(T,\mathscr{R})$ is non empty.

(vi) For the uniqueness of a fixed point, let y and y_2 be the arbitrary element of $C(I,\mathbb{R})$ and choose y_1 such that $y(t)\le y_1(t)$ and $y_2(t)\le y_1(t)$ for all $t\in I$. This implies $(y,y,y_1)\in\mathscr{R}$ and $(y_2,y_2,y_1)\in\mathscr{R}$. Therefore, the finite sequence $\{y,y_1,y_2\}$ describe a path which join y to y_2 in $\mathscr{R}$.

Operator T satisfies condition of Equation 3.1. Hence by Theorem 4.6 we have shown that the operator T has a fixed point $y(x)\in X$, which is a solution of (4.1).

5. Conclusions

We extended and generalized the results by Ahmadullah et al. [2] to self mappings on S-metric space under a binary relation via implicit contractive condition. In doing so, we corollaries several results in the existing literature (see Corollary 4.1). Illustrative example and an application to the integral equation provided to support Theorem 3.1.

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