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Characterization of quadratic independence polynomial of path graph, cycle graph and wheel graph

Sreeja K U[1]*, Vinodkumar P B [2] and Ramkumar P B [3]

Abstract
We present a combinatorial approach to the independence polynomial of quadratic graphs and fractals in this paper. This aids in overcoming the difficulties that mathematically rigorous self similar patterns present. Explanatory results show the various properties of various graph classes, such as energy, Hausdorff dimension, dynamics, and connectivity. The findings we obtained lay the groundwork for studying graphs from a fractal perspective.

Keywords
Energy, Hausdorff dimension, Independence polynomial, Fractals, Dynamics.

AMS Subject Classification
05C31, 05C69.

[1] *Department of Mathematics, KKTM Government College, Pullut, Thrissur, India.*
[2,3] *Department of Mathematics, Rajagiri School of Engineering and Technology, Kakkanad, India.*
*Corresponding author: [1] sreejaku@gmail.com; [2] vinod_kumar@rajagiritech.edu.in ; [3] ram_kumar@rajagiritech.edu.in
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1. Introduction

I. Gutman and F. Harary proposed the independence polynomial [2].The independence polynomial was introduced as an analog of the matching polynomial. The independence polynomial has found applications in chemistry and physics [1, 2]. Finding explicit and fairly simple theorems in terms of independence polynomial of graphs, on the other hand, is frequently difficult. The results obtained in this paper demonstrate how this can be accomplished by employing various polynomial manipulations.

Although independence polynomials can be found almost everywhere, determining the independence polynomial of a graph is an NP-complete problem. The independence polynomial is defined as follows:[3] Let s_k denote the number of independent sets of size k , which are induced sub graphs of G , then

$$I(G,x) = \sum_{k=0}^{\alpha(G)} s_k x^k$$

where $\alpha(G)$ is the independence number of G .

The next result gives a recurrence relation which help us to decompose the independence polynomial of a graph vertex by vertex.

Theorem 1.1. *[3] Let G be a simple graph .Let $v \in V(G)$ and $N[v]$ is the closed neighborhood of v. Then*

$$\boldsymbol{I}(G;x) = \boldsymbol{I}(G-v;x) + x\boldsymbol{I}(G-N[v];x).$$

Our work is structured as follows: In this section, we will review some fundamental definitions and results of the independence polynomial of cycle graphs, path graphs, and wheel graphs. The results for these graphs in terms of connectivity and dynamics are obtained in the main results section.

2. Preliminaries

In this section, we recall some definitions and basic results of recurrence relations and independence polynomial of Path graph, Cycle graph and Wheel graph which will be used throughout the paper.The following three standard graph lemmas are taken from the paper[3] and are relevant to our study in this paper.

Lemma 2.1. *For a Path graph of order n denoted by P_n, the recurrence relation is*

$$\boldsymbol{I}(P_n;z) = \boldsymbol{I}(P_{n-1};z) + z\boldsymbol{I}(P_{n-2};z)$$

and its independence polynomial is

$$\frac{1}{2^{n+1}}[(1+2z+s)(1+s)^n + (s-1-2z)(1-s)^n]$$

where $s = \sqrt{1+4z}$.

Lemma 2.2. *For a Cycle graph of order n denoted by C_n, the recurrence relation*

$$\boldsymbol{I}(C_n;z) = \boldsymbol{I}(P_{n-1};z) + z\boldsymbol{I}(P_{n-3};z)$$

is and its independence polynomial is

$$\frac{1}{2^{n+1}}[(1+2z+s)(1+s)^{n-2} + (1+2z-s)(1-s)^{n-2}]$$

where $s = \sqrt{1+4z}$

Lemma 2.3. *For a Wheel graph of order n denoted by W_n, the recurrence relation is*

$$\boldsymbol{I}(W_n;z) = \boldsymbol{I}(C_n;z) + z$$

and its independence polynomial is

$$\frac{1}{2^{n+1}}[(1+2z+s)(1+s)^{n-2} + (1+2z-s)(1-s)^{n-2}]$$

where $s = \sqrt{1+4z}$.

In this study, we examine the dynamicity of graphs in terms of their Julia set. The Julia set of a polynomial typically has a complicated, self similar structure. The dimension of a Julia set is ***Hausdorff dimension*** gives a reasonable way of assigning appropriate non integer dimension to such sets.

Definition 2.4. *[3] **Julia set** is defined on extended complex plane. The filled in Julia set of the polynomial f is defined as*

$$K(f) = \{z \in C : f^n(z) \nrightarrow \infty\}$$

.The Julia set is defined as the boundary of the filled in Julia set ie $J(f) = \partial K(f)$.

Reduced independence polynomial is an important term associated with independence polynomial, and it is directly related to polynomial conjugacy.

Definition 2.5. *The reduced independence polynomial of G is the function*

$$R(G,z) = \boldsymbol{I}(G,z) - 1$$

, since every independence polynomial has constant term 1.

The energy of a term is another relevant term associated with graphs, and we compare the energy with dimension in our paper. To define energy, we must first define the graph's adjacency matrix.

Definition 2.6. *Let G be a simple graph with n vertices and m edges. Adjacency matrix $A(G)$ of the graph G is given by ,*

$$(a_{i,j}) = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

. .

The zeros of the characteristic polynomial of the adjacency matrix are given by $\lambda_1, \lambda_2, \lambda_3, ...$ and are known as eigen values of G.

Definition 2.7. *[4]The Energy $E(G)$ of G is defined as the sum of the absolute values of the eigen values of an adjacency matrix of a graph.*

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

.

Definition 2.8. *[5] The quadratic family is the family of quadratic polynomials of the form $f(z) = z^2 + c$, where c is a complex constant called the parameter.*

Definition 2.9. *Two maps $f : C \to C$ and $g : C \to C$ are called conjugate if there exists a homeomorphism $h : C \to C$ such that $hof = goh$.*

The term quadratic conjugacy will be defined next.The independence polynomial of second degree can be reduced to the Julia set of the form $z^2 + c$ using quadratic conjugacy.

Theorem 2.10. *[5] Let $f(z) = az^2 + bz + c$ be a quadratic where $a, b, c \in C$ with $a \neq 0$. Then f is conjugate to some $g(z) = z^2 + k$ where $k \in C$.*

To understand how the dynamics of $f(z)$ change as c varies we have to solve the quadratic equation $z^2 + c = z$. Then we have the two roots namely ,

$$p_+ = \frac{1}{2}(1+\sqrt{1-4c}), p_- = \frac{1}{2}(1-\sqrt{1-4c})$$

Definition 2.11. *A fixed point x_0 for $F(x)$ is an attracting fixed point if $|F'(x_0)| < 1$, repelling fixed point if $|F'(x_0)| > 1$ and neutral or indifferent if $|F'(x_0)| = 1$.*

When the fixed or periodic point structure of a one parameter family of functions changes as c passes through a particular parameter value, bifurcation occurs, and the following results occur:

Proposition 2.12. *[5] The First Bifurcation.*
For the family of quadratic map $Q_c(z) = z^2 + c$:

1. *All orbits tend to infinity if $c > \frac{1}{4}$.*
2. *When $c = \frac{1}{4}, Q_c$ has a single fixed point at $p_+ = p_- = \frac{1}{2}$ that is neutral.*
3. *For $c < \frac{1}{4}$, Q_c has two fixed points at p_+ and p_-. The fixed point p_+ is always repelling. The fixed points p_- is*

a. *If $\frac{-3}{4} < c < \frac{1}{4}$, p_- is attracting.*

b. *If $c = \frac{-3}{4}, p_-$ is neutral.*

c. *If $c < \frac{-3}{4}, p_-$ is repelling.*

To study the cycle of period 2, we have the equation $Q_c^2(z) = z$ and its roots are p_+, p_- and

$$q\pm = \frac{1}{2}(-1 \mp \sqrt{-4c-3})$$

Proposition 2.13. *[5] The Second Bifurcation.*
For the family of quadratic maps $Q_c(z) = z^2 + c$:

1. *For $\frac{-3}{4} < c < \frac{1}{4}$, Q_c has an attracting fixed point at p_- and no 2-cycles.*
2. *For $c = \frac{-3}{4}$, Q_c has a neutral fixed point at $p_- = q_\pm$ and no 2 cycles.*
3. *For $\frac{-5}{4} < c < \frac{-3}{4}$ has repelling fixed points at $p_\pm$ and an an attracting 2 cycles at $q_\pm$.*

The Cycle graph is the next standard graph, and we can get some interesting results by analyzing its second degree independence polynomial.

3. Cycle graph

A simple graph with n vertices ($n \geq 3$) and n edges is called a cycle graph if all its edges form a cycle of length n and its degree of each vertex is two.We denote cycle graph by C_n.[**?**]

Theorem 3.1. *Independence polynomial of Cycle graph of order 5 is given by a $I(C_5, z) = 5z^2 + 5z + 1$ and it is conjugate to $z^2 + \frac{5}{4}$. .*

Theorem 3.2. *[5] For $c \in R$, $J(f_c)$ is connected if and only if $c \in [-2, \frac{1}{4}]$.Outside this interval $J(f_c)$ is a cantor set.*

Definition 3.3. *Graph G is called a Mandelbrot graph if $J(R(G;z))$ is connected.Mandelbrot graph is useful for the connectivity of a Julia set of independence polynomial.*

Corollary 3.4. *Julia set of independence polynomial of Cycle graph of order 5 is not connected and is not an element of Mandelbrot set.*

Proof.

$$J(I(C_5, z), z) = J(z^2 + \frac{5}{4}).$$

By using theorem 3.2 if $n \geq 2$, $J(z^2 + \frac{5}{4})$ is not connected, therefore only $J(I(C_5, z)) \notin Mandelbrotset$. ☐

Corollary 3.5. *Energy of $J(I(C_5, z))$ is greater than the Hausdorff dimension of $J(I(C_5, z))$.*

Proof. We have $J(I(C_5, z)) = J(z^2 + \frac{5}{4})$. Hausdorff dimension of $J(z^2 + \frac{5}{4}) = .4346$. The eigen values of C_5 is

$$2\sum_{j=1}^{5} |cos(\frac{\pi . j}{5})|$$

Therefore energy is 8.1484.Hence the result. ☐

Definition 3.6. *The orbit of independence polynomial is defined as the sequence of points*

$$x_0, x_1 = I(x_0), x_2 = I^2(x_0), \ldots, x_n = I^n(x_0), \ldots..$$

where $x_0 \in R$.

To understand the chaotic behaviour on Julia set of Cycle graph ,consider the fixed points of $I(C_5, z)$. It is given by the quadratic equation

$$I(C_5, z) = z^2 + \frac{5}{4} = z \Rightarrow z^2 - z + \frac{5}{4} = 0$$

. It gives $z_1 = \frac{1}{2} + i$ and $z_2 = \frac{1}{2} - i$ as fixed points and are repelling . All iterates of $I(C_5, z)$ tend to infinity.Hence we have the following theorem.

Theorem 3.7. *All orbits of independence polynomial of Cycle graph of order 5 tend to infinity .*

The Julia set fractal of $I(C_5)$ is visualized as:

Figure 1.

4. Path graph

The path graph is a tree with two nodes of vertex degree 1, and the other nodes of vertex degree 2.

4.1 Path graph of order 3

It is denoted by P_3.

Theorem 4.1. *Independence polynomial of path graph of order 3 is given by $I(P_3,z) = z^2 + 3z + 1$. It is same as that of independence polynomial of star graph of order 3, S_3. It is conjugate to $z^2 + \frac{1}{4}$.*

Corollary 4.2. *Julia set of independence polynomial of Path graph of order 3 is connected and is an element of Mandelbrot set.*

Proof. $J(I(P_3,z),z) = J(z^2 + 3z + 1)$. By using theorem 3.2, $J(I(P_3,z))$ is connected and therefore $J(I(C_5,z)) \in$ Mandelbrot set . □

Corollary 4.3. *Energy of $J(I(P_3,z))$ is greater than the Hausdorff dimension of $J(I(P_3,z))$.*

Proof. We have

$$J(I(P_3,z)) = J(z^2 + \frac{1}{4}).$$

Hausdorff dimension of $J(z^2 + \frac{1}{4}) = 1.0812$. The energy of P_3 is $2\sqrt{2}$ ie, 2.8285.Hence the result.

□

To discuss the nature of periodic points of path graph of order 3 , we have the following result:

Theorem 4.4. *$I(P_3,z)$ have neither attracting nor repelling fixed points.*

Proof. Fixed points of $I(P_3,z)$ is given by the quadratic equation: $I(P_3,z) = z^2 + \frac{1}{4} = z \Rightarrow z^2 - z + \frac{1}{4} = 0 \Rightarrow = \frac{1}{2}, \frac{1}{2}$.
Therefore, the only fixed point of $I(P_3,z)$ is at $z = \frac{1}{2}$.
$I'(P_3,z) = 1$, thus $I(P_3,z)$ has neither attractig nor repelling fixed point that is neutral. . □

Theorem 4.5. *$I(P_3,z)$ has repelling 2 cycles.*

Proof. The 2 cycles of $I(P_3,z)$ are given by
$I^2(P_3,z) = z \Rightarrow (z^2 + \frac{1}{4})^2 + \frac{1}{4} = z \Rightarrow z^4 + \frac{1}{2}z^2 - z + \frac{5}{16} = 0$.
By solving, $z = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \pm i$. But both $-\frac{1}{2} \pm i$ are not fixed points of $I(P_3,z)$. So $z = -\frac{1}{2} + i, -\frac{1}{2} - i$ is a 2 cycle of $I(P_3,z)$ and since $|I'(P_3,z)| > 1$ at both these points, the 2 cycle $-\frac{1}{2} \pm i$ is repelling. □

We have created a fractal diagram associated with $I(P_3)$ as

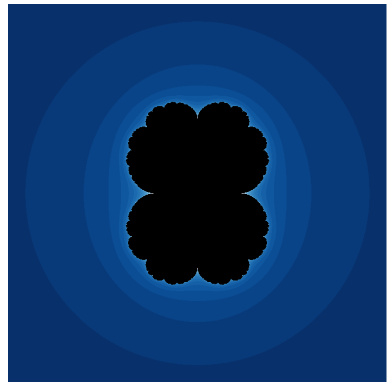

Figure 2.

4.2 Path graph of order 4

It is denoted by P_4.

Theorem 4.6. *Independence polynomial of path graph of order 4 is given by $4z^2 + 3z + 1$ and is conjugate to $z^2 + 1$.*

Corollary 4.7. *Julia set of independence polynomial of Path graph of order 4 is totally disconnected and is not an element of Mandelbrot set.*

Proof. $J(I(P_4,z),z) = J(z^2+1)$ By using theorem 3.2, $J(I(P_3,z))$ is disconnected and therefore $J(I(C_5,z)) \notin Mandelbrot set$. □

Corollary 4.8. *Energy of $J(I(P_4,z))$ is greater than the Hausdorff dimension of $J(I(P_4,z))$.*

Proof. We have $J(I(P_4,z)) = J(z^2+1)$. Hausdorff dimension of$J(z^2+1) = .6791$. The eigen values of P_4 is

$$2\sum_{j=1}^{4} |cos(\frac{2\pi.j}{4})|.$$

Therefore energy is 4.47206.Hence the result. □

Theorem 4.9. *The independence polynomial of path graph of order 4 has two fixed points at $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Both the fixed points are repelling since $|1 \pm \sqrt{3}i| > 1$.*

The fractal diagram associated with $I(P_4)$ as

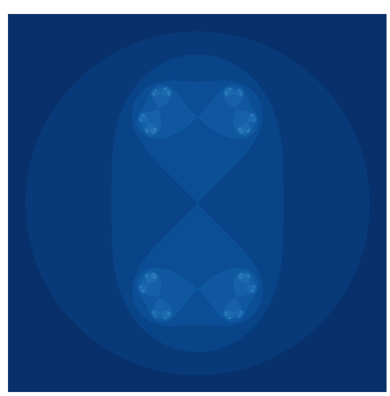

Figure 3.

5. Wheel graph

The order n wheel graph is a graph with $n+1$ vertices. This graph is created by copying C_n and adding a central vertex that is adjacent to every vertex in C_n. The wheel graph of order n is denoted by the symbol W_n. A second degree polynomial in z provides the independence polynomial for wheel graphs of order 5 and 6.

5.1 Wheel graph of order 5

It is denoted by W_5 The independence polynomial of a wheel graph is a quadratic polynomial, and we can prove the following theorem using conjugacy of quadratic maps.

Theorem 5.1. *The independence polynomial of a Wheel graph of order 5 is* $I(W_5,z) = 2z^2+5z+1$,*and it is conjugate to* $z^2-\frac{7}{4}$.

We can obtain the following result by analyzing the connectivity of these graphs.

Corollary 5.2. *The Julia set of independence polynomials of the Wheel graph of order 5 is connected and belongs to the Mandelbrot set.*

Proof.

$$J(I(W_5,z),z) = J(z^2-\frac{7}{4}).$$

By using theorem 5.3([5]),$J(I(W_5,z))$ is connected and therefore $J(I(W_5,z) \in Mandelbrotset$. ☐

Corollary 5.3. *Energy of* $J(I(W_5,z))$ *is greater than the Hausdorff dimension of* $J(I(W_5,z))$.

Proof. We have

$$J(I(W_5,z)) = J(z^2-\frac{7}{4}).$$

Hausdorff dimension of $J(z^2-\frac{7}{4}) = 1.1632$.The energy of W_5 is 9.37 . Hence the result. ☐

Theorem 5.4. *The independence polynomial of wheel graph of order 5 has two fixed points at*

$$p_+ = \frac{1}{2}+\sqrt{2}, p_- = \frac{1}{2}-\sqrt{2}$$

.Both the fixed points p_+ *and* p_- *are repelling.*

To help us understand these characteristics, we have a fractal diagram of $I(W_5)$.

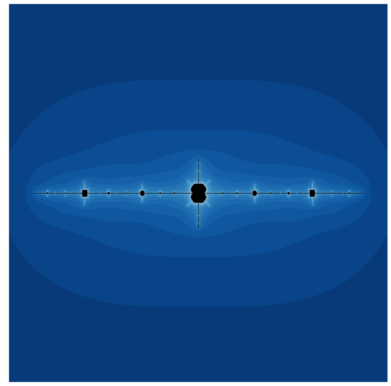

Figure 4.

5.2 Wheel graph of order 6

It is denoted by W_6.

Theorem 5.5. *Independence polynomial of Wheel graph of order 6 is given by* $I(W_6,z) = 5z^2+6z+1$ *and is conjugate to* z^2-1.

Corollary 5.6. *Julia set of independence polynomial of Wheel graph of order 6 is connected and is an element of Mandelbrot set.*

Proof.

$$J(I(W_5,z),z) = z^2-1.$$

By using theorem 3.2, $J(I(W_6,z))$ is connected and therefore $J(I(W_6,z)) \in Mandelbrotset$. ☐

Corollary 5.7. *Energy of* $J(I(W_6,z))$ *is greater than the Hausdorff dimension of* $J(I(W_6,z))$.

Proof. We have $J(I(W_6,z)) = J(z^2-1)$. Hausdorff dimension of $J(z^2-1) = 1.26835$. W_6 is 11.92.Hence the result. ☐

Theorem 5.8. *The independence polynomial of wheel graph of order 6 has two fixed points at*

$$p_+ = \frac{1+\sqrt{5}}{2}, p_- = \frac{1-\sqrt{5}}{2}.$$

Both the fixed points p_+ *and* p_- *are repelling.*

The fractal diagram of $I(W_6)$ is

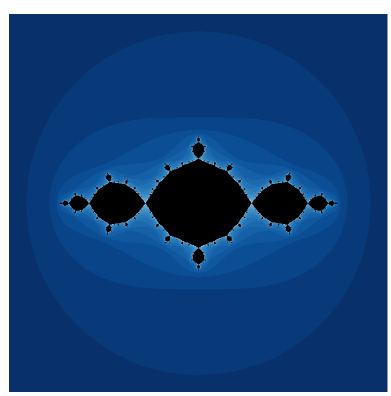

Figure 5.

6. Conclusion

Summarizing the research pertaining to the graphs, the salient observations are listed as follows:

- Energy of second degree independence polynomial of cycle graph,path graph and wheel graph is greater than the Hausdorff dimension of a Julia set of corresponding independence polynomial.
- Juia set of independence polynomial of P_3,W_5 & W_6 are all connected and therefore element of Mandelbrot set.
- Juia set of independence polynomial of C_5 & P_4 are not connected and therefore not an element of Mandelbrot set
- Connectivity of a graph does not depend on the connectivity of its fractal.
- The quadratic family of functions z^2+c where c is a constant have different dynamical properties and the behaviour of its orbit depends on the value of c.
- The number of fixed points of the independence polynomial of these graphs will differ depending on $c=\frac{1}{4}$ or $c\neq\frac{1}{4}$.
- Except the path graph of order 3, all other graphs mentioned above have fixed points which are repelling.

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