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The shared set and uniqueness of algebroid functions on annuli

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Abstract
In this paper, we discuss the shared set and uniqueness of algebroid function on annuli.

Keywords
Value Distribution Theory; algebroid functions; annuli.

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Contents

1. Introduction

Yang Tan [44], Yang Tan and Yue Wang [43] investigated some interesting results on the multiple values and uniqueness of algebroid functions on annuli and also others have proved several results for algebroid functions on annuli ([10, 11, 13–17, 19–38]. Therefore it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [42] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A} = \mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

2. Basic Notations and Definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [8, 9], [12] and [18]).

Let $A_\nu(z), A_{\nu-1}(z), \ldots, A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\psi(z,W) = A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + \ldots + A_1(z)W + A_0(z) = 0. \tag{2.1}$$

Then irreducible equation (2.1) defines a ν-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

In this paper, a algebroid function always mean a function which is algebroid in $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$. Let $W(z)$ and $M(z)$ be ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, $a \in \overline{\mathbb{C}}$. We say that W and M share the value a CM if $W(z) - a$ and $M(z) - a$ have the same zeros with the same multiplicities. We shall use standard notations of value distribution theory in annuli, $T_0(r,W)$, $m_0(r,W)$, $N_0(r,W)$, $\overline{N}_0(r,W)$,... ([43], [44]).

Let $W(z)$ and $M(z)$ be ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ share the finite value a IM (ignoring multiplicities), if $W(z) - a$ and $M(z) - a$ have the same zeros on annuli. If $W(z) - a$ and $M(z) - a$ have the same zeros with the same multiplicities, we say that $W(z)$ and $M(z)$ share the value a CM (counting multiplicities) on annuli. If $W(z) - a$ and $M(z) - a$ have the same zeros with different multiplicities, we say that $W(z)$ and $M(z)$ share the value a DM (different multiplicities) on annuli.

Next, let k be a positive integer, we denote by $N_0^{k)}\left(r,\frac{1}{W-a}\right)$ is the counting function of zeros of $W(z)-a$ with multiplicity $\leq k$ and $N_0^{(k+1}\left(r,\frac{1}{W-a}\right)$ is the counting function of zeros of $W(z)-a$ with multiplicity $>k$. Definitions of the terms $N_0^{k)}$ and $N_0^{(k+1}$ can be similarly formulated. Finally $N_0^2\left(r,\frac{1}{W}\right)$ denotes the counting function of zeros of W where a zero of multiplicity k is counted with multiplicity $min\{k,2\}$.

We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ to denote the extended complex plane, and $\mathbb{X}$ to denote the subset of $\mathbb{C}$. Let S be a set of distinct elements in $\overline{\mathbb{C}}$ and $\mathbb{X}\subseteq\mathbb{C}$. Define

$$E^{\mathbb{X}}(S,W)=\cup_{a\in S}\{z\in\mathbb{X}\mid W_a(z)=0,\ counting\ \ multiplicities\},$$
$$\overline{E}^{\mathbb{X}}(S,W)=\cup_{a\in S}\{z\in\mathbb{X}\mid W_a(z)=0,\ ignoring\ \ multiplicities\},$$

where $W_a(z)=W(z)-a$ if $a\in\mathbb{C}$ and $W_\infty(z)=\frac{1}{W(z)}$. We also define

$$\overline{E}_1^{\mathbb{X}}(S,W)=\cup_{a\in S}\{z\in\mathbb{X}: all\ \ the\ \ simple\ \ zeros\ \ of\ \ W_a(z)\}.$$

For $a\in\overline{\mathbb{C}}$, we say that two algebroid functions W_1 and W_2 share the value a CM(IM) in $\mathbb{X}$(*or* $\mathbb{C}$), if $W_1(z)-a$ and $W_2(z)-a$ have the same zeros with the same multiplicities (ignoring multiplicities) in $\mathbb{X}$ (or $\mathbb{C}$).

Definition 2.1. *[43] Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, the function*

$$T_0(r,W)=m_0(r,W)+N_0(r,W),\quad 1\leq r<R_0$$

is called Nevanlinna characteristic of $W(z)$.

Definition 2.2. *For positive integer k,m, we define that*

$$\delta_0^k(a,W)=1-\limsup_{r\to+\infty}\frac{N_0^k\left(r,\frac{1}{W-a}\right)}{T_0(r,W)},$$

$$\Theta_0(a,W)=1-\limsup_{r\to+\infty}\frac{\overline{N}_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)},$$

where $N_0^k\left(r,\frac{1}{W-a}\right)$ is counting function of a-points of $W(z)$ on $\mathbb{A}$ where a-points of multiplicity m is counted m times if $m\leq k$ and $1+k$ times if $m>k$. In particular, if $k=\infty$, then

$$\delta_0(a,W)=\liminf_{r\to+\infty}\frac{m_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)}=1-\limsup_{r\to+\infty}\frac{N_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)}$$

.

3. Some Lemmas

Lemma 3.1. *[43] (The first fundamental theorem on annuli) Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, $a\in\mathbb{C}$*

$$m_0(r,a)+N_0(r,a)=T_0(r,W)+O(1).$$

Lemma 3.2. *[43] (The second fundamental theorem on annuli). Let $W(z)$ be v-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, a_k $(k=1,2,..,p)$ are p distinct complex numbers (finite or infinite), then we have*

$$(p-2v)T_0(r,W)\leq\sum_{k=1}^{p}\overline{N}_0\left(r,\frac{1}{W-a_k}\right)+S_0(r,W).\quad(3.1)$$

Lemma 3.3. *[43] Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, if the following conditions are satisfied*

$$\liminf_{r\to\infty}\frac{T_0(r,W)}{\log r}<\infty,\quad R_0=+\infty,$$

$$\liminf_{r\to R_0^-}\frac{T_0(r,W)}{\log\frac{1}{(R_0-r)}}<\infty,\quad R_0<+\infty,$$

then $W(z)$ is an algebraic function.

The following result can be derived from the proof of Frank-Reinders' theorem in [46]

Lemma 3.4. *Let $n\geq 6$ and*

$$H(\omega)=\frac{(n-1)(n-2)}{2}\omega^{(n)}-n(n-2)\omega^{n-1}+\frac{n(n-1)}{2}\omega^{n-2},\quad(3.2)$$

Then $H(\omega)$ is a unique polynomial for admissible merommorphic functions, that is, for any two admissible meromorphic functions f and g on $\mathbb{A}$, $H(f)\equiv H(g)$ implies $f\equiv g$.

By similar process to the one in [47] we can obtain a stand and Valiron-Mohokotype result in $\mathbb{A}$ as follows

Lemma 3.5. [45] *Let f be a nonconstant meromorphic function in $\mathbb{A}$, $Q_1(f)$ and $Q_2(f)$ be two mutually prime polynomials in f with degree m and n, respectively. Then*

$$T_0\left(r,\frac{Q_1(f)}{Q_2(f)}\right)=max\{m,n\}T_0(r,f)+S_0(r,f)\qquad(3.3)$$

Lemma 3.6. *Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$. Then*

$$N_0\left(r,\frac{1}{W'}\right)=N_0\left(r,\frac{1}{W}\right)+\overline{N}_0(r,W)+S_0(r,W)\ (3.4)$$

where $S_0(r,W)$ as defined in Lemma 3.2.

Proof. Since

$$m_0\left(r,\frac{1}{W}\right) \leq m_0\left(r,\frac{1}{W'}\right)+m_0\left(r,\frac{W'}{W}\right) = m_0\left(r,\frac{1}{W'}\right)+S_0(r,W). \quad (3.5)$$

From Lemma 3.1, we have

$$T_0(r,W)-N_0\left(r,\frac{1}{W}\right) \leq T_0(r,W')-N_0\left(r,\frac{1}{W'}\right) + S_0(r,W). \quad (3.6)$$

That is,

$$N_0\left(r,\frac{1}{W'}\right) \leq T_0(r,W')T_0(r,W)+N_0\left(r,\frac{1}{W}\right) + S_0(r,W). \quad (3.7)$$

Since

$$\begin{aligned} T_0(r,W') &= m_0(r,W')+N_0(r,W') \\ &\leq m_0(r,W)+m_0\left(r,\frac{W'}{W}\right)+N_0(r,W)+\overline{N}_0(r,W) \\ &\leq T_0(r,W)+\overline{N}_0(r,W)+S_0(r,W). \quad (3.8) \\ &\leq 2\nu T_0(r,W)+S_0(r,W) \end{aligned}$$

Then from (3.7) and (3.8), we can get the conclusion of Lemma 3.6.

Lemma 3.7. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$ satisfying $E^{\mathbb{A}}(W_1,0)=E^{\mathbb{A}}(W_2,0)$ and let $c_1,c_2,c_3,...,c_q$ be $q(\geq 2)$ distinct nonzero complex numbers. If*

$$\begin{aligned} &\limsup_{r\to\infty,r\in I}\left(\left((2\nu+1)\overline{N}_0(r,W_1)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{W_1-c_j}\right)+\overline{N}_0\left(r,\frac{1}{W_1'}\right)\right)(T_0(r,W_1))^{-1}\right)<q, \\ &\limsup_{r\to\infty,r\in I}\left(\left((2\nu+1)\overline{N}_0(r,W_2)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{W_2-c_j}\right)+\overline{N}_0\left(r,\frac{1}{W_2'}\right)\right)(T_0(r,W_2))^{-1}\right)<q, \end{aligned} \quad (3.9)$$

where $\overline{N}_0^{(2)}(r,.)=\overline{N}_0(r,.)+\overline{N}_0^{(2}(r,.)$, $\overline{N}_0^{(2}(r,.)=\overline{N}_0(r,.)-\overline{N}_0^{1)}(r,.)$, and I is some set of r of infinite linear measure, then

$$W_1=\frac{aW_2+b}{cW_2+d}, \quad (3.10)$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$.

Proof.Set

$$H\equiv\frac{W_1''}{W_1'}-2\nu\frac{W_1'}{W_1}-\left(\frac{W_2''}{W_2'}-2\nu\frac{W_2'}{W_2}\right). \quad (3.11)$$

Supposing that $H\equiv 0$, we have

$$m_0(r,H)=S_0(r), \quad (3.12)$$

where $S_0(r)=o(T_0(r)),T_0(r)=max[T_0(r,W_1),T_0(r,W_2)]$. Since $E^{\mathbb{A}}(W_1,0)=E^{\mathbb{A}}(W_2,0)$, and by elementary calculation, we can conclude that if z_0 is a common simple zero of W_1 and W_2 in $\mathbb{A}$, then $H(z_0)=0$. Thus we have

$$\begin{aligned} N_0^{1)} &\leq N_0\left(r,\frac{1}{H}\right)\leq T_0(r,H)+O(1) \\ &\leq N_0(r,H)+S_0(r), \end{aligned} \quad (3.13)$$

where $N_0^{1)}(r)=N_0^{1)}(r,\frac{1}{W_1})=N_0^{1)}(r,\frac{1}{W_2})$. The poles of H in $\mathbb{A}$ can only occur at zeros of W_1' and W_2' in $\mathbb{A}$ or poles of W_1 and W_2 in $\mathbb{A}$. Moreover, H only has simple zeros in $\mathbb{A}$. Hence, from (3.13), we have

$$\begin{aligned} N_0^{1)}(r) \leq\ & \overline{N}_0(r,W_1)+\overline{N}_0(r,W_2)+\overline{N}_0^0\left(r,\frac{1}{W_1'}\right) \\ &+\overline{N}_0^0\left(r,\frac{1}{W_2'}\right)+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_1-c_j}\right) \\ &+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_2-c_j}\right)+S_0(r), \end{aligned} \quad (3.14)$$

where $\overline{N}_0^0\left(r,\frac{1}{W_1'}\right)$ is the reduced counting function for the zeros of W' in $\mathbb{A}$, where W_1 does not take one of the values $0,c_1,c_2,...,c_q$.
Since

$$\overline{N}_0\left(r,\frac{1}{W_1}\right)+\overline{N}_0\left(r,\frac{1}{W_2}\right) = 2\nu N_0^{1)}(r)+\overline{N}_0^{(2}\left(r,\frac{1}{W_1}\right) +\overline{N}_0^{(2}\left(r,\frac{1}{W_2}\right). \quad (3.15)$$

Then from (3.14) and (3.15), we have

$$\overline{N}_0\left(r,\frac{1}{W_1}\right)+\overline{N}_0\left(r,\frac{1}{W_2}\right)\leq 2\nu\overline{N}_0(r,W_1)+2\nu\overline{N}_0(r,W_2) +2\nu\overline{N}_0^0\left(r,\frac{1}{W_1'}\right)+2\nu\overline{N}_0^0\left(r,\frac{1}{W_2'}\right)+\overline{N}_0^{(2}\left(r,\frac{1}{W_1}\right)$$

$$+\overline{N}_0^{(2}\left(r,\frac{1}{W_2}\right)+2\nu\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_1-c_j}\right)$$
$$+2\nu\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_2-c_j}\right)+S_0(r). \quad (3.16)$$

From Lemma 3.2, we have

$$qT_0(r,W_1) \le \overline{N}_0(r,W_1)+\overline{N}_0\left(r,\frac{1}{W_1}\right)+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_1-c_j}\right)$$
$$-N_0^0\left(r,\frac{1}{W_1'}\right)+S_0(r),\ \ r\notin E,$$
$$qT_0(r,W_2) \le \overline{N}_0(r,W_2)+\overline{N}_0\left(r,\frac{1}{W_2}\right)+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_2-c_j}\right)$$
$$-N_0^0\left(r,\frac{1}{W_2'}\right)+S_0(r),\ \ r\notin E, \quad (3.17)$$

where E is a set of r of finite linear measure and it needs not to be the same at each occurrence. From (3.16) and (3.17), it follows that, for $r\notin E$,

$$q[T_0(r,W_1)+T_0(r,W_2)]$$
$$\le(2\nu+1)\overline{N}_0(r,W_1)+(2\nu+1)\overline{N}_0(r,W_2)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_1-c_j}\right)$$
$$+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_2-c_j}\right)+2\nu\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_1-c_j}\right)$$
$$+2\nu\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_2-c_j}\right)+\overline{N}_0(2\left(r,\frac{1}{W_1}\right)+\overline{N}_0(2\left(r,\frac{1}{W_2}\right)$$
$$+\overline{N}_0^0\left(r,\frac{1}{W_1'}\right)+\overline{N}_0^0\left(r,\frac{1}{W_2'}\right)+S_0(r). \quad (3.18)$$

Since

$$\sum_{j=1}^{q}\overline{N}_0^{(2}\left(r,\frac{1}{W_1-c_j}\right) + \overline{N}_0(2\left(r,\frac{1}{W_1}\right)+\overline{N}_0^0\left(r,\frac{1}{W_1'}\right)$$

$$=\overline{N}_0\left(r,\frac{1}{W_1'}\right), \quad (3.19)$$

From (3.18) and (8.18), we can get that, for $r\notin E$,

$$q[T_0(r,W_1)+T_0(r,W_2)]$$
$$\le(2\nu+1)\overline{N}_0(r,W_1)+(2\nu+1)\overline{N}_0(r,W_2)$$
$$+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_1-c_j}\right)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_2-c_j}\right)$$
$$+\overline{N}_0\left(r,\frac{1}{W_1'}\right)+\overline{N}_0\left(r,\frac{1}{W_2'}\right)+S_0(r). \quad (3.20)$$

From (3.9) and (3.20), Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, we can get that

$$[T_0(r,W_1)+T_0(r,W_2)]\le o[T_0(r,W_1)+T_0(r,W_2)],\ r\notin E,R\in I \quad (3.21)$$

Thus we can get a contradiction. Therefore $H\equiv 0$; that is

$$\frac{W_1''}{W_1'}-2\nu\frac{W_1'}{W_1}\equiv\frac{W_2''}{W_2'}-2\nu\frac{W_2'}{W_2}. \quad (3.22)$$

For the above equality, by integration, we can get

$$W_1\equiv\frac{aW_2+b}{cW_2+d}, \quad (3.23)$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$.

Lemma 3.8. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ satisfying $E_1^{\mathbb{A}}(W_1,0)=E_1^{\mathbb{A}}(W_2,0)$ and let $c_1,c_2,c_3,...,c_q$ be $q(\ge 2)$ distinct nonzero complex numbers. If*

$$\limsup_{r\to\infty,r\in I}\left(\left((2\nu+1)\overline{N}_0(r,W_1)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{W_1-c_j}\right)+\overline{N}_0\left(r,\frac{1}{W_1'}\right)+2\nu\overline{N}_0^{(2}\left(r,\frac{1}{W_1}\right)\right)(T_0(r,W_1))^{-1}\right)<q,$$
$$\limsup_{r\to\infty,r\in I}\left(\left((2\nu+1)\overline{N}_0(r,W_2)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{W_2-c_j}\right)+\overline{N}_0\left(r,\frac{1}{W_2'}\right)+2\nu\overline{N}_0^{(2}\left(r,\frac{1}{W_2}\right)\right)(T_0(r,W_2))^{-1}\right)<q, \quad (3.24)$$

where $\overline{N}_0^{(2)}(r,.)=\overline{N}_0(r,.)+\overline{N}_0^{(2}(r,.)$, $\overline{N}_0^{(2}(r,.)=\overline{N}_0(r,.)-\overline{N}_0^{1)}(r,.)$, and I is some set of r of infinite linear measure, then

$$W_1=\frac{aW_2+b}{cW_2+d}, \quad (3.25)$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$.

Proof. Let H be stated as in the proof of Lemma 3.7, since $E_1^{\mathbb{A}}(W_1,0)=E_1^{\mathbb{A}}(W_2,0)$, we can get that

$$N_0^{1)}(r)\le\overline{N}_0(r,W_1)+\overline{N}_0(r,W_2)+\overline{N}_0^0\left(r,\frac{1}{W_1'}\right)$$
$$+\overline{N}_0^0\left(r,\frac{1}{W_2'}\right)+\overline{N}_0^{(2}\left(r,\frac{1}{W_1}\right)+\overline{N}_0^{(2}\left(r,\frac{1}{W_2}\right)$$

$$+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_1-c_j}\right)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_2-c_j}\right). \quad (3.26)$$

Similar to the argument in Lemma 3.7, we can get that, for $r\notin E$

$$\begin{aligned}&q[T_0(r,W_1)+T_0(r,W_2)]\\ \leq&(2\nu+1)\overline{N}_0(r,W_1)+(2\nu+1)\overline{N}_0(r,W_2)\\ &+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_1-c_j}\right)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{W_2-c_j}\right)\\ &+2\nu\overline{N}_0^{(2}\left(r,\frac{1}{W_1}\right)+2\nu\overline{N}_0^{(2}\left(r,\frac{1}{W_2}\right)\\ &+\overline{N}_0\left(r,\frac{1}{W_1'}\right)+\overline{N}_0\left(r,\frac{1}{W_2'}\right)+S_0(r). \quad (3.27)\end{aligned}$$

From (3.24) and (3.27), let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, we can get that

$$[T_0(r,W_1)+T_0(r,W_2)]\leq o[T_0(r,W_1)+T_0(r,W_2)],\ r\notin E, R\in I. \quad (3.28)$$

Thus we can get a contradiction. Therefore $H\equiv 0$; that is

$$\frac{W_1''}{W_1'}-2\nu\frac{W_1'}{W_1}\equiv\frac{W_2''}{W_2'}-2\nu\frac{W_2'}{W_2}. \quad (3.29)$$

For the above equality, by integration, we can get

$$W_1\equiv\frac{aW_2+b}{cW_2+d}, \quad (3.30)$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. □

4. Main Results

In this paper, we will focus our attention on the uniqueness problem of shared set of algebroid functions on annuli. In fact, we will prove the uniqueness of algebroid functions on annuli sharing one set $S=\{\omega\in\mathbb{A}:P_1(\omega)=0\}$, where

$$\begin{aligned}P_1(\omega)=&\frac{(n-1)(n-2\nu)}{2\nu}\omega^n-n(n-2\nu)\omega^{n-1}\\ &+\frac{n(n-1)}{2\nu}\omega^{n-2}-c, \quad (4.1)\end{aligned}$$

and c is a complex number satisfying $c\neq 0,\ 1$.

Our main theorems of this paper are listed as follows

Theorem 4.1. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$. If $E^{\mathbb{A}}(S,W_1)=E^{\mathbb{A}}(S,W_2)$ and n is an integer $\geq 10\nu+1$, then $W_1\equiv W_2$.*

Proof. From the definition of $P_1(\omega)$, we can get that $P_1=1-c:=c_1\neq 0$, $P_0=-c:=c_2\neq 0$ and

$$P_1'(\omega)=\frac{n(n-1)(n-2\nu)}{2\nu}(\omega-1)^2\omega^{n-3}, \quad (4.2)$$

$$P_1(\omega)-c_1=(\omega-1)^3Q_1(\omega),\ Q_1(1)\neq 0, \quad (4.3)$$

$$P_1(\omega)-c_2=\omega^{n-2}Q_2(\omega),\ Q_2(0)\neq 0, \quad (4.4)$$

where Q_1,Q_2 are polynomials of degree $n-(2\nu+1)$ and 2ν, respectively. We also see that $Q_i(i=1,2)$ and P_1 have only simple zeros.

Let F and G be defined as $F=P_1(W_1)$ and $G=P_1(W_2)$. Since $E^{\mathbb{A}}(W_1,S)=E^{\mathbb{A}}(W_2,S)$, we have $E^{\mathbb{A}}(F,0)=E^{\mathbb{A}}(G,0)$. From (4.3) and (4.4), we have

$$\begin{aligned}&\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_1}\right)=\overline{N}_0\left(r,\frac{1}{F-c_1}\right)+\overline{N}_0^{(2}\left(r,\frac{1}{F-c_1}\right)\\ \leq&2\nu\overline{N}_0\left(r,\frac{1}{W_1-1}\right)+\sum_{i=1}^{n-(2\nu+1)}N_0\left(r,\frac{1}{W_1-a_i}\right)\\ \leq&(n-1)T_0(r,W_1)+S_0(r),\end{aligned}$$

$$\begin{aligned}\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_2}\right)&=\overline{N}_0\left(r,\frac{1}{F-c_2}\right)+\overline{N}_0^{(2}\left(r,\frac{1}{F-c_2}\right)\\ &\leq 2\nu\overline{N}_0\left(r,\frac{1}{W_1}\right)+\sum_{i=1}^{2\nu}N_0\left(r,\frac{1}{W_1-b_j}\right)\\ &\leq 4\nu T_0(r,W_1)+S_0(r), \quad (4.5)\end{aligned}$$

where $a_i(i=1,2,...,n-(2\nu+1))$ and $b_j(j=1,2\nu)$ are the zeros of $Q_1(\omega)$ and $Q_2(\omega)$ in $\mathbb{A}$, respectively. From (4.2), we have

$$\begin{aligned}N_0\left(r,\frac{1}{F'}\right)\leq&\overline{N}_0\left(r,\frac{1}{W_1}\right)\\ &+\overline{N}_0\left(r,\frac{1}{W_1-1}\right)+\overline{N}_0\left(r,\frac{1}{W_1'}\right). \quad (4.6)\end{aligned}$$

From Lemma 3.5, we have $T_0(r,F)=nT_0(r,W_1)+S_0(r)$. Thus, combining (4.5) and (4.6), by Lemmas 3.6 and 3.7 and $n\geq(10\nu+1)$, we have

$$\begin{aligned}&\limsup_{r\to\infty,r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,F)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_j}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\right)(T_0(r,F))^{-1}\right)\\ &\leq\limsup_{r\to\infty,r\notin I}\frac{4\nu\overline{N}_0(r,W_1)+(n+6\nu)T_0(r,W_1)}{nT_0(r,W_1)}<2\nu. \quad (4.7)\end{aligned}$$

Similarly, we have

$$\limsup_{r\to\infty, r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,G)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(r,\frac{1}{G-c_j}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)\right)(T_0(r,G))^{-1}\right)$$
$$\leq \limsup_{r\to\infty, r\notin I}\frac{4\nu\overline{N}_0(r,W_2)+(n+6\nu)T_0(r,W_2)}{nT_0(r,W_2)}<2\nu. \tag{4.8}$$

Thus by Lemma 3.7, we have

$$\frac{F''}{F'}-2\nu\frac{F'}{F}\equiv\frac{G''}{G'}-2\nu\frac{G'}{G}. \tag{4.9}$$

From the previous equality, by integration, we get

$$F\equiv\frac{aG+b}{cG+d}, \tag{4.10}$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. Since $E^{\mathbb{A}}(W_1,S)$ is non-empty and $E^{\mathbb{A}}(W_1,S)=E^{\mathbb{A}}(W_2,S)$, we have $b=0, a\neq 0$. Hence

$$F\equiv\frac{aG}{cG+d}\equiv\frac{G}{AG+B}, \tag{4.11}$$

where $A=\frac{c}{a}$, $B=\frac{d}{a}\neq 0$.
Two cases will be considered as follows
Case 1 ($A\neq 0$): From the definition of $P_1(\omega)$ and (4.11), we see that every zero of $P_1(W_2)+\frac{B}{A}$ in $\mathbb{A}$ has a multiplicity of at least n. Here, three following subcases will be discussed.
Subcase 1 ($\frac{B}{A}=-c_1$): From (4.3), we have

$$P_1(W_2)+\frac{B}{A}=(W_2-1)^3(W_2-a_1)(W_2-a_{2\nu})\ldots (W_2-a_{n-(2\nu+1)}), \tag{4.12}$$

where $a_i\neq 0,1$, are distinct values. It follows that

$$\Theta_0(a_i,W_1) = 1-\limsup_{r\to\infty}\frac{\overline{N}_0(r,a)}{T_0(r,W_1)} \geq 1-\limsup_{r\to\infty}\frac{\overline{N}_0(r,a)}{N_0(r,W_1)}\geq\frac{1}{2\nu}. \tag{4.13}$$

We can see that $P_1(W_2)+\frac{B}{A}$ has $n-2\nu$ values satisfying the above inequality. Thus, from Lemma 3.2 and $n\geq(10\nu+1)$, we can get a contradiction.
Subcase 2 ($\frac{B}{A}=-c_2$): From (4.3), we have

$$P_1(W_2)+\frac{B}{A}=W_2)^{(n-2)}(W_2-b_1)(W_2-b_{2\nu}), \tag{4.14}$$

where $b_1\neq b_2, b_i\neq 0,1 (i=1,2)$. It follows that every zero of W_2 in $\mathbb{A}$ has a multiplicity of at least 2ν and every zero of $W_2-b_i (i=1,2)$ in $\mathbb{A}$ has multiplicity of at least n. Then by Lemma 3.2, we have

$$\begin{aligned} T_0(r,W_2) &\leq \overline{N}_0\left(r,\frac{1}{W_2}\right)+\overline{N}_0\left(r,\frac{1}{W_2-b_1}\right) +\overline{N}_0\left(r,\frac{1}{W_2-b_2}\right)+S_0(r)\\ &\leq \frac{1}{2\nu}N_0\left(r,\frac{1}{W_2}\right)+\frac{1}{n}N_0\left(r,\frac{1}{W_2-b_1}\right) +\frac{1}{n}N_0\left(r,\frac{1}{W_2-b_2}\right)+S_0(r) \qquad (4.15)\\ &\leq \left(\frac{1}{2\nu}+\frac{2\nu}{n}\right)T_0(r,W_2)+S_0(r). \end{aligned}$$

Since $W_2(z)$ be an ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$ and $n\geq(10\nu+1)$, we can get a contradiction.
Subcase 3 ($\frac{B}{A}\neq -c_1,-c_2$) : By using the same argument as in Subcases 1 or 2, we can get a contradiction.
Case 2 ($A=0$): If $B\neq 1$, from (4.11), we have $F=\frac{G}{B}$; that is

$$P_1(W_1)=\frac{1}{B}P_1(W_2). \tag{4.16}$$

From (4.4) and (4.16), we get

$$\begin{aligned} P_1(W_1)-\frac{c_2}{B} &= \frac{1}{B}(P_1(W_2)-c_2)\\ &= \frac{1}{B}W_2^{n-2}(W_2-b_1)(W_2-b_2). \qquad (4.17)\end{aligned}$$

Since $\frac{c_2}{B}\neq c_2$, from (4.2), it follows that $P_1(W_1)-\frac{c_2}{B}$ has at least $n-2\nu$ distinct zeros $e_1,e_2,\ldots,e_{n-2}$. Then, by applying Lemma 3.2, we get

$$\begin{aligned} (n-4\nu)T_0(r,W_1) &\leq \sum_{i=1}^{n-2\nu}\overline{N}_0\left(r,\frac{1}{W_1-e_i}\right)+S_0(r)\\ &\leq \overline{N}_0\left(r,\frac{1}{W_2}\right)+\overline{N}_0\left(r,\frac{1}{W_2-b_1}\right) +\overline{N}_0\left(r,\frac{1}{W_2-b_2}\right)+S_0(r)\\ &\leq (2\nu+1)T_0(r,W_2)+S_0(r). \qquad (4.18)\end{aligned}$$

By applying Lemma 3.7 to (4.16) and from (4.18), since $n\geq(10\nu+1)$ and Since $W_1(z)$ be an ν-valued algebroid function

which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, we can get a contradiction.
Thus, we have $A=0$ and $B=1$; that is, $P_1(W_1)\equiv P_1(W_2)$. Noting the form of $P_1(\omega)$; we can get that $P_1(W_1)\equiv P_1(W_2)$, that is,

$$\frac{(n-1)(n-2\nu)}{2\nu}W_1^n-n(n-2\nu)W_1^{n-1}+\frac{n(n-1)}{2\nu}W_1^{n-2}\equiv\frac{(n-1)(n-2\nu)}{2\nu}W_2^n-n(n-2\nu)W_2^{n-1}+\frac{n(n-1)}{2\nu}W_2^{n-2}. \tag{4.19}$$

Since $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, then it follows by Lemma 3.4 that $W_1\equiv W_2$. Therefore, the proof of Theorem 4.1 is completed.

A set S is called a unique range set for algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$. Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}$. If $E^{\mathbb{A}}(S,W_1)=E^{\mathbb{A}}(S,W_2)$ implies $W_1\equiv W_2$. We denote λ the cardinality of a set S. □

Thus from Theorem 4.1 we can get the following corollary.

Corollary 4.2. *There exists one finite set S with $\lambda=(6\nu+1)$, such that any two ν-valued algebroid functions $W_1(z)$ and $W_2(z)$ which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ must be identical if $E^{\mathbb{A}}(S,W_1)=E^{\mathbb{A}}(S,W_2)$.*

Theorem 4.3. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$. If $E^{\mathbb{A}}(S,W_1)=E^{\mathbb{A}}(S,W_2)$, $\Theta_0(\infty,W_1)>\frac{(2\nu+1)}{4\nu}$, $\Theta_0(\infty,W_2)>\frac{(2\nu+1)}{4\nu}$ and n is an integer $\ge 6\nu+1$, then $W_1\equiv W_2$.*

Proof. Since $\Theta_0(\infty,W_1)>\frac{(2\nu+1)}{4\nu}$, $\Theta_0(\infty,W_2)>\frac{(2\nu+1)}{4\nu}$ it follows that

$$\limsup_{r\to\infty}\frac{\overline{N}_0(r,W_1)}{T_0(r,W_1)}<\frac{1}{4\nu},\quad \limsup_{r\to\infty}\frac{\overline{N}_0(r,W_1)}{T_0(r,W_1)}<\frac{1}{4\nu}. \tag{4.20}$$

By applying (4.20), from (4.7) and (4.8), and since $n\ge 6\nu+1$, we get

$$\begin{aligned}
&\limsup_{r\to\infty,r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,F)+\sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_j}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\right)(T_0(r,F))^{-1}\right)\\
&\le\limsup_{r\to\infty,r\notin I}\frac{4\nu\overline{N}_0(r,W_1)+(n+6\nu)T_0(r,W_1)}{nT_0(r,W_1)}<2\nu.\\
&\limsup_{r\to\infty,r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,G)+\sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{G-c_j}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)\right)(T_0(r,G))^{-1}\right)\\
&\le\limsup_{r\to\infty,r\notin I}\frac{4\nu\overline{N}_0(r,W_2)+(n+6\nu)T_0(r,W_2)}{nT_0(r,W_2)}<2\nu.
\end{aligned} \tag{4.21}$$

Then from Lemma 3.6, we have $F\equiv\frac{aG+b}{cG+d}$, $a,b,c,d\in\mathbb{C}$ and $ad-bc\ne0$. Thus, by using the same argument as that in Theorem 4.1, we can prove the conclusion of Theorem 4.3. □

Corollary 4.4. *There exists one finite set S with $\lambda=(6\nu+1)$, such that any two ν-valued algebroid functions $W_1(z)$ and $W_2(z)$ which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ must be identical if $E_1^{\mathbb{A}}(S,W_1)=E_1^{\mathbb{A}}(S,W_2)$.*

Theorem 4.5. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$. If $E_1^{\mathbb{A}}(S,W_1)=E_1^{\mathbb{A}}(S,W_2)$ and n is an integer $\ge 14\nu+1$, then $W_1\equiv W_2$.*

Proof. Since $E_1^{\mathbb{A}}(S,W_1)=E_1^{\mathbb{A}}(S,W_2)$, we have $E_1^{\mathbb{A}}(F,0)=E_1^{\mathbb{A}}(G,0)$ From (4.2) and (4.4), we get

$$\overline{N}_0^{(2}\left(r,\frac{1}{F}\right)=\sum_{i=1}^{n}\overline{N}_0\left(r,\frac{1}{W_1-d_i}\right)\overline{N}_0\left(r,\frac{1}{W_1'}\right), \tag{4.22}$$

where $d_i(i=1,2,...,n)$ are distinct zeros of $P_1(\omega)$. And from (4.6) (4.22) , Lemma 3.5, we have

$$\begin{aligned}
&\overline{N}_0\left(r,\frac{1}{F'}\right)+2\nu\overline{N}_0^{(2}\left(r,\frac{1}{F}\right)\\
&\le\overline{N}_0\left(r,\frac{1}{W_1}\right)+\overline{N}_0\left(r,\frac{1}{W_1-1}\right)\\
&+(2\nu+1)\overline{N}_0\left(r,\frac{1}{W_1}\right)\\
&+(2\nu+1)\overline{N}_0(r,W_1)\\
&\le(4\nu+1)T_0(r,W_1)+(2\nu+1)\overline{N}_0(r,W_1)+S_0(r).
\end{aligned} \tag{4.23}$$

Then from (4.5) and (4.23), since $T_0(r,F) = nT_0(r,W_1) + S_0(r)$ and $n \geq 14\nu + 1$, we have

$$\limsup_{r\to\infty, r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,F) + \sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_j}\right) + \overline{N}_0\left(r,\frac{1}{F'}\right) + 2\nu\overline{N}_0^{(2}\left(r,\frac{1}{F}\right)\right)(T_0(r,F))^{-1}\right)$$
$$\leq \limsup_{r\to\infty, r\notin I}\frac{6\nu\overline{N}_0(r,W_1) + (n+8\nu)T_0(r,W_1)}{nT_0(r,W_1)} < 2\nu. \tag{4.24}$$

Similarly, we get

$$\limsup_{r\to\infty, r\notin I}\left(\left((2\nu+)\overline{N}_0(r,G) + \sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{G-c_j}\right) + \overline{N}_0\left(r,\frac{1}{G'}\right) + 2\nu\overline{N}_0^{(2}\left(r,\frac{1}{G}\right)\right)(T_0(r,G))^{-1}\right)$$
$$\leq \limsup_{r\to\infty, r\notin I}\frac{6\nu\overline{N}_0(r,W_2) + (n+8\nu)T_0(r,W_2)}{nT_0(r,W_2)} < 2\nu. \tag{4.25}$$

Thus by Lemma 3.7, we have

$$F \equiv \frac{aG+b}{cG+d}, \tag{4.26}$$

where $a,b,c,d \in \mathbb{C}$ and $ad - bc \neq 0$. By using arguments similar to those in the proof of Theorem 4.1, we can get that $W_1 \equiv W_2$.
This completes the proof of Theorem 4.5. □

A set S is called a unique range set with weight 1 for algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$. Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}$. If $E_1^{\mathbb{A}}(S,W_1) = E_1^{\mathbb{A}}(S,W_2)$ implies $W_1 \equiv W_2$. Thus from Theorem 4.5 we can get the following corollary.

Corollary 4.6. *There exists one finite set S with $\lambda = (14\nu + 1)$, such that any two ν-valued algebroid functions $W_1(z)$ and $W_2(z)$ which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ must be identical if $E_1^{\mathbb{A}}(S,W_1) = E_1^{\mathbb{A}}(S,W_2)$.*

Theorem 4.7. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and $S = \{\omega \in \mathbb{A} : P_1(\omega) = 0\}$, where $P_1(\omega)$ and c are stateed as in Theorem 4.1. If $E_1^{\mathbb{A}}(S,W_1) = E_1^{\mathbb{A}}(S,W_2)$, $\Theta_0(\infty,W_1) > \frac{(4\nu+1)}{6\nu}$, $\Theta_0(\infty,W_2) > \frac{(4\nu+1)}{6\nu}$ and n is an integer $\geq 8\nu+1$, then $W_1 \equiv W_2$.*

Proof. Since $\Theta_0(\infty,W_1) > \frac{(4\nu+1)}{6\nu}$, $\Theta_0(\infty,W_2) > \frac{(4\nu+1)}{6\nu}$, it follows that

$$\limsup_{r\to\infty, r\notin E}\frac{\overline{N}_0(r,W_1)}{T_0(r,W_1)} < \frac{1}{6\nu},$$
$$\limsup_{r\to\infty, r\notin E}\frac{\overline{N}_0(r,W_1)}{T_0(r,W_1)} < \frac{1}{6\nu}. \tag{4.27}$$

From (4.27), (4.24) and (4.25), since $(n \geq 6nu + 1)$, we can get

$$\limsup_{r\to\infty, r\notin I}\left(\left((2\nu+1)\overline{N}_0(r,F) + \sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{F-c_j}\right) + \overline{N}_0\left(r,\frac{1}{F'}\right) + 2\nu\overline{N}_0^{(2}\left(r,\frac{1}{F}\right)\right)(T_0(r,F))^{-1}\right)$$
$$\leq \limsup_{r\to\infty, r\notin I}\frac{6\nu\overline{N}_0(r,W_1) + (n+8\nu)T_0(r,W_1)}{nT_0(r,W_1)} < 2\nu. \tag{4.28}$$

Similarly, we get

$$\limsup_{r\to\infty, r\notin I}\left(\left((2\nu+)\overline{N}_0(r,G) + \sum_{j=1}^{2\nu}\overline{N}_0^{(2)}\left(r,\frac{1}{G-c_j}\right) + \overline{N}_0\left(r,\frac{1}{G'}\right) + 2\nu\overline{N}_0^{(2}\left(r,\frac{1}{G}\right)\right)(T_0(r,G))^{-1}\right)$$
$$\leq \limsup_{r\to\infty, r\notin I}\frac{6\nu\overline{N}_0(r,W_2) + (n+8\nu)T_0(r,W_2)}{nT_0(r,W_2)} < 2\nu.$$

Thus by Lemma 3.7, we have

$$F \equiv \frac{aG+b}{cG+d}, \tag{4.30}$$

where $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$. By using arguments similar to those in the proof of Theorem 4.1, we can get that $W_1 \equiv W_2$.
This completes the proof of Theorem 4.7. □

From Theorem 4.7, we can get the following corollary as follows:

Corollary 4.8. *There exists one finite set S with $\lambda = (8\nu+1)$, such that any two ν-valued algebroid functions $W_1(z)$ and $W_2(z)$ which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ must be identical if $E_1^{\mathbb{A}}(S,W_1) = E_1^{\mathbb{A}}(S,W_2)$.*

Compliance with ethical standards

Conflict of interest : The authors declare that there are no conflicts of interest regarding the publication of this paper.

Human/animals participants : The author declare that there is no research involving human participants and/or animals in the contained in this paper.

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