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Uniqueness and value sharing of meromorphic functions on annuli

Ashok Rathod[1*] and Praveen Basavanneppa Burji[2]

Abstract
In this paper, we study meromorphic functions that share only one value on annuli and prove the following results.

1. Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 6$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.
2. Let $f(z)$ and $g(z)$ be two non constant entire functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 1$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.

Using the results (1) and (2) we prove, let $f(z)$ and $g(z)$ two non constant meromorphic functions on annuli and For $n \geq 11$, if $f^n f'$ and $g^n g'$ share the same nonzero and finite value a with the same multiplicities on annuli, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where d is an $(n+1)^{th}$ root of unity, c, c_1 and c_2 being constants.

Keywords
Value Distribution Theory; meromorphic functions; annuli.

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[1] *Department of Mathematics, K.L.E Society's G. I. Bagewadi Arts Science and Commerce College, Nipani, Karnataka- 591237, India.*
[2] *Department of Mathematics, S. S. Arts College and T. P. Science Institute, Sankeshwar, Karnataka-591313, India.*
****Corresponding author**: [1] ashokmrmaths@gmail.com; [2]prasnemath@gmail.com
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Contents

1. Introduction and Mani Results

In this paper, a meromorphic function always mean a function which is meromorphic in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Let $f(z)$ and $g(z)$ be non constant meromorphic in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $a \in \overline{\mathbb{C}}$. We say that f and g share the value a CM if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. We shall use standard notations of value distribution theory in annuli, $T_0(R, f)$, $m_0(R, f)$, $N_0(R, f)$, $\overline{N}_0(R, f)$,...(see [6],[7]).

In this paper, we shall show that certain types of differential polynomials on annuli when they share only one value.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non constant meromorphic fuctions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = a^{-2}$.*

Remark 1.2. *The following example shows that $a \neq 0$ is necessary. For $f = e^{e^z}$ and $g = e^z$, we see that $f^n f'$ and $g^n g'$ share 0 CM for any integer n, but f and g do not satisfy the conclusion of Theorem 1.*

In order to prove the above result, we shall first prove the following two theorems.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 6$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two non constant entire fuctions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, $n \geq 1$. If $f^n f' g^n g' = 1$, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

2. Some Basic Theorems and Lemmas

Theorem 2.1. [7] *(Lemma on the Logarithmic Derivative). Let f be a nonconstant meromorphic function in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, and $\alpha \geq 0$. Then*
1. In the case, $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log(RT_0(R,f))\right)$$

for $R \in (1,+\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\alpha-1}dR < +\infty$;
2. In the case, $R_0 < +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for $R \in (1,R_0)$ except for the set $\triangle'_R$ such that $\int_{\triangle'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty$.

Lemma 2.2. *Let f and g be two non constant entire functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. Then for any $1 < R < R_0$, we have*

$$N_0\left(R, \frac{f}{g}\right) - N_0\left(R, \frac{g}{f}\right) = N_0(R,f) + N_0\left(R, \frac{1}{g}\right) - N_0(R,g) - N_0\left(R, \frac{1}{f}\right).$$

In studying on uniqueness theorems of meromorphic functions, the following lemma plays an important role.

Lemma 2.3. *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$ satisfying the following identity*

$$\sum_{j=1}^{n} f_j \equiv 1 \tag{2.1}$$

Then for $1 \leq j \leq n$, we have

$$T_0(R,f) \leq \sum_{k=1}^{n} N_0\left(R, \frac{1}{f_k}\right) + N_0(R,f_j) + N_0(R,D) - \sum_{k=1}^{n} N_0(R,f_k) - N_0\left(R, \frac{1}{D}\right) + S(R,f). \tag{2.2}$$

Where D is the Wronskian determinant $W(f_1, f_2, \ldots, f_n)$, $S(r,f) = o(T_0(R,f))$ and $T_0(R,f) = max_{1 \leq k \leq n}\{T_0(R,f_k)\}$, for every R such that $1 < R < R_0$, $R \notin E$ and E is the set of finite linear measure.

First of all, we prove a lemma which is a essentially generalization of Borel's theorem.

Lemma 2.4. *Let $g_j(z)$ (j=1,2,...,n) be an entire functions and $a_j(z)$ (j=0,1,2,...,n) be a meromorphic functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$, satisfying $T_0(R,aj) = o\left(\sum_{k=1}^{n} T_0(R, e^{gk})\right)$, for every R such that $1 < R < R_0$, $R \notin E$, $(j = 0,1,2,...,n)$. If*

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)} \equiv a_0(z) \tag{2.3}$$

then there exists constant c_j (j=1,2,...,n) at least one of them is not zero such that

$$\sum_{j=1}^{n} c_j a_j(z)e^{g_j(z)} \equiv 0. \tag{2.4}$$

Lemma 2.5. *Let $f(z)$ and $g(z)$ be two non constant entire functions in $\mathbb{A}(R_0)$, where $1 < R_0 \leq +\infty$. If f and g share 1 CM, one of the following three cases holds:*

$$(i)\ T_0(R,f) \leq \overline{N}_0(R,f) + \overline{N}_0^{(2}(R,f) + \overline{N}_0(R,g) + \overline{N}_0^{(2)}(R,g) + \overline{N}_0\left(R,\frac{1}{f}\right) + \overline{N}_0^{(2}\left(R,\frac{1}{f}\right) + \overline{N}_0\left(R,\frac{1}{g}\right) + \overline{N}_0^{(2}\left(R,\frac{1}{g}\right) + S(R,f) + S(R,g)$$

the same inequality holding for $T_0(R,g)$;

(ii) $f \equiv dg$;

(iii) $fg \equiv 1$,

where

$$\overline{N}_0^{(2}(R,1/f) = \overline{N}_0\left(R,\frac{1}{f}\right) - N_0^{1)}\left(R,\frac{1}{f}\right)$$

and $N_0^{1)}\left(R,\frac{1}{f}\right)$ is the counting function of the zeros of f in $\{z : |z| \leq R\}$.

3. Proof of Lemmas

1. Proof of Lemma 2.2: By Jensen's formula in annuli, we have

$$N_0\left(R,\frac{1}{f}\right) - N_0(R,f) = \int_0^{2\pi} \log\frac{1}{|f(Re^{i\theta})|}\frac{d\theta}{2\pi} + \int_0^{2\pi} \log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{\pi}$$

for every R such that $1 < R < R_0$. Consider,

$$\begin{aligned}
&N_0\left(R,\frac{f}{g}\right)-N_0\left(R,\frac{g}{f}\right)\\
&=\int_0^{2\pi}\log\left|\frac{f(Re^{i\theta})}{g(Re^{i\theta})}\right|\frac{d\theta}{2\pi}+\int_0^{2\pi}\log\left|\frac{g(Re^{i\theta})}{f(Re^{i\theta})}\right|\frac{d\theta}{2\pi}\\
&\quad+\int_0^{2\pi}\log\left|\frac{g(e^{i\theta})}{f(e^{i\theta})}\right|\frac{d\theta}{\pi}\\
&=\left\{\int_0^{2\pi}\log\left|\frac{1}{g(Re^{i\theta})}\right|\frac{d\theta}{2\pi}+\int_0^{2\pi}\log\left|g(Re^{i\theta})\right|\frac{d\theta}{2\pi}\right.\\
&\quad\left.-\int_0^{2\pi}\log\left|g(e^{i\theta})\right|\frac{d\theta}{\pi}\right\}\\
&\quad-\left\{\int_0^{2\pi}\log\left|\frac{1}{f(Re^{i\theta})}\right|\frac{d\theta}{2\pi}+\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|\frac{d\theta}{2\pi}\right.\\
&\quad\left.-\int_0^{2\pi}\log\left|f(e^{i\theta})\right|\frac{d\theta}{\pi}\right\}\\
&=N_0(R,f)+N_0\left(R,\frac{1}{g}\right)-N_0(R,g)-N_0\left(R,\frac{1}{f}\right).
\end{aligned}$$

This completes the proof of Lemma 2.2.

2. Proof of Lemma 2.3: Taking the derivative in both sides of identity (2.1), we get

$$\sum_{j=1}^{n} f_j^{(k)} = 0 \quad (k=1,2,...,n-1) \tag{3.1}$$

Since $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent, we see that $D \not\equiv 0$. (2.1) and (3.1) imply

$$D = D_j \ \ (j=1,2,...,n), \tag{3.2}$$

where D_j is algebraic cofactor of f_j in D. Hence

$$f_1 = \frac{\frac{D_1}{f_2 f_3 \cdots f_n}}{\frac{D}{f_1 f_2 \cdots f_n}} = \frac{\Delta_1}{\Delta}, \tag{3.3}$$

where

$$\Delta = \begin{vmatrix} 1 & 1\cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2}\cdots & \frac{f_n'}{f_n} \\ \cdots & \cdots\cdots & \cdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2}\cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix}$$

and Δ is the algebraic cofactor of the elements at the first column and the first row in Δ. From (3.3), we have

$$\begin{aligned}
m_0(R,f_1) &\le m_0(R,\Delta_1)+m_0\left(R,\frac{1}{\Delta}\right)\\
&\le m_0(R,\Delta_1)+m_0(R,\Delta)+N_0(R,\Delta)-N_0\left(R,\frac{1}{\Delta}\right)
\end{aligned} \tag{3.4}$$

since $\Delta = \frac{D}{f_1 f_2 \cdots f_n}$, which leads to

$$\begin{aligned}
N_0(R,\Delta)-N_0\left(R,\frac{1}{\Delta}\right) &= \sum_{k=1}^{n} N_0\left(R,\frac{1}{f_k}\right)-\sum_{k=1}^{n} N_0(R,f_k)\\
&\quad+N_0(R,D)-N_0\left(R,\frac{1}{D}\right)
\end{aligned} \tag{3.5}$$

Note that $m_0\left(R,\frac{f_j^{(k)}}{f_j}\right) = S(R,f_j) = S(R,f)$, (j=1,2,...,n and k=1,2,...,n-1). We have

$$m_0(R,\Delta_1)+m_0(R,\Delta) = S(R,f) \tag{3.6}$$

From (3.4), (3.5) and (3.6), we get

$$\begin{aligned}
T_0(R,f_1) &= m_0(R,f_1)+N_0(R,f_1)\\
&\le \sum_{k=1}^{n} N_0\left(R,\frac{1}{f_k}\right)+N_0(R,f_1)+N_0(R,D)\\
&\quad-\sum_{k=1}^{n} N_0(R,f_k)-N_0\left(R,\frac{1}{D}\right)+S(R,f)
\end{aligned} \tag{3.7}$$

By the same method, we can prove other results similar to (3.7) for f_j, $(2 \le j \le n)$. Hence (2.2) holds.

3. Proof of Lemma 2.4: If $a_0(z) \equiv 0$, Lemma 2.4 is obviously true. In the following, we assume that $a_0(z) \not\equiv 0$. From (2.3), we have $\sum_{j=1}^{n} \frac{a_j(z)}{a_0(z)} e^{g_j(z)} \equiv 1$. Let $G_j(z) = \frac{a_j(z)}{a_0(z)} e^{g_j(z)}$ (j=1,2,...,n). Then $\sum_{j=1}^{n} \equiv 1$.

If $G_1(z), G_2(z), \ldots, G_n(z)$ are linearly independent, then from Lemma 2.2 we have

$$T_0(R,G) \le \sum_{j=1}^{n} N_0\left(R,\frac{1}{G_j}\right)+N_0(R,D)+S(R,f), \tag{3.8}$$

where D is Wronskian $W(G_1, G_2, ..., G_n)$, and $S(r,f) = o(T_0(R,f))$ and $T_0(R,f) = max_{1\le k\le n}\{T_0(R,f_k)\}$, as $1 < R < R_0$, $R \notin E$.
E is the set of finite linear measure.
Note that

$$\begin{aligned}
N_0\left(R,\frac{1}{G_j}\right) &\le N_0\left(R,\frac{1}{a_j}\right)+N_0(R,a_0) \le T_0(R,a_j)\\
&\quad+T_0(R,a_0)\\
&= o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right),\ \ (1<R<R_0, R\notin E).
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
N_0(R,G_j) &\le N_0(R,a_j)+N_0\left(R,\frac{1}{a_0}\right) \le T_0(R,a_j)\\
&\quad+T_0(R,a_0)\\
&= o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right),\ \ (1<R<R_0, R\notin E).
\end{aligned}$$

We have

$$N_0(R,D) \le n\sum_{j=1}^{n} N_0(R,G_j) = o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right),\ (1<R<R_0, R\notin E). \tag{3.10}$$

From (3.8), (3.9) and (3.10), we get

$$T_0(R,G_j) < o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right)+S(R,f),$$
$$(1<R<R_0,\ R\notin E),\ j=1,2,...,n.$$

On the other hand, we have

$$T_0(R,G_j) = T_0(R,e^{g_k})+o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right)\ (R\notin E),$$
$$S(R,f)=o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right)\ (R\notin E).$$

Hence for $j=1,2,...,n$ we have

$$T_0(R,e^{g_k}) = o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right)\ (R\notin E).$$

Therefore

$$\sum_{k=1}^{n} T_0(R,e^{g_k}) = o\left(\sum_{k=1}^{n} T_0(R,e^{gk})\right)\ (R\notin E).$$

This is a contradiction. Hence $G_1(z),G_2(z),\dots,G_n(z)$ are linearly dependent. This completes the proof of Lemma 2.4.

4. Proof of Lemma 2.5: Set

$$\phi = \frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1} \tag{3.11}$$

Since f and g share 1 CM, a simple computation on local expansions shows that $\phi(z_0)=0$ if z_0 is a simple zero of $f-1$ and $g-1$. Next we consider two cases $\phi\not\equiv 0$ and $\phi\equiv 0$.

If $\phi\not\equiv 0$, then

$$\begin{aligned} N_0^{1)}\left(R,\frac{1}{f-1}\right) &= N_0^{1)}\left(R,\frac{1}{g-1}\right)\le N_0\left(R,\frac{1}{\phi}\right)\\ &\le T_0(R,\phi)+O(1)\le N_0(R,\phi)\\ &\quad +S(R,f)+S(R,g) \end{aligned} \tag{3.12}$$

where $N_0^{1)}(R,1/f-1)$ is the counting function of the simple zeros of $f-1$ in $\{z:|z|\le R\}$. Since f and g share 1 CM, any root of $f(z)=1$ can not be a pole of $\phi(z)$. In addition, we can easily see from (3.11) that any simple pole of f and g is not a pole of ϕ. Therefore, by (3.11), the poles of ϕ only occur at zeros of f' and g' and the multiple poles of f and g. If $f'(z_0)=f(z_0)=0$, then z_0 is a multiple zero of f. We denote by $N_0(R,1/f')$ the counting function of those zeros of f' but not that of $f(f-1)$. From (3.11), (3.12) and the above observation that

$$\begin{aligned} N_0^{1)}\left(R,\frac{1}{f-1}\right) &\le \overline{N}_0^{(2}(R,f)+\overline{N}_0^{(2}(R,g)+N_0\left(R,\frac{1}{f'}\right)\\ &\quad+N_0\left(R,\frac{1}{g'}\right)+N_0^{(2}\left(R,\frac{1}{f'}\right)\\ &\quad+N_0^{(2}\left(R,\frac{1}{g'}\right)+S(R,f)+S(R,g) \end{aligned} \tag{3.13}$$

On the other hand, by the second fundamental theorem we have

$$\begin{aligned} T_0(R,f) \quad\le\quad & \overline{N}_0(R,f)+N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-1}\right)\\ &-\overline{N}_0\left(R,\frac{1}{f'}\right)+S(R,f) \end{aligned} \tag{3.14}$$

and by the first fundamental theorem on annuli, we have

$$\begin{aligned} N_0\left(R,\frac{1}{g'}\right)-N_0\left(R,\frac{1}{g}\right) &= N_0\left(R,\frac{g}{g'}\right)\le T_0\left(R,\frac{g}{g'}\right)+O(1)\\ &= \overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,g). \end{aligned}$$

This implies that

$$N_0\left(R,\frac{1}{g'}\right)=\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,g).$$

It is easy to see from the definition of $N_0^{(0)}\left(R,\frac{1}{g'}\right)$ that

$$\overline{N}_0^{(0)}\left(R,\frac{1}{g'}\right)+\overline{N}_0^{(2}(R,\frac{1}{g-1})+\overline{N}_0^{(2}\left(R,\frac{1}{g}\right)-\overline{N}_0^{(2}\left(R,\frac{1}{g}\right)\le N_0\left(R,\frac{1}{g'}\right).$$

The above two inequalities yield

$$\overline{N}_0^{(0)}\left(R,\frac{1}{g'}\right)+\overline{N}_0^{(2}(R,\frac{1}{g-1})\le N_0(R,g)+N_0\left(R,\frac{1}{g}\right)+S(R,g). \tag{3.15}$$

Since $f(z)$ and $g(z)$ share 1 CM, we have

$$\overline{N}_0\left(R,\frac{1}{f-1}\right)\le \overline{N}_0^{1)}\left(R,\frac{1}{f-1}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{g-1}\right). \tag{3.16}$$

Combining (3.13) to (3.16), we obtain (i). If $\phi(z)\equiv 0$, we deduce from (3.11) that

$$f\equiv\frac{Ag+B}{Cg+D}, \tag{3.17}$$

where A, B, C and D are finite complex numbers satisfying $AD-BC\neq 0$.
Then, by the first fundamental theorem,

$$T_0(R,f)=T_0(R,g)+S(R,f). \tag{3.18}$$

Next we consider three respective subcases.

Subcase 1. $AC\neq 0$. Then

$$f-\frac{A}{C}=\frac{B-AD/C}{Cg+D}.$$

By the second fundamental theorem on annuli, we have

$$\begin{aligned} T_0(R,f)\leq&\overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f-(A/C)}\right)\\ &+\overline{N}_0\left(R,\frac{1}{f}\right)+S(R,f)\\ =&\overline{N}_0(R,f)+\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{f}\right)+S(R,f). \end{aligned} \tag{3.19}$$

we get (i).

Subcase 2. $A\neq 0$, $C=0$ Then $f\equiv(Ag+B)/D$. If $B\neq 0$, by the second fundamental theorem on annuli, we have

$$\begin{aligned} T_0(R,f)\leq&\overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-(B/D)}\right)\\ &+S(R,f)\\ =&\overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,f). \end{aligned} \tag{3.20}$$

we get (i). If $B=0$, then $f\equiv Ag/D$. If $A/D=1$, then $f\equiv g$; this is (ii). If $A/D\neq 1$, then by the assumption that f and g share 1 CM, it is easy to see that $f\neq 1$ and $g\neq 1$, which yields $f\neq 1, A/D$. By the second fundamental theorem on annuli, we have

$$T_0(R,f)\leq\overline{N}_0(R,f)+S(R,f),$$

and (i) follows.

Subcase 3. $A=0$, $C\neq 0$ Then $f\equiv B/(Cg+D)$. if $D\neq 0$, by the second fundamental theorem on annuli, we have

$$\begin{aligned} T_0(R,f)\leq&\overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-(B/D)}\right)\\ &+S(R,f)\\ =&\overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{g}\right)+S(R,f). \end{aligned} \tag{3.21}$$

we get (i). If $D=0$, then $f\equiv B/Cg$. If $B/C=1$, then $fg\equiv 1$ and we obtain (iii). If $B/C\neq 1$, by the assumption that f and g share 1 CM, we have $f\neq 1$, B/C. By the second fundamental theorem on annuli, we get

$$T_0(R,f)\leq\overline{N}_0(R,f)+S(R,f).$$

This implies (i). Thus the proof of Lemma 2.5 is complete.

4. Proof of Theorems

1. Proof of Theorem 1.3: We prove the theorem step by step as follows.

Step 1. We prove that

$$f\neq 0,\qquad g\neq 0. \tag{4.1}$$

In fact, suppose that f has a zero z_0 with order m. Then z_0 is a pole of g (with order p, say) by

$$f^nf'g^ng'=1. \tag{4.2}$$

Thus, $nm+m-1=np+p+1$, i.e.,$(m-p)(n+1)=2$. This impossible since $n\geq 6$ and $m,\ p$ are integers.

Step 2. We claim that

$$N_0(R,f)+N_0(R,g)\leq 2m_0\left(R,\frac{1}{fg}\right)+O(1). \tag{4.3}$$

By step 1 and (4.2) we deduce that

$$(n+1)N_0(R,g)+\overline{N}_0(R,g)=N_0\left(R,\frac{1}{f'}\right). \tag{4.4}$$

From Lemma 2.2 we have

$$\begin{aligned} &N_0\left(R,\frac{f}{f'}\right)-N_0\left(R,\frac{f'}{f}\right)\\ =\ &N_0\left(R,f\right)+N_0\left(R,\frac{1}{f'}\right)-N_0\left(R,f'\right)-N_0\left(R,\frac{1}{f}\right)\\ =\ &N_0\left(R,\frac{1}{f'}\right)-\overline{N}_0\left(R,f\right). \end{aligned}$$

By the first fundamental theorem on annuli, the left side is $m_0(R,f'/f)-m_0(R,f/f')+O(1)$, so we have

$$N_0\left(R,\frac{1}{f'}\right)=\overline{N}_0\left(R,f\right)+m_0\left(R,\frac{f}{f'}\right)-m_0\left(R,\frac{f'}{f}\right)+O(1). \tag{4.5}$$

Now we rewrite (4.2) in the form $g'/g=(f'/f)(1/fg)^{n+1}$. Then

$$m_0\left(R,\frac{f}{f'}\right)\geq m_0\left(R,\frac{g'}{g}\right)-(n+1)m_0\left(R,\frac{1}{fg}\right)-O(1).$$

combining this, (4.4) and (4.5), we get

$$\begin{aligned} &(n+1)N_0\left(R,g\right)+\overline{N}_0\left(R,g\right)\\ &\leq\overline{N}_0\left(R,f\right)+m_0\left(R,\frac{f'}{f}\right)-m_0\left(R,\frac{g'}{g}\right)\\ &+(n+1)m_0\left(R,\frac{1}{fg}\right)+O(1). \end{aligned}$$

By symmetry,

$$\begin{aligned}&(n+1)N_0(R,f)+\overline{N}_0(R,f)\\&\le \overline{N}_0(R,g)+m_0\left(R,\frac{g'}{g}\right)-m_0\left(R,\frac{f'}{f}\right)\\&+(n+1)m_0\left(R,\frac{1}{fg}\right)+O(1).\end{aligned}$$

By adding above two inequalities we obtain (4.3).

Step 3. We prove that fg is constant. Let $h=1/fg$. Then h is entire by Step 1, and (4.2) can be written as

$$\left(\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}\right)^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-h^{n+1}.$$

Let

$$\alpha=\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}$$

The above equation becomes

$$\alpha^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-h^{n+1}. \tag{4.6}$$

If $\alpha\equiv 0$, then $h^{n+1}=\frac{1}{2}\left(h'/h\right)^2$. Combining this with Step 1 we obtain $T_0(R,h)=m_0(R,h)=S(R,h)$; thus h is a constant. Next we assume that $\alpha\not\equiv 0$. Differentiating (4.6) yields

$$2\alpha\alpha'=\frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)'-(n+1)h'h^n.$$

From this and (4.6) it follows that

$$h^{n+1}\left((n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)=\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{\alpha'}{\alpha}\frac{h'}{h}\right) \tag{4.7}$$

If $(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\equiv 0$, then there exists a constant c such that $\alpha^2=ch^{n+1}$. This and (4.6) give

$$(c+1)h^{n+1}=\frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c=-1$, then $h'\equiv 0$, and so h is constant. If $c\neq -1$, we have $T_0(R,h)=S(R,h)$, and h is constant. Next we suppose that

$$(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\not\equiv o.$$

Then, by (4.7) and the fact that h is entire,

$$\begin{aligned}(n+1)T_0(R,h)=&(n+1)m_0(R,h)\\ \le& m_0\left(R,h^{n+1}\left((n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)\right)\\ &+m_0\left(R,\frac{1}{(n+1)h'/h-2\alpha'/\alpha}\right)+O(1)\\ \le& m_0\left(R,\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\right)\right)\\ &+T_0\left(R,(n+1)\frac{h'}{h}-2\frac{\alpha'}{\alpha}\right)\\ \le&\overline{N}_0(R,f)+\overline{N}_0(R,g)+\overline{N}_0\left(R,\frac{1}{\alpha}\right)\\ &+S(R,h)+S(R,\alpha).\end{aligned}$$

Now by (4.6) and (4.3) we have

$$T_0(R,\alpha)\le\frac{1}{2}(n+3)T_0(R,h)+S(R,h),$$

and

$$N_0(R,f)+N_0(R,g)\le 2m_0(R,h)+O(1).$$

Combining the above three inequalities we obtain

$$\frac{1}{2}(n-5)T_0(R,h)\le S(R,h).$$

Thus h must be a constant.

Step 4. We prove our conclusion. By Step 3, h is constant. Then, by (4.2),

$$\frac{g'}{g}=c,\quad c=ih^{(n+1)/2.}$$

Thus

$$g(z)=c_1e^{cz},\quad f=c_2e^{-cz}$$

where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2=-1$ by (4.2). This completes the proof of the theorem.

2. Proof of Theorem 1.4: From

$$f^nf'g^ng'=1$$

and the assumption that f and g are entire we immediately see that f and g have no zeros. Thus there exists two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$f(z)=e^{\alpha(z)},\quad g(z)=e^{\beta(z)}.$$

Inserting these in the above equality, we get

$$\alpha'\beta'e^{(n+1)(\alpha+\beta)}\equiv 1.$$

Thus α' and β' have no zeros and we may set

$$\alpha'=e^{\delta(z)},\quad \beta'=e^{\gamma(z)}.$$

Differentiating this gives

$$(n+1)(e^{\delta}+e^{\gamma})+\delta'+\gamma'\equiv 0.$$

By Lemma 2.4, $\delta=\gamma+(2m+1)\pi i$ for some integer m. Inserting this in the above equality we deduce that $\delta'\equiv\gamma'\equiv 0$, and so δ and γ are constants, i.e., α' and β' are constants. From this we can easily obtain the desired result.

3. Proof of Theorem 2.1: Let $F=f^{n+1}/a(n+1)$ and $G=g^{n+1}/a(n+1)$. Then condition that f^nf' and g^ng' share the value a CM implies that F' and G' share the value 1 CM. Obviously,

$$\begin{aligned}N_0(R,F')=&(n+1)N_0(R,f)+\overline{N}_0(R,f),\\ N_0(R,G')=&(n+1)N_0(R,g)+\overline{N}_0(R,g),\end{aligned} \tag{4.8}$$

$$\begin{aligned}\overline{N}_0(R,F')=&\overline{N}_0^{(2}(R,F')=\overline{N}_0(R,f)\\ \le&\frac{1}{n+2}T_0(R,F')+O(1),\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&\overline{N}_0\left(R,\frac{1}{F'}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{F'}\right)\\
&=2\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f'}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{f'}\right)\\
&\leq 2\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f'}\right)\\
&\leq 2T_0(R,f)+\overline{N}_0\left(R,\frac{1}{f'}\right)+O(1).
\end{aligned}
\tag{4.10}
$$

Since

$$
\begin{aligned}
&n\,m_0(R,f)\\
&=m_0\left(R,a\frac{F'}{f'}\right)\leq m_0\left(R,F'\right)+m_0\left(R,\frac{1}{f'}\right)+O(1)\\
&=m_0\left(R,F'\right)+T_0(R,f)-N_0\left(R,\frac{1}{f'}\right)+O(1)\\
&\leq m_0\left(R,F'\right)+T_0(R,f)+\overline{N}_0(R,f)-N_0\left(R,\frac{1}{f'}\right)\\
&+m_0\left(R,\frac{f'}{f}\right)+O(1)\\
&\leq m_0\left(R,F'\right)+T_0(R,f)+\overline{N}_0(R,f)-N_0\left(R,\frac{1}{f'}\right)\\
&+m_0\left(R,\frac{F'}{F}\right)+O(1),
\end{aligned}
$$

it follows from this,(4.8), and Theorem 4 that

$$
(n-1)T_0(R,f)\leq T_0(R,F')-N_0(R,f)-N_0\left(R,\frac{1}{f'}\right)+S(R,F').
$$

This and Theorem 4 imply that

$$
\begin{aligned}
&2T_0(R,f)+N_0\left(R,\frac{1}{f'}\right)\\
&=\frac{2}{n-1}\left\{(n-1)T_0(R,f)+N_0\left(R,\frac{1}{f'}\right)\right\}+\frac{n-3}{n-1}N_0\left(R,\frac{1}{f'}\right)\\
&\leq\frac{2}{n-1}\left\{T_0(R,F')+N_0\left(R,f\right)\right\}+\frac{n-3}{n-1}\left\{T_0(R,f)+\overline{N}_0\left(R,f\right)\right\}\\
&+m_0\left(R,\frac{f'}{f}\right)+O(1)\\
&\leq\left(\frac{2}{n-1}+\frac{n-3}{(n-1)^2}\right)T_0(R,F')+\left(\frac{n-5}{n-1}+\frac{n-3}{(n-1)^2}\right)\\
&\quad N_0\left(R,f\right)+S(R,F').
\end{aligned}
$$

Combining this (4.9), and (4.10), we obtain

$$
\begin{aligned}
\overline{N}_0\left(R,\frac{1}{F'}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{F'}\right)\leq&\frac{4n^2-6n-2}{(n-1)^2(n+2)}T_0(R,F')\\
&+S(R,F').
\end{aligned}
\tag{4.11}
$$

We similarly derive for G' that

$$
\overline{N}_0(R,G')=\overline{N}_0^{(2}(R,G')=\overline{N}_0(R,g)\leq\frac{1}{n+2}T_0(R,G')+S(R,G'),
\tag{4.12}
$$

$$
\overline{N}_0\left(R,\frac{1}{G'}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{G'}\right)\leq\frac{4n^2-6n-2}{(n-1)^2(n+2)}T_0(R,G')+S(R,G').
\tag{4.13}
$$

Without loss of generality, we suppose that there exists a set $I\subset[0,\infty)$ such that $T_0(R,G')\leq T_0(R,F')$. Next we apply Lemma 2.5 to F' and G', it follows that there are three cases to be considered.

Case (i).

$$
\begin{aligned}
T_0(R,F')\leq&\overline{N}_0(R,F')+\overline{N}_0^{(2}(R,F')+\overline{N}_0(R,G')\\
&+\overline{N}_0^{(2}(R,G')+\overline{N}_0\left(R,\frac{1}{F'}\right)\\
&+\overline{N}_0^{(2}\left(R,\frac{1}{F'}\right)+\overline{N}_0\left(R,\frac{1}{G'}\right)\\
&+\overline{N}_0^{(2}\left(R,\frac{1}{G'}\right)+S(R,F')+S(R,G').
\end{aligned}
$$

Setting (4.9), (4.11), (4.12), and (4.13) into the above inequality and keeping in mind that $T_0(R,G')\leq T_0(R,F')$, we get

$$
\frac{n^3-12n^2+17n+2}{(n+1)^2(n+2)}T_0(R,F')\leq S(R,F').
\tag{4.14}
$$

We denote by $p(n)$ the numerator of the coefficient on the left hand side above. Then $p'(n)=3n^2-24n+17>0$ for $n\leq 8$. Note that $p(11)=68$; thus $p(n)$ is positive for $n\leq 11$. It follows from (4.14) that F' must be rational function. But then, by the above derivatives, $S(R,F')=O(1)$. Using (4.14) again, F' must be a constant, which is impossible.

Case (ii). $F'=G'$. Then we deduce that $f^{n+1}=g^{n+1}+c$ ($c\in\mathbb{C}$). Let $f=hg$, and we have

$$
(h^{n+1}-1)g^{n+1}=c.
\tag{4.15}
$$

If $h^{n+1}\equiv 1$, then h is $(n+1)^{th}$ unit root and we obtain the desired result. If $h^{n+1}\not\equiv 1$, then by (4.15),

$$
g^{n+1}=\frac{c}{h^{n+1}-1.}
$$

Thus h is not constant. We write this in the form

$$
g^{n+1}=\frac{c}{(h-u_1)\ldots(h-u_{n+1})},
$$

where $u_1,\ldots,u_{n+1}$ are different $(n+1)^{th}$ roots of unity. Thus h has at least $n+1(\geq 14)$ multiple values. However, from Nevanlinna's second fundamental theorem on annuli, we know that h has at most 4 multiple values, a contradiction.

Case (iii). $F'G'\equiv 1$, i.e., $a^{-2}f^nf'g^ng'\equiv 1$. Let $\widehat{f}=a^{-1/(n+1)f}$ and $\widehat{g}=a^{-1/(n+1)g}$. Then $\widehat{f}^n\,f'\widehat{g}^n\,g'=1$. The conclusion follows follows from Theorem 2.

References

[1] T. B. Cao, Z. S. Deng. On the uniqueness of meromorphic functions that share three or two finite sets on annuli, *Proceeding Mathematical Sciences*. 122(2012), 203-220.

[2] T. B. Cao, H. X. Yi, Uniqueness theorems of meromorphic functions share sets IM on Annili, *Acta Math.Sinica (Chin.Ser.)*, 54(2011), 623-632.

[3] Y. X. Chen, Z. J. Wu. Exceptional values of meromorphic functions and of their derivatives on annuli, *Ann. Polon.Math.* 105(2012), 154-165.

[4] H. Y. Xu, and Z. X. Xuan. The uniqueness of analytic functions on annuli sharing some values, *Abstract and Applied analysis*. 2012, Article ID 896596, 13 pages,2012.

[5] M. L. Fang. A note on a result of Singh and Kulkarni, *Internat. J Math Sci.* 23(4)(2000), 285-288.

[6] A. Ya. Khrystiyanyn, A. A. Kondratyuk. On the Nevanlinna Theory for meromorphic functions on annuli. I, *Mathematychin Studii.* 23(1)(2005), 19-30.

[7] A. Ya. Khrystiyanyn, A. A. Kondratyuk. On the Nevanlinna Theory for meromorphic functions on annuli. II, *Mathematychin Studii.* 24(2)(2005), 57-68.

[8] T.B.Cao, H. X. Yi, H. Y. Xu. On the multiple values and uniqueness of meromorphic functions on annuli, *Compute. Math. Appl.* 58(7)(2009), 1457-1465.

[9] Yu-Zan He and Xiu-Zhi Xiao. *Algebroid functions and Ordinarry Difference Equations*, Beijing; Science Press, (1988).

[10] A. Fernandez. On the value distribution of meromorphic function in the punctured plane, *Mathematychin Studii.* 34(2)(2010), 136-144.

[11] Meili Liang. On the value distribution of algebroid functions, *Sop transactions on applied mathematics*. 1(1)(2014).

[12] Hayman W. K. *Meromorphic functions,* Oxford: Oxford University Press; (1964).

[13] R. S. Dyavanal, Ashok Rathod. Uniqueness theorems for meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences.* 12(1)(2016), 1-10.

[14] R. S. Dyavanal, Ashok Rathod. Multiple values and uniqueness of meromorphic functions on annuli, *International Journal Pure and Applied Mathematics.* 107(4)(2016), 983-995.

[15] R. S. Dyavanal, Ashok Rathod. On the value distribution of meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences*. 12(2)(2016), 203-217.

[16] R. S. Dyavanal, Ashok Rathod. Some generalisation of Nevanlinna's five-value theorem for algebroid functions on annuli, *Asian Journal of Mathematics and Computer Research,* 20(2)(2017), 85-95.

[17] R. S. Dyavanal, Ashok Rathod. Nevanlinna's five-value theorem for derivatives of meromorphic functions sharing values on annuli, *Asian Journal of Mathematics and Computer Research*. 20(1)(2017), 13-21.

[18] R. S. Dyavanal, Ashok Rathod. Unicity theorem for algebroid functions related to multiple values and derivatives on annuli, *International Journal of Fuzzy Mathematical Archive.* 13(1)(2017), 25-39.

[19] R. S. Dyavanal, Ashok Rathod. General Milloux inequality for algebroid functions on annuli, *International Journal of Mathematics and applications.* 5(3)(2017), 319-326.

[20] Ashok Rathod. The multiple values of algebroid functions and uniqueness, *Asian Journal of Mathematics and Computer Research.* 14(2)(2016), 150-157.

[21] Ashok Rathod. The uniqueness of meromorphic functions concerning five or more values and small functions on annuli, *Asian Journal of Current Research.* 1(3)(2016), 101-107.

[22] Ashok Rathod. Uniqueness of algebroid functions dealing with multiple values on annuli, *Journal of Basic and Applied Research International*. 19(3)(2016), 157-167.

[23] Ashok Rathod. On the deficiencies of algebroid functions and their differential polynomials, *Journal of Basic and Applied Research International.* 1(1)(2016), 1-11.

[24] Ashok Rathod. The multiple values of algebroid functions and uniqueness on annuli, *Konuralp Journal of Mathematics*. 5(2)(2017), 216-227.

[25] Ashok Rathod. Several uniqueness theorems for algebroid functions, *J. Anal.* 25(2)(2017), 203-213.

[26] Ashok Rathod. Nevanlinna's five–value theorem for algebroid functions, *Ufa Mathematical Journal.* 10(2)(2018), 127-132.

[27] Ashok Rathod. Nevanlinna's five–value theorem for derivatives of algebroid functions on annuli, *Tamkang Journal of Mathematics.* 49(2)(2018), 129-142.

[28] S. S. Bhoosnurmath, R. S. Dyavanal, Mahesh Barki, Ashok Rathod. Value distribution for n'th difference operator of meromorphic functions with maximal deficiency sum, *J. Anal.* (2018). https://doi.org/10.1007/s41478–018–0130–5.

[29] Ashok Rathod. Characteristic function and deficiency of algebroid functions on annuli, *Ufa Mathematical Journal*. 11(1)(2019), 121-132.

[30] Ashok Rathod. Value distribution of a algebroid function and its linear combination of derivatives on annuli, *Electronic Journal of Mathematical analysis and applications*. 8(1)(2020), 129-142.

[31] Ashok Rathod. Uniqueness theorems for functions on annuli, *Ufa Mathematical Journal*. 12(1)(2020), 115-121.

[32] Ashok Rathod. Uniqueness of algebroid functions in connection to Nevanlinna's five-value theorem, *Jnanabha*. 50(2)(2020), 160-116.

[33] A. A. Kondratyuk, I. Laine. Meromorphic functions in multiply connected domains,Ilpo laine(ed), *Fourier Series Methods in Complex Analysis, Report Series,* 10, University of Joensuu, 1-111(2006).

[34] Ashok Rathod, Naveenkumar S. H. On the uniqueness and value distribution of entire functions with their deriva-

tives, *International J. Math. Combin.* 2(2020), 33-42.

[35] Ashok Rathod. On the order of a meromorphic matrix valued function on annuli, *International J. Math. Combin.* 4(2020), 36-46.

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