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(a,d)-distance antimagic labeling for some regular graphs

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Abstract
Let $G=(V,E)$ be a graph of order n. Let $f:V(G)\longrightarrow\{1,2,3,...,n\}$ be a bijective function. For every vertex v in G, we define its weight $w(v)$ as the sum $\sum_{u\in N(v)} f(u)$, where $N(v)$ is the open neighborhood of v. If the set of all vertex weights forms an arithmetic progression $\{a,a+d,a+2d,\dots,a+(n-1)d\}$, then f is called (a,d)-distance antimagic labeling and the graph G is called (a,d)-distance antimagic graph. In this paper we prove circulant graph Circ$(2n,\{1,n\})$ for odd n and nK_{2n+1} for odd n are (a,d)-distance antimagic graphs. We also give some necessary conditions for mK_n to be (a,d)-distance antimagic graph for $d=2k$, where k is some positive integer.

Keywords
Distance magic graph, distance antimagic graph, (a,d)-distance antimagic graph, circulant graph.

AMS Subject Classification
05C78.

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Contents

1. Introduction and Definitions

By a graph G we mean simple, finite, and connected graph with V as vertex set and E as edge set respectively. For the graph theoretical terminology and notations, we refer to Gross and Yellen[2]. Throughout the paper, $N(v)$ denotes an open neighborhood of a vertex v.

Definition 1.1. *Let G be a graph of order n. A bijective function $f:V(G)\longrightarrow\{1,2,3,...,n\}$ is said to be distance magic labeling if there is a constant k such that $\sum_{u\in N(v)} f(u)=k$ for every vertex v in V. The constant k is called magic constant of the labeling f.*

The concept of distance magic labeling was introduced by many researchers under the different names [7], [8], [9]. For more details about distance magic labeling reader may refer to the Gallian survey [1]. A distance antimagic labeling is a natural variant of distance magic labeling. Arumugam and Kamatchi[3] observed that set of all vertex weight $\left\{\frac{n(n+1)}{2}-k-i|1\le i\le n\right\}$ of G^c is in arithmetic progression if G is distance magic graph of order n with magic constant k. So, they motivated to introduce (a,d)-distance antimagic labeling.

Definition 1.2. *Let G be a graph of order n. A bijective function $f:V(G)\longrightarrow\{1,2,3,...,n\}$ is said to be an (a,d)-distance antimagic labeling if the set of all vertex weights is $\{a,a+d,a+2d,\dots,a+(n-1)d\}$, where a and d are fixed positive integers and G is called (a,d)-distance antimagic graph.*

Arumugam and Kamatchi[3] investigate (a,d)-distance antimagic labeling for C_n and $C_n\times K_2$. In [4], R. Simanjuntak and K. Wijaya proved: Wheel graph W_n is (a, d)-distance antimagic if and only if $3\le n\le 5$; the fan graph $F_n=P_n\times K_1$ is (a, d)-distance antimagic if and only if $n=2$ or $n=4$; the friendship graph f_n is (a, d)-distance antimagic if and only if $n=1$ or $n=2$. In [5], M. Nalliah proved that graph mC_n is (a, d)-distance antimagic if and only if mn is odd and $d=1$; the path P_n of order upto 15 except $n=3,4$ and 5 is (a, d)-distance antimagic. Patel and Vasava[6] proved that circulant graph Circ$(2n,1,n)$ is $(2n+2,1)$-distance antimagic for all even n; mK_{2n} is $(n(2mn-2m+1),1)$-distance antimagic for all m and n . They also proved that $2K_{2n+1}$, the Helm graph H_n, the book graph B_n and the graph $K_n \odot K_1$ are not (a, d)-distance antimagic. In this paper we show that circulant graph Circ$(2n,\{1,n\})$ is $(2n+2,1)$-distance antimagic for all odd n and nK_{2n+1} is $\left(\frac{4n^3+n+1}{2},1\right)$-distance antimagic for all odd n. We give some necessary conditions for mK_n to be a (a,d)-distance antimagic for even values of d.

2. Main Results

Definition 2.1. *Let $s_1,s_2,\dots,s_m,n$ be positive integers such that $1 \le s_1 < s_2 < \dots < s_m < n$. Then the circulant graph Circ$(n,s_1,s_2,\dots,s_m)$ is the graph with vertex set $\{v_1,v_2,\dots,v_n\}$ and whose edges are of the type $v_iv_{i+s_j}$ for $i=1,2,\dots,n$, $j=1,2,\dots,m$; where $i+s_j$ is taken modulo n.*

Patel and Vasava[6] proved that the circulant graph Circ($2n,\{1,n\})$ is $(2n+2,1)$-distance antimagic for all even n. Here, we prove it is also $(2n+2,1)$-distance antimagic for all odd n.

Theorem 2.2. *The circulant graph Circ$(2n,\{1,n\})$ is $(2n+2,1)$-distance antimagic for all odd n.*

Proof. Let G denote the graph Circ$(2n,\{1,n\})$ where n is odd. Let $u_1,u_2,\dots,u_{2n}$ be the vertices of the graph G. We define $f : V(G) \longrightarrow \{1,2,3,\dots,2n\}$ as follows.

$$f(u_i)=\begin{cases} \frac{i+1}{2}, & i \text{ is odd }, 1 \le i \le n \\ \frac{5n+1}{2}-\frac{i+1}{2}, & i \text{ is odd }, n+2 \le i \le 2n-3 \\ 2n, & i=2n-1 \\ \frac{n+1}{2}+\frac{i}{2}, & i \text{ is even }, 2 \le i \le n-3 \\ 2n-\frac{i}{2}, & i \text{ is even }, n-1 \le i \le 2n \end{cases}$$

Vertex weights are given by

$$w(u_i)=\begin{cases} 3n+1, & i=1 \\ \frac{5n+3}{2}+\frac{i-1}{2}, & i=3,5,7,\dots,n-4 \\ \frac{7n+1}{2}, & i=n-2 \\ 4n, & i=n \\ 4n-\frac{i-1}{2}, & i=n+2,n+4,n+6,\dots,2n-3 \\ \frac{7n+3}{2}, & i=2n-1 \\ 2n+1+\frac{i}{2}, & i=2,4,6,\dots,n-3 \\ 3n, & i=n-1 \\ \frac{5n+1}{2}, & i=n+1 \\ \frac{9n+1}{2}-\frac{i}{2}, & i=n+3,n+5,\dots,2n-4 \\ 4n+1, & i=2n-2 \\ \frac{5n+3}{2}, & i=2n \end{cases}$$

Here, all the weights are in arithmetic progression with $d=1$ in following sequence, $w(u_2),w(u_4),w(u_6),\dots,w(u_{n-3})$, $w(u_{n+1}),w(u_{2n}),w(u_3),w(u_5),w(u_7),\dots,w(u_{n-4}),w(u_{n-1})$, $w(u_1)$, $w(u_{2n-3}),w(u_{2n-1}),\dots,w(u_{n+2}),w(u_{n-2}),w(u_{2n-1})$, $w(u_{2n-4}),w(u_{2n-2}),\dots,w(u_{n+3}),w(u_n),w(u_{2n-4})$.

Hence the graph Circ$(2n,\{1,n\})$ is $(2n+2,1)$-distance antimagic for all odd n. □

Illustration 1. $(28,1)$-distance antimagic labeling of Circ$(26,\{1,13\})$ is shown in Figure 1. In this figure the vertex label indicated in the usual font and its weight in the bold font.

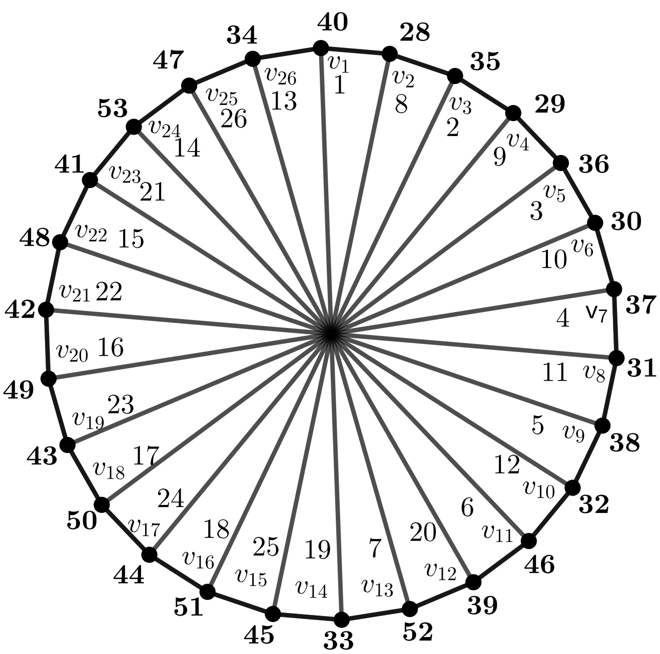

Figure 1. $(28,1)$-distance antimagic labeling of Circ$(26,\{1,13\})$.

Theorem 2.3. *The graph nK_{2n+1} is $\left(\frac{4n^3+n+1}{2},1\right)$-distance antimagic for all odd n.*

Proof. Let $G=nK_{2n+1}$, where n is odd. Let $u_1^j,u_2^j,u_3^j,\dots,u_{2n+1}^j$ be the vertices of j^{th} copy of K_{2n+1} for $j=1,2,\dots,n$. First we define two functions $\delta_l(m)$ and $\Delta_l(m)$ as follows.

$$\delta_l(m)=\begin{cases} 0, & l<m \\ 1, & l \ge m \end{cases}$$

$$\Delta_l(m)=\begin{cases} 1, & l \le m \\ 0, & l>m \end{cases}$$

Now we define $f : V(G) \longrightarrow \{1,2,3,\dots,2n^2+n\}$ as follows.

$f(u_1^j)=j, 1 \le j \le n$

$f(u_i^j)=j+(n+1)(i-1), 2 \le i \le n, j=1$

$f(u_i^j)=j+(n+1)(i-1), 2 \le i \le n, 2 \le j \le n-1$ where $i \ne n+2-j$, $j=2,3,\dots,n-1$

$f(u_i^j) = n(n-1)+1-n(n-i), i = n+2-j, j = 2,3,\ldots,n-1$

$$f(u_i^j) = \begin{cases} n^2 + \frac{n+3}{2} - \frac{j+1}{2}, & j \text{ is odd} \\ n(n+1)+1-\frac{j}{2}, & j \text{ is even} \end{cases} \quad i = n+1$$

$$f(u_i^j) = \begin{cases} n(n+2)+1-\frac{j+1}{2}, & j \text{ is odd} \\ n(n+1)+\frac{n+1}{2}-\frac{j}{2}, & j \text{ is even} \end{cases} \quad i = n+2$$

$f(u_i^j) = (n+1)^2 + (i-n-3)(n+1),\ n+3 \le i \le 2n+3,\ j = 1$

$f(u_i^j) = (n+1)^2 + (n+1-j) + \Delta_{n+3+j}(i) + \delta_{n+2+j}(i)(i-n-3)(n+1) + \Delta_{n+3+j}(i)(i-n-4)(n+1)$, $n+4 \le i \le 2n+3$, where $i \ne n+2+j$, for $j = 2,3,\ldots,n-1$

$f(u_i^j) = (n+1)^2 + n + n(j-2), i = n+2+j, 2 \le j \le n-1$

$f(u_i^j) = (n+1)^2 + 1 + (i-n-3)(n+1), n+3 \le i \le 2n+3, j = n$

In the graph G,

$$\sum_{i=1}^{2n^2+n} f(u_i^j) = \frac{4n^4+4n^3+3n^2+n}{2}$$

So, weight of u_i^j is given by $w_i^j = \frac{4n^3+4n^2+3n+1}{2} - f(u_i^j)$, which is in arithmetic progression with $a = \frac{4n^3+n+1}{2}$ and $d = 1$.

Hence, nK_{2n+1} is $\left(\frac{4n^3+n+1}{2}, 1\right)$-distance antimagic for all odd n. □

Theorem 2.4. *For even value of d, the graph mK_n is not (a,d)-distance antimagic if m is odd.*

Proof. Let $G = mK_n$ and $d = 2k$ for some positive integer k. Suppose m is any odd number.

Since d is even, all the weights in graph G are either even or odd. Therefore, the labels of the vertices in each copy of K_n are either all even or all odd. For, if there is a copy of K_n having vertices with even labels and vertices with odd labels together, then in this copy there are some vertices with even weights and some vertices with odd weights which is not possible.

In the graph G, there are $\lceil \frac{mn}{2} \rceil$ vertices with odd labels and $\lfloor \frac{mn}{2} \rfloor$ vertices with even labels. After labeling mn vertices by keeping parity of labels of vertices in each copy, there must be $\frac{m-1}{2}$ copies having vertices with even labels and $\frac{m-1}{2}$ copies having vertices with odd labels as m is odd number. Hence the remaining one copy of must contains vertices having even labels as well as vertices having odd labels, which is not possible. Hence, G is not (a,d)-distance antimagic for $d = 2k$ if m is odd. □

In [3], Arumugam and Kamatchi observed that every r regular (a,d)-distance antimagic graph with cardinality v satisfies the equation

$$2a + (v-1)d = r(v+1) \tag{2.1}$$

Theorem 2.5. *For even value of d, the graph mK_n is not (a,d)-distance antimagic if n is even.*

Proof. Let $G = mK_n$ and $d = 2k$ for some positive integer k. If m is odd then by Theorem 2.4, G is not (a,d)-distance antimagic.

So, we assume m is even. Let n be any even number. For G, $|V(G)| = mn$ and $r = n-1$. So, Equation 2.1 becomes

$$\begin{aligned} & 2a + (mn-1)(2k) = (n-1)(mn+1) \\ \Rightarrow\ & 2a + 2kmn - 2k = mn^2 + n - mn - 1 \\ \Rightarrow\ & 2a = mn^2 - (2k+1)mn + n + 2k - 1 \\ \Rightarrow\ & a = \frac{mn^2 - (2k+1)mn}{2} + \frac{n+2k-1}{2} \end{aligned}$$

Since n is even, a can not be an integer. It follows that G is not (a,d)-distance antimagic. □

Theorem 2.6. *For even value of d, the graph mK_n is not (a,d)-distance antimagic if a is odd.*

Proof. Let a be any odd integer. If possible suppose $G = mK_n$ is (a,d)-distance antimagic for $d = 2k$ where k is positive integer. So by Theorems 2.4 and 2.5, m must be even and n must be odd.

Since a is odd integer and $d = 2k$, all the weights in the graph G are odd. As per the reason given in Theorem 2.4, there does not exist any copy having vertices with even labels as well as vertices with odd labels. Since m is even, there are $\frac{m}{2}$ copies with even labeled vertices and $\frac{m}{2}$ copies with odd labeled vertices. But then vertex weights of copies of K_n with even labeled vertices must be even which is a contradiction to the fact that all the vertex weights in G are odd. Therefore, the graph mK_n is not (a,d)-distance antimagic for even d if a is odd. □

Note: From Theorems 2.4, 2.5 and 2.6, it follows that if the graph mK_n is (a,d)-distance antimagic for even value of d then m is even, n is odd and a is even integers.

Theorem 2.7. *If the graph mK_n is (a,d)-distance antimagic for $d = 2k$ where k is some positive integer, then*
(i) $n \not\equiv 1 (\bmod 4)$ and $d \not\equiv 2 (\bmod 4)$
(ii) $n \not\equiv 3 (\bmod 4)$ and $d \not\equiv 0 (\bmod 4)$

Proof. Let graph $G = mK_n$ be (a,d)-distance antimagic for $d = 2k$. Then by Theorems 2.4 and 2.6, m and a are even numbers and

$$a = \frac{mn^2 - (2k+1)mn}{2} + \frac{n+2k-1}{2}$$

(i) If possible suppose $n \equiv 1 (\bmod 4)$ and $d \equiv 2 (\bmod 4)$. Let $n = 4q+1$ and $d = 2k = 4p+2$, where p and q are integers. So, we have

$$a = \frac{m(4q+1)^2-(4p+3)m(4q+1)}{2}+\frac{4q+1+4p+1}{2}$$

Since $(4q+1)^2 = 4t+1$ for some integer t and $(4q+1)(4p+3) = 4s+1$ for some integer s, a can be written as

$$a = \frac{m(4t+1)-m(4s+3)}{2}+\frac{4(p+q)+2}{2}$$
$$\Rightarrow a = \frac{m(4t-4s-2)}{2}+2(p+q)+1$$
$$\Rightarrow a = m(2t-2s-1)+2(p+q)+1$$

This shows that, if m is even integer then a is odd integer which is not possible. So, our assumptions is wrong. Therefore, $n \not\equiv 1 (\text{mod } 4)$ and $d \not\equiv 2 (\text{mod } 4)$.
(ii) If possible suppose $n \equiv 3 (\text{mod } 4)$ and $d \equiv 0 (\text{mod } 4)$. Let $n = 4q+3$ and $d = 2k = 4p$, where p and q are integers. So, we have

$$a = \frac{m(4q+3)^2-(4p+1)m(4q+3)}{2}+\frac{4q+3+4p-1}{2}$$

Since $(4q+3)^2 = 4t+1$ for some integer t and $(4q+1)(4p+3) = 4s+3$ for some integer s, a can be written as

$$a = \frac{m(4t+1)-m(4s+3)}{2}+\frac{4(p+q)+2}{2}$$
$$\Rightarrow a = \frac{m(4t-4s-2)}{2}+2(p+q)+1$$
$$\Rightarrow a = m(2t-2s-1)+2(p+q)+1$$

This shows that, if m is even integer then a is odd integer which is not possible. So, our assumptions is wrong. Therefore, $n \not\equiv 3($ mod $4)$ and $d \not\equiv 0 (\text{mod } 4)$.

□

Theorem 2.8. *For $d = 4$, the graph mK_n is not (a,d)-distance antimagic if $m \equiv 2 (\text{mod } 4)$.*

Proof. Let $G = mK_n$ and $m \equiv 2 (\text{mod } 4)$. If possible suppose G is a (a,d)-distance antimagic graph for $d = 4$. Then by Theorems 2.5 and 2.6, n is odd and a is even.
Since $d = 4$ and a is even, for all the weights w_i in the graph G, either $w_i \equiv 0 (\text{mod } 4)$ or $w_i \equiv 2 (\text{mod } 4)$. By theorem 2.7, $n \equiv 1 (\text{mod } 4)$. Hence, $|V(G)| = mn = 4s+2$ for some integer s. It is clear that there are $2s+1$ vertices with even labels and $2s+1$ vertices with odd labels. As per the reason given in Theorem 2.4, there are exactly $\frac{m}{2}$ copies of K_n having even labels of vertices and exactly $\frac{m}{2}$ copies of K_n having odd labels of vertices. Now we focus on $\frac{m}{2}$ copies of K_n having only even labels of vertices. Since $m \equiv 2 (\text{mod } 4)$, $\frac{m}{2}$ is an odd number. Let $\frac{m}{2} = 2t+1$ for some integer t. Observe that among $2t+1$ copies there is no copy of K_n having vertices with labels congruent to 0 mod 4 as well as no vertices with labels congruent to 2 mod 4. For, if there exists such a copy then it contains both the types of weights which is not possible. Now the number of vertices of $2t+1$ copies of K_n are $(2t+1)n$. After labeling $(2t+1)n$ vertices by keeping parity of labels of vertices in a copy, there must be t copies having vertices with labels congruent to 0 mod 4 and t copies having vertices with labels congruent to 2 mod 4. Remaining one copy of K_n must contain both the types of vertices and hence it contains both the types of weights w_i that is, $w_i \equiv 0 (\text{mod } 4)$ and $w_i \equiv 2 (\text{mod } 4)$ which gives a contradiction. So, our assumption is wrong. Hence, G is not (a,d)-distance antimagic if $m \equiv 2 (\text{mod } 4)$. □

3. Conclusion

Here, we have shown that circulant graph $\text{Circ}(2n, \{1,n\})$ is (a,d)-distance antimagic for all odd n and nK_{2n+1} is (a,d)-distance antimagic for all odd n. we gave some necessary conditions for mK_n to be a (a,d)-distance antimagic for even values of d. One can investigate sufficient conditions for mK_n to be a (a,d)-distance antimagic for even values of d.

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References

[1] J. A. Gallian, *A dynamic Survey of Graph labeling*, The Electronic Journal of Combinatorics, (2019), #DS6.

[2] J. Gross and J. Yellen, *Graph theory and its applications*, CRC Press, (2005).

[3] S. Arumugam and N. Kamatchi, *On (a,d)-distance antimagic graphs*, Australasian Journal of Combinatorics, 54(2012), 279-287.

[4] R. Simanjuntak and K. Wijaya, *On distance antimagic graphs*, Dec. 2013, arXiv:1312.7405v1.

[5] N. Nalliah, *Antimagic labelings of graphs and digraphs*, Ph.D. Thesis Kalasalingam University, Faculty of Science and Humanities, 2013.

[6] S. Patel and J. Vasava, *Some results on (a,d)-distance antimagic labeling*, Proyecciones Journal of Mathematics, 39(2)(2020), 361-381

[7] V.Vilfred, *Σ-labeled graphs and circulant graphs*, Ph.D. Thesis, University of Kerala, Trivandrum, India(1994)

[8] K. Sugeng, D. Fronĉek, J. Ryan, M. Miller and J. Walker, *On distance magic labeling of graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, 71(2009), 39-48.

[9] M. Miller, C. Rodger and R. Simanjuntak, *Distance magic labeling of graphs*, Australasian Journal of Combinatorics, 28(2003), 305-315.

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