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Sharp sufficient conditions for oscillation of second-order general noncanonical difference equations

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Abstract

We derive new oscillation conditions for the second-order noncanonical difference equation with deviating argument of the form

$$\Delta(r(\xi)(\Delta x(\xi))^{\gamma})+q(\xi)x^{\delta}(\xi+\kappa)=0; \quad \xi\geq\xi_0,$$

where γ and δ are quotients of odd positive integers and κ is an integer. Examples are provided to illustrate our established results.

Keywords

Oscillation, nonoscillation, second-order, canonical, noncanonical, delay, advanced, difference equations.

AMS Subject Classification

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Contents

1. Introduction

Recently, there is a great attention to research the oscillation and asymptotic behaviour of difference equations, one may refer to [2, 5, 11, 16–18]. One may refer [1, 8, 9, 14] to the general theory of difference equations. Solutions to second-order difference equations may have a variety of dynamical behaviours. Here, we consider only sufficient conditions which ensures that the equation (1.1) is oscillatory.

We investigate the second-order noncanonical difference equations given by

$$\Delta(r(\xi)(\Delta x(\xi))^{\gamma})+q(\xi)x^{\delta}(\xi+\kappa)=0; \quad \xi\geq\xi_0, \quad (1.1)$$

where Δ is called the forward difference operator and it is given by $\Delta x(\xi)=x(\xi+1)-x(\xi)$.
Throughout paper, the constraints given below are presumed to be hold:

(C_1) $\{q(\xi)\}_{\xi=\xi_0}^{\infty}$ is a nonnegative real sequence which is not identically zero eventually;

(C_2) $\{r(\xi)\}_{\xi=\xi_0}^{\infty}$ is a positive sequence of real numbers;

(C_3) κ is an integer;

(C_4) $\gamma,\ \delta\in\{\ \frac{a}{b}$: a and b are odd positive integers $\}$.

"Let ξ_0 be a nonnegative integer that is fixed. A real sequence $\{x(\xi)\}$ defined for $\xi\geq min\{\xi_0,\xi_0+\sigma\}$ and satisfies the (1.1) for $\xi\geq\xi_0$ is called a solution of (1.1). An oscillatory solution $\{x(\xi)\}$ of (1.1) is a solution of (1.1) if $K>0$ is a positive integer, then there exists an integer $\xi\geq K$ with the property that $x(\xi)x(\xi+1)\leq 0$. Otherwise $\{x(\xi)\}$ is called a nonoscillatory solution. If every solution of the equation (1.1) are oscillatory, then the (1.1) said to be oscillatory" [12] .

We say that (1.1) is canonical if

$$R(\xi) := \sum_{s=\xi_1}^{\xi-1} \frac{1}{r^{\frac{1}{\gamma}}(s)} \to \infty \quad as \quad \xi \to \infty$$

and it is said to be noncanonical in the opposite case. We define the sequence $\{\phi(\xi)\}$ for noncanonical difference equation by

$$\phi(\xi) := \sum_{s=\xi}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)}. \tag{1.2}$$

Li and Cheng [10], analyze the second-order difference equation

$$\Delta(p(\xi)(\Delta(y(\xi)))^{\kappa}) + q(\xi+1)f(y(\xi+1)) = 0 \quad \xi = 0,1,2,\dots. \tag{1.3}$$

where κ is ratio of odd positive integers and derived conditions which are sufficient for the oscillation of the above equation.

Zhang and Li [19], investigated the functional difference equation of second-order with advanced argument

$$\Delta(a(\xi)\Delta x(\xi)) + p(\xi)(x(g(\xi))) = 0 \tag{1.4}$$

and developed sufficient conditions for oscillations (1.4) under the conditions

$$\sum_{s=n_0}^{\infty} p(s) = \infty.$$

Peng et al. [15] investigated oscillatory properties of the second-order difference equation

$$\Delta(r(\xi-1)(\Delta(y(\xi-1)))^{\kappa}) + q(\xi)f(y(\xi)) = R(\xi) \tag{1.5}$$

and given sufficient conditions for oscillation and asymptotic properties of solutions of (1.5).

Dinakar et al. [6] established sufficient conditions under which all solutions of the second-order half-linear advanced difference equation

$$\Delta(r(\xi)(\Delta x(\xi))^{\gamma}) + q(\xi)x^{\gamma}(\kappa(\xi)) = 0, \quad \xi \geq \xi_0 \tag{1.6}$$

are either oscillatory or tending to zero.

Chandrasekaran et al. [4] used a new improved method and established oscillation criteria for the solutions to the second-order advanced difference equation

$$\Delta(r(\xi)\Delta x(\xi)) + q(\xi)x(\kappa(\xi)) = 0, \quad \xi \geq \xi_0 \tag{1.7}$$

using the difference equation

$$\Delta(r(\xi)\Delta x(\xi)) + q(\xi)x(\xi+1) = 0, \quad \xi \geq \xi_0. \tag{1.8}$$

We [13] also derived new oscillatory conditions for the second-order noncanonical delay and advanced difference equations of the form

$$\Delta(r(\xi)\Delta x(\xi)) + q(\xi)x(\xi+\kappa) = 0, \quad \xi \geq \xi_0 \tag{1.9}$$

by creating monotonical properties of nonoscillatory solutions.

Grace et al. [7] established new oscillation conditions for all solutions of the following nonlinear second-order neutral difference equations.

$$\Delta(a(\xi)(\Delta u(\xi))^{\gamma}) + b(\xi)y^{\gamma}(\xi-\tau+1) + c(\xi)y^{\mu}(\xi+\kappa+1) = 0 \tag{1.10}$$

and

$$\Delta(a(\xi)(\Delta u(\xi))^{\gamma}) = b(\xi)y^{\gamma}(\xi-\tau+1) + c(\xi)y^{\mu}(\xi+\kappa+1), \tag{1.11}$$

where $u(\xi) = y(\xi) + q_1(\xi)y^{\beta}(\xi-k) - q_2(\xi)y^{\delta}(\xi-k)$.

In this paper, we establish sufficient conditions for oscillation of all solutions to the equation (1.1).

In the following section, we presume that if a functional inequality is written without a domain of validity, it holds eventually, for the sake of convenience.

2. Preliminaries

For the set of eventually positive solutions of (1.1), the following structure can be seen:

Lemma 2.1. *An eventually positive solution $\{x(\xi)\}$ of (1.1) fulfils one of the following criteria.*

(A_1) : $r(\xi)(\Delta x(\xi))^{\gamma} > 0, \Delta(r(\xi)(\Delta x(\xi))^{\gamma}) < 0.$

(A_2) : $r(\xi)(\Delta x(\xi))^{\gamma} < 0, \Delta(r(\xi)(\Delta x(\xi))^{\gamma}) < 0.$

The following result demonstrates that the case (A_*) is the most significant.

Lemma 2.2. *If*

$$\sum_{s=\xi_1}^{\infty} q(s) = \infty, \tag{2.1}$$

then every eventually positive solution $\{x(\xi)\}$ of (1.1) satisfies (A_*) of Lemma 2.1 and moreover $\{\frac{x(\xi)}{\phi(\xi)}\}$ is an increasing sequence.

Proof. We may suppose, on the contrary, that $\{x(\xi)\}$ is an eventually positive solution of (1.1) which satisfies the condition (A_1) for $\xi \geq \xi_1 \geq \xi_0$. Summing the (1.1) from ξ_1 to ∞, we get

$$r(\xi_1)(\Delta x(\xi_1))^{\gamma} \geq \sum_{s=\xi_1}^{\infty} q(s)x^{\delta}(s+\kappa).$$

Since $\{x(\xi)\}$ is a positive increasing sequence, then there is a constant $\nu > 0$ with $x(\xi) \geq \nu$ and $x^{\delta}(\xi+\kappa) \geq \nu^{\delta}$ eventually. Therefore, we obtain

$$r(\xi_1)(\Delta x(\xi_1))^{\gamma} \geq \nu^{\delta} \sum_{s=\xi_1}^{\infty} q(s),$$

which contradicts (2.1) and this implies that $\{x(\xi)\}$ satisfies (A_*). By the decreasing nature of $\{r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi)\}$, we get

$$-x(\xi) \leq r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi)\sum_{s=\xi}^{\infty}\frac{1}{r}^{\frac{1}{\gamma}}(s) = r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi)\phi(\xi). \quad (2.2)$$

Compute

$$\Delta\left(\frac{x(\xi)}{\phi(\xi)}\right) = \frac{r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi)\phi(\xi)+x(\xi)}{r^{\frac{1}{\gamma}}(\xi)\phi(\xi)\phi(\xi+1)} \geq 0$$

and hence proved.

□

Lemma 2.3. *If $\{x(\xi)\}$ is an eventually positive solution of (1.1) and*

$$\sum_{u=\xi_1}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(u)}\left(\sum_{s=\xi_1}^{u-1}q(s)\right)^{\frac{1}{\gamma}} = \infty, \quad (2.3)$$

holds, then

$$\lim_{\xi\to\infty} x(\xi) = 0. \quad (2.4)$$

Proof. We can easily analyze that (2.3) gives (2.1) and by Lemma 2.2, we ensure that the eventually positive solution $\{x(\xi)\}$ satisfies (A_*) of Lemma 2.1. Hence $\{x(\xi)\}$ is decreasing sequence and we conclude that there exists a finite number μ with $\lim_{\xi\to\infty} x(\xi) = \mu \geq 0$. We claim that $\mu = 0$. If not, then $\lim_{\xi\to\infty} x(\xi) \geq \mu > 0$ and $(x(\xi+\kappa))^{\delta} \geq \mu^{\delta} > 0$ eventually. A summation of (1.1) from ξ_1 to $\xi - 1$ gives

$$-r(\xi)(\Delta x(\xi))^{\gamma} \geq \mu^{\delta}\sum_{s=\xi_1}^{\xi-1}q(s).$$

Sum the above inequality from ξ_1 to ∞, we attain

$$x(\xi_1) \geq \mu^{\frac{\delta}{\gamma}}\sum_{u=\xi_1}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(u)}\left(\sum_{s=\xi_1}^{u-1}q(s)\right)^{\frac{1}{\gamma}},$$

which contradicts (2.3) and implies that $x(\xi) \to 0$ as $\xi \to \infty$. □

Lemma 2.4. *If $\{x(\xi)\}$ is an eventually positive solution of (1.1) with (2.3) and*

$$\sum_{s=\xi_1}^{\infty}q(s)\phi^{\delta}(s+\kappa) = \infty, \quad (2.5)$$

holds, then $\lim_{\xi\to\infty}\frac{x(\xi)}{\phi(\xi)} = \infty$.

Proof. We can easily analyze that (2.3) gives (2.1). Then by Lemma 2.2, we see that the eventually positive solution $\{x(\xi)\}$ satisfies (A_*) of Lemma 2.1. Applying Discrete L'Hospital's rule, we get

$$\lim_{\xi\to\infty}\frac{x(\xi)}{\phi(\xi)} = \lim_{\xi\to\infty}(-r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi))$$

and so it is sufficient to show that $\lim_{\xi\to\infty}(-r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi))=\infty$. By contrary, assume that the positive increasing sequence $(-r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi))$ has a finite limit. Hence there exists a constant $M > 0$ with the property that

$$-r^{\frac{1}{\gamma}}(\xi)\Delta x(\xi) \leq M < \infty \quad (2.6)$$

Summing (1.1) from ξ_1 to $\xi - 1$, we have

$$\begin{aligned} M^{\gamma} &\geq -r(\xi)(\Delta x(\xi))^{\gamma} \geq \sum_{s=\xi_1}^{\xi-1}q(s)x^{\delta}(s+\kappa) \\ &\geq \frac{x^{\delta}(\xi_1+\kappa)}{\phi^{\delta}(\xi_1+\kappa)}\sum_{s=\xi_1}^{n-1}q(s)\phi^{\delta}(s+\kappa) \end{aligned} \quad (2.7)$$

where, we used that $\{\frac{x(\xi)}{\phi(\xi)}\}$ is a positive increasing sequence. The above inequality (2.7) contradicts (2.5) which implies that $\frac{x(\xi)}{\phi(\xi)} \to \infty$ as $\xi \to \infty$. □

We now present oscillation conditions for (1.1).

3. Delay Equation

Theorem 3.1. *Let $\kappa \leq -1$. Assume that $\gamma > \delta$, and (2.3) holds. If*

$$\limsup_{\xi\to\infty}\phi^{\gamma}(\xi)\sum_{s=\xi_1}^{\xi-1}q(s) > 0, \quad (3.1)$$

then (1.1) oscillates.

Proof. Let us suppose, on the contrary, that $\{x(\xi)\}$ is a nonoscillatory solution of (1.1). Without lacking generality, we assume that $x(\xi) > 0$ for $\xi \geq \xi_1$. The Condition (2.3) gives (2.1) which assures that $\{x(\xi)\}$ is from class (A_*). Suppose

$$\limsup_{\xi\to\infty}\phi^{\gamma}(\xi)\sum_{s=\xi_1}^{\xi-1}q(s) > 0.$$

Then there is a positive constant $K > 0$ with the property that

$$\phi^{\gamma}(\xi)\sum_{s=\xi_1}^{\xi-1}q(s) > \frac{1}{K^{\gamma}}. \quad (3.2)$$

Summing (1.1) from ξ_1 to $\xi - 1$, we obtain

$$-\Delta x(\xi) \geq \frac{1}{r^{\frac{1}{\gamma}}(\xi)}\left(\sum_{s=\xi_1}^{\xi-1}q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}}.$$

Sum the previous inequality from ξ to ∞, we attain

$$\begin{aligned} x(\xi) &\geq \sum_{u=\xi}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}}(u)\left(\sum_{s=\xi_1}^{u-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &\geq \left(\sum_{u=\xi}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}}(u)\right)\left(\sum_{s=\xi_1}^{\xi-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &= \phi(\xi)\left(\sum_{s=\xi_1}^{\xi-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi)x^{\frac{\delta}{\gamma}}(\xi+\kappa-1)\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi)x^{\frac{\delta}{\gamma}}(\xi+\kappa)\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}}. \end{aligned} \tag{3.3}$$

Since $\{x(\xi)\}$ is positive decreasing sequence and $\lim_{\xi\to\infty} x(\xi) = 0$, $x(\xi) \leq K^{\frac{\gamma}{\delta-\gamma}}$ and $x^{\frac{\delta}{\gamma}-1}(\xi+\kappa) \geq K$ for $\xi \geq \xi_1$. By employing the above estimate in the inequality (3.3), we get

$$\begin{aligned} x(\xi) &\geq \phi(\xi)x(\xi+\kappa)K\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi)x(\xi)K\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \end{aligned}$$

and

$$\frac{1}{K^{\gamma}} \geq \phi^{\gamma}(\xi)\sum_{s=\xi_1}^{\xi-1} q(s),$$

which contradicts (3.2) and thus we have (1.1) is oscillatory. □

Theorem 3.2. *Let $\kappa \leq -1$. Assume that $\gamma < \delta$, (2.3) and (2.5) holds. If*

$$\limsup_{\xi\to\infty} \phi^{\gamma}(\xi)\phi^{\delta-\gamma}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > 0, \tag{3.4}$$

then (1.1) oscillates.

Proof. Assume on the contrary that $\{x(\xi)\}$ is a nonoscillatory solution of (1.1) . We may suppose,without lacking generality, that $\{x(\xi)\}$ is an eventually positive solution of (1.1) with $x(\xi) > 0$ for $\xi \geq \xi_1$. By condition (2.1), we see that $\{x(\xi)\}$ is from class (A_*). By

$$\lim_{\xi\to\infty} \phi^{\gamma}(\xi)\phi^{\delta-\gamma}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > 0,$$

we arrive at the conclusion that there exists a constant $L > 0$ with

$$\phi^{\gamma}(\xi)\phi^{\delta-\gamma}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > \frac{1}{L^{\gamma}}. \tag{3.5}$$

Since $\{\frac{x(\xi)}{\phi(\xi)}\}$ is positive increasing sequence and $\lim_{\xi\to\infty}\frac{x(\xi)}{\phi(\xi)} = \infty$, we have

$$\frac{x(\xi)}{\phi(\xi)} \geq L^{\frac{\gamma}{\delta-\gamma}}$$

and

$$x^{\frac{\delta}{\gamma}-1}(\xi+\kappa) \geq \phi^{\frac{\delta}{\gamma}-1}(\xi+\kappa)L \quad for \quad \xi \geq \xi_1.$$

Summing (1.1) from ξ_1 to $\xi - 1$ as followed in the proof of Theorem 3.1, one get (3.3). Using the above estimate in the (3.3), we get

$$\begin{aligned} x(\xi) &\geq \phi(\xi)x(\xi+\kappa)\phi^{\frac{\delta}{\gamma}-1}(\xi+\kappa)L\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi)x(\xi)\phi^{\frac{\delta}{\gamma}-1}(\xi+\kappa)L\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \end{aligned}$$

and

$$\frac{1}{L^{\gamma}} \geq \phi^{\gamma}(\xi)\phi^{\delta-\gamma}(\xi+\kappa)\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right),$$

which contradicts (3.5). Thus, (1.1) is oscillatory. □

4. Advanced Equation

Theorem 4.1. *Let $\kappa \geq 1$. Assume that (2.3) holds and $\gamma > \delta$. If*

$$\limsup_{\xi\to\infty} \phi^{\gamma}(\xi)\sum_{s=\xi_1}^{\xi-1} q(s) > 0, \tag{4.1}$$

then (1.1) oscillates.

Proof. We may suppose, on the contrary and without lacking generality, that $\{x(\xi)\}$ is an eventually positive solution of (1.1). We conclude that there is an integer $\xi_1 \geq \xi_0$ such that $x(\xi) > 0$ for $\xi \geq \xi_1$. By condition (2.3), we have (2.1) and we assures that $\{x(\xi)\}$ is belongs to the class (A_*). If

$$\limsup_{\xi\to\infty} \phi^{\gamma}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > 0,$$

then there is a positive constant $k > 0$ with

$$\phi^{\gamma}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > \frac{1}{K^{\gamma}}. \tag{4.2}$$

Summing (1.1) from ξ_1 to $\xi+\kappa-1$, we obtain, we have

$$-\Delta x(\xi+\kappa) \geq \frac{1}{r^{\frac{1}{\gamma}}(\xi+\kappa)}\left(\sum_{s=\xi_1}^{\xi+\kappa-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}}.$$

Sum the above inequality from ξ to ∞,one get

$$\begin{aligned} x(\xi+\kappa) &\geq \sum_{u=\xi}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(u+\kappa)}\left(\sum_{s=\xi_1}^{\xi+\kappa-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &\geq \left(\sum_{u=\xi}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}}(u+\kappa)\right)\left(\sum_{s=\xi_1}^{\xi+\kappa-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &= \phi(\xi+\kappa)\left(\sum_{s=\xi_1}^{\xi+\kappa-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi+\kappa)\left(\sum_{s=\xi_1}^{\xi-1} q(s)x^{\delta}(s+\kappa)\right)^{\frac{1}{\gamma}} \\ &\geq \phi(\xi+\kappa)x^{\frac{\delta}{\gamma}}(s+\kappa)\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}}. \qquad (4.3) \end{aligned}$$

Since $\{x(\xi)\}$ is positive decreasing sequence and $\lim_{\xi\to\infty} x(\xi) = 0$,

$$x(\xi) \leq K^{\frac{\gamma}{\delta-\gamma}}$$

and

$$x^{\frac{\delta}{\gamma}-1}(\xi+\kappa) \geq K, \quad for \quad \xi \geq \xi_1$$

Employing the previous estimate in the inequality (4.3), we obtain

$$x(\xi+\kappa) \geq \phi(\xi+\kappa)x(\xi+\kappa)K\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}}$$

and

$$\frac{1}{k^{\gamma}} \geq \phi^{\gamma}(\xi+\kappa)\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}},$$

which contradicts (4.2) and thus, we have (1.1) is oscillatory. □

Theorem 4.2. *Let $\kappa \geq 1$. Assume that $\gamma < \delta$, (2.3) and (2.5) holds. If*

$$\limsup_{\xi\to\infty} \phi^{\delta}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > 0, \qquad (4.4)$$

then (1.1) oscillates.

Proof. We may suppose, on the contrary and without lacking generality, that $\{x(\xi)\}$ is an eventually positive solution of (1.1). We conclude that there is an integer $\xi_1 \geq \xi_0$ such that $x(\xi) > 0$ for $\xi \geq \xi_1$. By the condition (2.3), we have (2.1) which assures that $\{x(\xi)\}$ is from class (A_*). Since

$$\lim_{\xi\to\infty} \phi^{\delta}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > 0,$$

then we find a constant $L > 0$ with

$$\phi^{\delta}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s) > \frac{1}{L^{\gamma}}. \qquad (4.5)$$

Since $\{\frac{x(\xi)}{\phi(\xi)}\}$ is positive increasing sequence and $\lim_{\xi\to\infty}\{\frac{x(\xi)}{\phi(\xi)}\} = \infty$,

$$\frac{x(\xi)}{\phi(\xi)} \geq L^{\frac{\gamma}{\delta-\gamma}}$$

and

$$x^{\frac{\delta}{\gamma}-1}(\xi+\kappa) \geq \phi^{\frac{\delta}{\gamma}-1}(\xi+\kappa)L \quad for \quad \xi \geq \xi_1.$$

As followed in the proof of Theorem 4.1, sum (1.1) from ξ_1 to $\xi+\kappa-1$, one have (4.3). By applying the above estimate in (4.3), we derive

$$\begin{aligned} x(\xi+\kappa) &\geq \phi(\xi+\kappa)x(\xi+\kappa)\phi^{\frac{\delta}{\gamma}-1}(\xi+\kappa)L\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \\ &\geq \phi^{\frac{\delta}{\gamma}}(\xi+\kappa)x(\xi+\kappa)L\left(\sum_{s=\xi_1}^{\xi-1} q(s)\right)^{\frac{1}{\gamma}} \end{aligned}$$

and

$$\frac{1}{L^{\gamma}} \geq \phi^{\delta}(\xi+\kappa)\sum_{s=\xi_1}^{\xi-1} q(s)$$

which contradicts (4.5) and thus, we have (1.1) is oscillatory. □

5. Examples

Example 5.1. *Let us investigate the oscillation of the following second-order difference equation*

$$\Delta\left(2^{\frac{\xi}{3}}(\Delta x(\xi))^{\frac{1}{3}}\right) + 2^{\xi}x^{\frac{1}{5}}(\xi-1) = 0; \quad \xi = 1,2,\ldots. \quad (5.1)$$

We have $r(\xi) = 2^{\frac{\xi}{3}}$, $q(\xi) = 2^{\xi}$, $\kappa = -1$, $\gamma = \frac{1}{3}$, and $\delta = \frac{1}{5}$.
Also, $\phi(\xi) = \frac{1}{2^{\xi-1}}$. We can easily show that

$$\sum_{u=1}^{\infty}\frac{1}{r^{\frac{1}{\gamma}(u)}}\left(\sum_{s=1}^{u-1} q(s)\right)^{\frac{1}{\gamma}} = \sum_{u=1}^{\infty}\left(1-\frac{1}{2^{u-1}}\right)^{3} = \infty.$$

and

$$\phi^{\gamma}(\xi)\sum_{s=1}^{\xi-1} q(s) = 2^{\frac{2\xi+1}{3}} + \frac{16}{2^{\xi}}.$$

So, by Theorem 3.1, (5.1) oscillates.

Example 5.2. *Let us investigate the second-order difference equation*

$$\Delta\left(2^{\frac{\xi}{5}}(\Delta x(\xi))^{\frac{1}{5}}\right)+2^{\xi}x^{\frac{1}{3}}(\xi+1)=0; \quad \xi=0,1,2\dots. \quad (5.2)$$

Here, $\kappa=1$, $r(\xi)=2^{\frac{\xi}{5}}$, $q(\xi)=2^{\xi}$, $\gamma=\frac{1}{5}$ *and* $\delta=\frac{1}{3}$. *Also, we have* $\phi(\xi)=\frac{1}{2^{\xi-1}}$. *We can easily show that*

$$\sum_{u=0}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(\xi)}\left(\sum_{s=0}^{u-1}q(s)\right)^{\frac{1}{\gamma}} = \sum_{u=0}^{\infty}\left(1-\frac{1}{2^{\xi-1}}\right)^{5} = \infty$$

and

$$\limsup_{\xi\to\infty}\phi^{\frac{1}{3}}(\xi+1)\sum_{s=0}^{\xi-1}2^{s} > 0.$$

So, all the constraints of the Theorem 4.2 are verified and thus, the equation (5.2) is oscillatory.

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