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Modified Newton method for solution of nonlinear equations

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Abstract
In this paper, we present a new modified Newton method for solving non-linear equations. This new method do not require the use of the second-order derivative. It is shown that the new method is cubically convergent. Furthermore, an unified method has been designed by generalizing the modified Newton method. Some numerical experiments are conducted to establish our theoretical findings.

Keywords
Newton method; Haar wavelet; Iterative Method; Third-order convergence; Non-linear equations; Root-finding.

AMS Subject Classification
65H05.

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Contents

1. Introduction

In numerical analysis, finding a solution of non-linear equation is one of the most attractive problems. In this paper, we emphasize on finding an iterative method to find a simple root α of a non-linear equation $f(x) = 0$, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Here, we are less concerned about multiple roots. Newton's method [1],[6] is the well known algorithm to solve nonlinear equation. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \cdots \tag{1.1}$$

and it converges quadratically.

Earlier, [3]-[5] and [8]-[13] derived third-order convergence methods based on integral interpretation of Newton's method. Where, Newton's method was modified using different quadrature formulas for the indefinite integral arising from Newton's theorem [2]

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{1.2}$$

Weerakoon et al.[3] have approximated the integral part of (1.2) by trapezoidal rules and derived a variant of Newton's method. It is, further, shown that this method converges cubically. Subsequently, Frontini et al. [4] have proposed a third order convergent method by approximating the integral by the midpoint rule. In [10], Homeier has developed a cubically convergent iteration scheme by considering Newton's theorem for the inverse function. Further, in [11], [12] modified Newton methods are derived for multivariate case. Kou et al. in [13] have applied a new interval of integration on Newton's theorem and arrived a third-order convergent iterative scheme.

Recently, Islam et al.[7] have applied Haar wavelet function to derived quadrature rules for indefinite integration. In this paper, we modified Newton's theorem by using the quadrature rule proposed by Islam in [7] and a modified Newton method is proposed. The proposed mthod is then generalized to obtain a family of modified Newton method.It is shown that all these new methods are cubically convergent. Further, the new methods did not evaluate second derivative of f. The efficiency of the new method is demonstrated by numerical examples.

This paper is organized as follows. In Section 2, we develop a modified Newton method and the generalized unified method. Section 3, we establish convergence analysis for the new methods. finally in section 4, various numerical experiments are conducted to affirm our theoretical finding.

2. Main results: Modified Newton method

To derive the new method, we consider Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\tau)d\tau \tag{2.1}$$

We use the Haar wavelet function to approximate the integral term of (2.1) as

$$\int_{x_n}^{x} f'(\tau)d\tau = \frac{(x-x_n)}{2M}\sum_{k=1}^{2M} f'\big(x_n + \frac{(x-x_n)(k-0.5)}{2M}\big). \tag{2.2}$$

where $M = 2^{J_1}$ and J_1 is the maximum level of resolution of Haar wavelets, see [7].
Substitute (2.2) in (2.1) to obtain

$$f(x) = f(x_n) + \frac{(x-x_n)}{2M}\sum_{k=1}^{2M} f'\big(x_n + \frac{(x-x_n)(k-0.5)}{2M}\big). \tag{2.3}$$

Now, looking for $f(x) = 0$ we arrive at

$$x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'\big(x_n + \frac{(x-x_n)(k-0.5)}{2M}\big)} \tag{2.4}$$

Further, substitute $x_{n+1} = x$ in (1.1) and replace $x - x_n$ in (2.4) we obtain the new method as

$$x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'\big(x_n - \frac{f(x_n)}{f'(x_n)}\frac{(k-0.5)}{2M}\big)} \tag{2.5}$$

In particular choosing $M = 1$ and using $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ we have

$$x_{n+1} = x_n - \frac{2f(x_n)}{\left[f'(\frac{3x_n+y_n}{4}) + f'(\frac{x_n+3y_n}{4})\right]}, \quad n = 0,1,2,\cdots \tag{2.6}$$

The scheme(2.6) is named as **MNMH** method.
Unified Method Let us construct the unified scheme as follows

$$x_{k+1} = x_k - \frac{f(x_k)}{\sum_{i=1}^{2} h_i f'\left(\alpha_i x_k + \beta_i y_k\right)}, \tag{2.7}$$

where $\alpha_i + \beta_i = 1 \forall i = 1,2$, $\sum_i^2 h_i = 1$ and

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)} \tag{2.8}$$

3. Convergence Analysis

Theorem 3.1. *Let the function $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ has a simple root $\alpha \in \mathbb{D}$, where $\mathbb{D}$ is an open interval. Assume $f(x)$ has first, second and third derivatives in the interval $\mathbb{D}$. If the initial guess x_0 is closed to α, then the method defined by (2.5) converges cubically to α.*

Proof. Let α is the simple root of $f(x)$ and $x_n = \alpha + e_n$. A use of Taylor expansion with $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)\big(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\big), \tag{3.1}$$

where $C_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}$. Again

$$f'(x_n) = f'(\alpha)\big(1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)\big). \tag{3.2}$$

Further, dividing (3.1) by (3.2) yields

$$\frac{f(x_n)}{f'(x_n)} = \big(e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)\big). \tag{3.3}$$

Now,

$$x_n - M_k \frac{f(x_n)}{f'(x_n)} = x_n - M_k\big(e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)\big), \tag{3.4}$$

where $M_k = \frac{(k-0.5)}{2M}$.
Equation (3.4) rewrite as

$$\begin{aligned} x_n - M_k \frac{f(x_n)}{f'(x_n)} &= x_n - M_k\big(e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4)\big),\\ &= x_n - M_k e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4),\\ &= x_n - (1 - 1 + M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4),\\ &= x_n - e_n + (1 - M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4),\\ &= \alpha + (1 - M_k)e_n + M_k C_2 e_n^2 - 2(C_2^2 - C_3)M_k e_n^3 + O(e_n^4). \end{aligned} \tag{3.5}$$

From (3.5) we can easily find that

$$f'(x_n - M_k \frac{f(x_n)}{f'(x_n)}) = f'(\alpha)\big(1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)\big). \tag{3.6}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{N} f'(x_n - M_k \frac{f(x_n)}{f'(x_n)}) &= \sum_{k=1}^{N} f'(\alpha)\big(1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)\big),\\ &= f'(\alpha)\sum_{k=1}^{N}\big(1 + 2C_2(1 - M_k)e_n + (2C_2^2 M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)\big),\\ &= f'(\alpha)\big(\sum_{k=1}^{N} 1 + 2C_2 e_n \sum_{k=1}^{N}(1 - M_k) + 2C_2^2 e_n^2 \sum_{k=1}^{N} M_k + 3C_3 e_n^2 \sum_{k=1}^{N}(1 - M_k)^2 + O(e_n^3)\big),\\ &= f'(\alpha)\big(N + 2C_2 N e_n - 2C_2 e_n N/2 + 2C_2^2 e_n^2 N/2 + 3C_3 e_n^2 N\\ &+ 3C_3 e_n^2(\frac{N}{3} - \frac{1}{12N}) - 6C_3 e_n^2 N/2 + O(e_n^3)\big)\\ &= f'(\alpha)\big(N + 2C_2 N e_n + \big(NC_2^2 + NC_3 - \frac{C_3}{4N}\big)e_n^2 + O(e_n^3)\big) \end{aligned} \tag{3.7}$$

substitute (3.7) and (3.1) in(2.5) we obtain

$$\begin{aligned} x_{n+1} &= x_n - \frac{\big(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\big)}{\big(N + 2C_2 N e_n + \big(NC_2^2 + NC_3 - \frac{C_3}{4N}\big)e_n^2 + O(e_n^3)\big)},\\ &= x_n - \big(e_n + \big(-C_2^2 + \frac{C_3}{4N^2}\big)e_n^3 + O(e_n^4)\big). \end{aligned} \tag{3.8}$$

Substract α from both side of (3.8), then we have

$$e_{n+1} = \big(C_2^2 - \frac{C_3}{4N^2}\big)e_n^3 + O(e_n^4) \tag{3.9}$$

this completes the rest of the proof. □

Convergence Analysis for Unified Method

Theorem 3.2. *Let the function $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ has a simple root $\alpha \in \mathbb{D}$, where $\mathbb{D}$ is an open interval. Assume $f(x)$ is a smooth function on $\mathbb{D}$. If the initial guess x_0 is closed to α, then the method defined by (2.7) converges cubically to α.*

Proof. Now

$$\begin{aligned} f'\Big(\alpha_i x_k + \beta_i y_k\Big) &= f'\Big(x_k + (y_k - x_k) + \alpha_i x_k + (\beta_i - 1)y_k\Big) = f'\Big(x_k + (y_k - x_k) + (-\alpha_i)y_k + \alpha_i x_k\Big) \\ &= f'\Big(x_k + (y_k - x_k) + (-\alpha_i)(y_k + x_k)\Big) = f'(x_k + \beta_i(y_k - x_k)) \\ &= f'(x_k) + f''(x_k)\beta_i(y_k - x_k) + \frac{f'''(x_k)}{2!}(\beta_i(y_k - x_k))^2 + \frac{f^{iv}(x_k)}{3!}(\beta_i(y_k - x_k))^3 + O(\|e_n\|^4) \end{aligned} \tag{3.10}$$

From (2.8) and (3.3), we find that

$$y_k - x_k = \big(-e_k + C_2 e_k^2 - 2(C_2^2 - C_3)e_k^3 + O(e_k^4)\big). \tag{3.11}$$

Substituting (3.11) in (3.10) yields

$$\begin{aligned} f'\Big(\alpha_i x_k + \beta_i y_k\Big) &= f'(x_k) - \beta_i f''(x_k)e_k + \Big(\beta_i C_2 f''(x_k) + \frac{\beta_i^2}{2} f'''()x_k\Big)e_k^2 \\ &\quad - \Big(\beta_i f'(x_k) + 3\beta_i^2 C_3 f''(x_k) + \frac{\beta_i^3}{6} f^{iv}(x_k)\Big)e_k^3 + O(e_k^4). \end{aligned} \tag{3.12}$$

Substract α form both hand side of (2.7) to obtain

$$e_{k+1} = e_k - \frac{f(x_k)}{h_1 f'\Big(\alpha_1 x_k + \beta_1 y_k\Big) + h_2 f'\Big(\alpha_2 x_k + \beta_2 y_k\Big)}. \tag{3.13}$$

Further,

$$\Big(h_1 f'(\alpha_1 x_k + \beta_1 y_k) + h_2 f'(\alpha_2 x_k + \beta_2 y_k)\Big)e_{k+1} = \Big(h_1 f'(\alpha_1 x_k + \beta_1 y_k) + h_2 f'(\alpha_2 x_k + \beta_2 y_k)\Big)e_k - f(x_k). \tag{3.14}$$

substitute (3.12) on right-hand side of (3.14) yields

$$\begin{aligned} \Big(h_1 f'(\alpha_1 x_k + \beta_1 y_k) + h_2 f'(\alpha_2 x_k + \beta_2 y_k)\Big)e_{k+1} &= h_1 f'(x_k)e_k - h_1\beta_1 f''(x_k)e_k^2 + h_1\beta_1 C_2 f''(x_k)e_k^3 \\ &\quad + \frac{h_1\beta_1^2}{2} f'''(x_k)e_k^3 - h_1\beta_1 C_3 f''(x_k)e_k^4 - 3h_1\beta_1^2 C_3 f''(x_k)e_k^4 - \frac{h_1\beta_1^3}{6} f^{iv}(x_k)e_k^4 + h_2 f'(x_k)e_k \\ &\quad - h_2\beta_2 f''(x_k)e_k^2 + h_2\beta_2 C_2 f''(x_k)e_k^3 + \frac{h_2\beta_2^2}{2} f'''(x_k)e_k^3 - h_2\beta_2 C_3 f''(x_k)e_k^4 - 3h_2\beta_2^2 C_3 f''(x_k)e_k^4 \\ &\quad - \frac{h_2\beta_2^3}{6} f^{iv}(x_k)e_k^4 + O(e_k^5) - f(x_k). \end{aligned} \tag{3.15}$$

By replacing $f(x_k) = f(\alpha) + f'(x_k)e_k + 1/2 f''(x_k)e_k^2 + 1/6 f'''(x_k)e_k^3 + O(e_k^4)$ on the right hand side of (3.15) to obtain

$$\begin{aligned} \Big(h_1 f'(\alpha_1 x_k + \beta_1 y_k) + h_2 f'(\alpha_2 x_k + \beta_2 y_k)\Big)e_{k+1} &= (h_1 + h_2 - 1)f'(x_k)e_k - (h_1\beta_1 + h_2\beta_2 - 1/2)f''(x_k)e_k^2 \\ &\quad + (h_1\beta_1 + h_2\beta_2)C_2 f''(x_k)e_k^3 + (\frac{h_1\beta_1^2}{2} + \frac{h_2\beta_2^2}{2} - 1/6)f'''(x_k)e_k^3 - (h_1\beta_1 C_3 + h_2\beta_2 C_3 \\ &\quad + 3h_1\beta_1^2 C_3 + 3h_2\beta_2^2 C_3)f''(x_k)e_k^4 - (\frac{h_1\beta_1^3}{6} + \frac{h_1\beta_1^3}{6} - 1/24)f^{iv}(x_k)e_k^4 + O(e_k^5). \end{aligned} \tag{3.16}$$

If $h_1 + h_2 \neq 0$ and $f(x_k) \neq 0$ then $\Big(h_1 f'(\alpha_1 x_k + \beta_1 y_k) + h_2 f'(\alpha_2 x_k + \beta_2 y_k)\Big) \neq 0$ which guranted that (2.7) is a valid method. For cubically convergent ,we need to choose $h_1, h_2, \beta_1 and \beta_2$ so that $h_1 + h_2 = 1$ and $h_1\beta_1 + h_2\beta_2 = 1/2$.

4. Numerical Examples

In this section, we present some numerical results for various third order convergent iterative methods. The following methods were compared: Modified Newton's Method in Weerakoon and Fernando [3] :$x_{n+1} = x_n - \frac{2f(x_n)}{f'\left(xn - \frac{f(x_n)}{f'(x_n)}\right) + f'(x_n)}$.

Modified Newton's Method proposed by Frontini et al. [4] :$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - 2\frac{f(x_n)}{f'(x_n)}\right)}$.

Modified Newton's Method derived by Ozbal et al. [5] :$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}\right)$.

Modified Newton's Method derived by Kou et al. [13] :$x_{n+1} = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)}$.

and method proposed by (2.6)[MNMH] :$x_{n+1} = x_n - \frac{2f(x_n)}{\left[f'(\frac{3x_n+y_n}{4}) + f'(\frac{x_n+3y_n}{4})\right]}$.

The computational results are displayed in Table 1.

Function	x_0	Various Method	IT	NFE	x_n
f_1	2	MNM([3])			Diverse
		MNM([4])			Diverse
		MNM([5])	13	39	-1.16730397826142
		MNM([13])	17	51	-1.16730397826142
		MNMH	9	36	-1.16730397826142
f_2	1.2	MNM([3])	4	12	0.739085133215161
		MNM([4])			Diverse
		MNM([5])			Diverse
		MNM([13])	4	12	0.739085133215161
		MNMH	4	16	0.739085133215161
f_3	3	MNM([3])			Diverse
		MNM([4])			Diverse
		MNM([5])			Diverse
		MNM([13])	4	12	0.000000000000015
		MNMH	4	16	0.0
f_4	2.5	MNM([3])	4	12	1.67963061042845
		MNM([4])	7	21	1.67963061042845
		MNM([5])	4	12	0.101025848315685
		MNM([13])	6	18	1.67963061042845
		MNMH	5	20	1.67963061042845
f_5	1.3	MNM([3])	3	9	7.68481808334733E-021
		MNM([4])	4	12	1.7197167733818E-028
		MNM([5])	4	12	5.12759588393657E-030
		MNM([13])	3	9	3.82180552357862E-019
		MNMH	3	12	4.53468286561001E-017
f_6	2	MNM([3])	5	15	0.77288295914921
		MNM([4])	4	12	0.77288295914921
		MNM([5])	4	12	0.77288295914921
		MNM([13])	5	15	0.77288295914921
		MNMH	4	16	0.77288295914921
f_7	2	MNM([3])	3	9	1.29269571937339
		MNM([4])	3	9	1.29269571937339
		MNM([5])	4	12	1.29269571937339
		MNM([13])	4	12	1.29269571937339
		MNMH	3	12	1.29269571937339

Table 1. Comparison of various third order convergent iterative methods with the modified Newton method (MNMH)
.

The numerical experiments carried over the following equation:

$$
\begin{aligned}
f_1(x) &= x^5 - x + 1, \\
f_2(x) &= cosx - x, \\
f_3(x) &= arctanx, \\
f_4(x) &= 10xe^{-x^2} - 1, \\
f_5(x) &= e^{-x}sinx + log(x^2+1), \\
f_6(x) &= x^3 - e^{-x}, \\
f_7(x) &= e^{-x} - cosx,
\end{aligned}
\tag{4.1}
$$

The numerical results presented in Table 1 show that the proposed method(MNMH) has performed efficiently as compared with the other methods of the same order. Thus, the new methods can be seen as an alternative to other third-order methods in literature.

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