Malaya Journal of Matematik, Vol. 9, No. 1, 1153-1159, 2021
https://doi.org/10.26637/MJM0901/0200

Oscillation result for half-linear delay difference equations of second-order

C. Jayakumar [1]* and A. Murugesan [2]

Abstract
We obtained new single-condition criteria for the oscillation of second-order half-linear delay difference equation

$$\Delta(\phi(\zeta)(\Delta x(\zeta))^{\nu})+\rho(\zeta)x^{\nu}(\zeta-\eta)=0; \quad \zeta\geq\zeta_0.$$

Even in the linear case, the sharp result is new and, to our knowledge, improves all previous results. Furthermore, our method has the advantage of being simple to prove, as it relies just on sequentially improved monotonicities of a positive solution. Examples are provided to illustrate our results.

Keywords
Oscillation, non-oscillation, second-order, delay, half-linear, difference equations.

AMS Subject Classification
39A12, 39A13, 39A21.

[1] *Department of Mathematics, Mahendra Arts & Science College (Autonomous), Kalipatti-637501, Tamil Nadu, India.*
[2] *Department of Mathematics, Government Arts College (Autonomous)-636007, Tamil Nadu, India.*
***Corresponding author**: [1]jaya100maths@gmail.com; [2] amurugesan3@gmail.com
Article History: Received **24** December **2020**; Accepted **27** March **2021**

Contents

1. Introduction

We take into consideration a second-order half-linear delay difference equation of the type

$$\Delta(\phi(\zeta)(\Delta x(\zeta))^{\nu})+\rho(\zeta)x^{\nu}(\zeta-\eta)=0; \quad \zeta\geq\zeta_0. \tag{1.1}$$

The forward difference operator Δ is defined by $\Delta x(\zeta)=x(\zeta+1)-x(\zeta)$.

The following conditions are assumed throughout the paper:

(A_1) η is a non-negative integer;

(A_2) $\{\phi(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a positive real sequence;

(A_3) $\{\rho(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of non-negative real numbers and $\rho(\zeta)\not\equiv 0$ for infinitely many values of ζ;

(A_4) $\nu\in\{\frac{a}{b} : a$ and b are odd integers$\}$;

(A_5) The equation (1.1) is in so-called noncanonical form, i.e.,

$$\theta(\zeta):=\sum_{s=\zeta}^{\infty}\frac{1}{\phi^{\frac{1}{\nu}}(s)}<\infty. \tag{1.2}$$

A solution of (1.1) is a real sequence $\{x(\zeta)\}$ which is defined for $\zeta\geq-\eta$ and satisfies (1.1) for $\zeta\geq\zeta_0$. A solution $\{x(\zeta)\}$ is said to be oscillatory, if the terms $\{x(\zeta)\}$ of the solution are not eventually positive or eventually negative. Otherwise the solution is called nonoscillatory.

The oscillation theory of delay differential equations has been significantly developed in recent decades. In recent years, the oscillation theory of discrete analogues of delay differential equations has received much interest (see, for example, [1, 10, 12, 15, 20, 21]) and the references referenced therein are recommended to the reader.

For the second-order difference equations, oscillation and nonoscillation problems have recently received considerable attention. This is likely due to the similarity of such phenomena to equivalent differential equations. Furthermore, these equations have a wide range of applications in physics and other domains (see [3–5, 11, 14]), including neutral and delay terms equations [17–19, 22]. They're often used, for example, in the research of distributed networks with lossless transmission lines. The reader is referred to [2, 3, 6, 8] for a basic idea of these equations.

In [16], we established that the second-order noncanonical advanced difference equation

$$\Delta(\phi(\zeta)(\Delta x(\zeta))^{\nu})+\rho(\zeta)x^{\nu}(\zeta+\eta)=0; \quad \zeta\geq\zeta_0 \quad (1.3)$$

is oscillating if

$$\sum_{\zeta=\zeta_0}^{\infty}\left(\frac{1}{\phi(\zeta)}\sum_{s=\zeta_0}^{\zeta-1}\theta^{\nu}(s+\eta)\rho(s)\right)^{\frac{1}{\nu}}=\infty.$$

The goal of this study is to find the best single-condition oscillation criterion possible for (1.1). The concepts are based on some of the most recent ones from [9] for linear equations.

$$\Delta(\phi(\zeta)\Delta x(\zeta))+\rho(\zeta)x(\zeta-\eta)=o; \quad \zeta\geq\zeta_0 \quad (1.4)$$

Theorem A. [9] Assume that

$$\sum_{\zeta=\zeta_0}^{\infty}\theta(s+1)\rho(s)=\infty, \quad (1.5)$$

and there exists a $\delta_0>0$ such that

$$\rho(\zeta)\theta(\zeta+1)\theta(\zeta)r(\zeta)\geq\delta_0 \quad (1.6)$$

eventually. Suppose that there exists a $k\in\mathbb{N}$ such that $\delta_j<1$ for $j=0,1,2,\ldots,k-1$, and

$$\liminf_{\zeta\to\infty}\sum_{s=\zeta-\eta}^{\zeta-1}\frac{\rho(s)\theta(s+1)}{1-\delta_k}>\left(\frac{\eta}{\eta+1}\right)^{\eta+1}, \quad (1.7)$$

where

$$\delta_k=\delta_0\frac{\lambda^{\delta_{k-1}}}{1-\delta_{k-1}}.$$

Then (1.4) is oscillatory.

In this study, we derive new single-condition constraints for the oscillation of all unimprovable constant solutions to (1.1). Even in the linear situation, this sharp conclusion is unique and, to our knowledge, improves all previous results in the literature. Moreover, in the linear case, we can express comparable results for canonical equations.

Our established results are discrete analogues of some well-known results due to J. Džurina and I. Jadlovská [7].

The following is the paper's structure: We proved some auxiliary lemmas in section 2. Then, the paper's main results are stated and established in section 3. Finally, two examples are offered in section 4 to demonstrate our results.

2. Some useful lemmas

Let us define

$$\delta_*=\liminf_{\zeta\to\infty}\frac{1}{\nu}\phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\rho(\zeta) \quad (2.1)$$

and

$$\mu_*=\liminf_{\zeta\to\infty}\frac{\theta(\zeta-\eta)}{\theta(\zeta)}<\infty. \quad (2.2)$$

The proofs rely on the existence of positivity δ_*, which is also required for Theorems 3.1 and 3.4 to be valid. Then there is a $\zeta_1\geq\zeta_0$ for every arbitrary fixed $\delta\in(0,\delta_*)$ and $\mu\in[1,\mu_*)$ such that

$$\frac{1}{\nu}\rho(\zeta)\phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\geq\delta$$

and

$$\frac{\theta(\zeta-\eta)}{\theta(\zeta)}\geq\mu, \quad \zeta\geq\zeta_0. \quad (2.3)$$

In the following section, we presume that all functional inequalities are satisfied; eventually, that is, for all ζ large enough.

Using the procedure used in [[8], Theorem 2], one can prove the following result.

Lemma 2.1. *Suppose that*

$$\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(\zeta)}\left(\sum_{s=\zeta_0}^{\zeta-1}\rho(s)\right)^{\frac{1}{\nu}}=\infty. \quad (2.4)$$

If $\{x(\zeta)\}$ *is eventually positive solution of* (1.1)*, then* $\Delta x(\zeta)<0$ *and* $\lim_{\zeta\to\infty}x(\zeta)=0$.

Lemma 2.2. *Let* $\delta_*>0$. *If* (1.1) *has an eventually positive solution* $\{x(\zeta)\}$*, then*

(*i*) $\{x(\zeta)\}$ *is eventually decreasing with* $\lim_{\zeta\to\infty}x(\zeta)=0$*;*

(*ii*) $\{\frac{x(\zeta)}{\theta(\zeta)}\}$ *is eventually nondecreasing.*

Proof. (*i*) By using (1.2), (2.3) and the decreasing nature of $\{\theta(\zeta)\}$, we have

$$\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\zeta_1}^{u-1}\rho(s)\right)^{\frac{1}{\nu}}$$
$$\geq\sqrt[\nu]{\delta}\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\zeta_1}^{u-1}\frac{\nu}{r^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)}\right)^{\frac{1}{\nu}}$$
$$\geq\sqrt[\nu]{\delta}\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(-\nu\sum_{s=\zeta_1}^{u-1}\frac{\theta(s)}{\theta^{\nu+1}(s+1)}\right)^{\frac{1}{\nu}}$$

$$\geq \sqrt[\nu]{\delta}\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\frac{1}{\theta^{\nu}(u)}-\frac{1}{\theta^{\nu}(n_1)}\right)^{\frac{1}{\nu}}.$$

Since $\theta^{-\nu}(\zeta)\to\infty$ as $\zeta\to\infty$, for any $l\in(0,1)$ and ζ large enough, we have $\theta^{-\nu}(\zeta)-\theta^{-\nu}(\zeta_1)\geq l^{\nu}\theta^{-\nu}(\zeta)$ and hence

$$\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\zeta_1}^{u-1}\rho(s)\right)^{\frac{1}{\nu}}\geq l\sqrt[\nu]{\delta}\sum_{u=\zeta_1}^{\zeta-1}\frac{1}{r^{\frac{1}{\nu}}(u)\theta(u)}$$

$$\geq l\sqrt[\nu]{\delta}\ln\frac{\theta(\zeta_1)}{\theta(\zeta)}\to\infty \text{ as } \zeta\to\infty.$$

By Lemma 2.1, the conclusion follows.

(ii) Using the fact that $\{r^{\frac{1}{\nu}}(n)\Delta x(n)\}$ is nonincreasing, we obtain

$$\begin{aligned}x(\zeta)&\geq-\sum_{s=\zeta}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(s)}r^{\frac{1}{\nu}}(s)\Delta x(\zeta)\\&\geq -r^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\sum_{s=\zeta}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(s)}\\&=-r^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta),\end{aligned}$$

i.e.,

$$\Delta\left(\frac{x(\zeta)}{\theta(\zeta)}\right)=\frac{r^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta)+x(\zeta)}{r^{\frac{1}{\nu}}(\zeta)\theta(\zeta)\theta(\zeta+1)}\geq 0.$$

The proof is complete. □

To develope the (i) - part of Lemma 2.2, let us define a sequence $\{\delta_k\}$ by

$$\begin{aligned}\delta_0&=\sqrt[\nu]{\delta_*}\\ \delta_k&=\frac{\delta_0\mu_*^{\delta_{k-1}}}{\sqrt[\nu]{1-\delta_{k-1}}},\quad k\in\mathbb{N}.\end{aligned}\tag{2.5}$$

We can easily show by induction that if for any $k\in\mathbb{N}$, $\delta_i<1$, $i=0,1,2,\ldots,k,$. then δ_{k+1} exists and

$$\delta_{k+1}=\xi_k\delta_k>\delta_k,\tag{2.6}$$

where l_k is defined by

$$\xi_0=\frac{\mu_*^{\delta_0}}{\sqrt[\nu]{1-\delta_0}}\tag{2.7}$$

$$\xi_{k+1}=\mu_*^{\delta_0(\xi_k-1)}\sqrt[\nu]{\frac{1-\delta_k}{1-\xi_k\delta_k}},\quad k\in\mathbb{N}_0.\tag{2.8}$$

Lemma 2.3. *Let $\delta_*>0$ and $\mu_*<\infty$. If (1.1) has an eventually positive solution $\{x(\zeta)\}$, then for any $k\in\mathbb{N}$, $\{\frac{x(\zeta)}{\theta^{\delta_k}(\zeta)}\}$ is eventually decreasing.*

Proof. Let $\{x(\zeta)\}$ be an eventually positive solution of (1.1). Then there exists a $\zeta_1\geq\zeta_0$ such that $x(\zeta-\eta)>0$ for $\zeta\geq\zeta_1$. Summing (1.1) from ζ_1 to $\zeta-1$, we have

$$-\phi(\zeta)(\Delta x(\zeta))^{\nu}=-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}+\sum_{s=\zeta_1}^{\zeta-1}\rho(s)x^{\nu}(s-\eta).\tag{2.9}$$

By (i) of Lemma 2.2, $\{x(\zeta)\}$ is decreasing and $x(\zeta-\eta)\geq x(\zeta)$ for $\zeta\geq\zeta_1$. Therefore,

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^{\nu}&\geq-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}+\sum_{s=\zeta_1}^{\zeta-1}\rho(s)x^{\nu}(s-\eta)\\&\geq-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}+x^{\nu}(\zeta)\sum_{s=\zeta_1}^{\zeta-1}\rho(s).\end{aligned}$$

Using (2.3) in the above inequality, we get

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^{\nu}\geq&-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}\\&+\delta x^{\nu}(\zeta)\sum_{s=\zeta_1}^{\zeta-1}\frac{c}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)}\end{aligned}$$

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^{\nu}\geq&-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}\\&+\delta\frac{x^{\nu}(\zeta)}{\theta^{\nu}(\zeta)}-\delta\frac{x^{\nu}(\zeta)}{\theta^{\nu}(\zeta_1)}.\end{aligned}\tag{2.10}$$

From (i)-part of Lemma 2.2, we have that $\lim_{\zeta\to\infty}x(\zeta)=0$. Hence, there is a $\zeta_2\geq\zeta_1$ such that

$$-\phi(\zeta_1)(\Delta x(\zeta_1))^{\nu}-\delta\frac{x^{\nu}(\zeta)}{\theta^{\nu}(\zeta_1)}>0,\quad \zeta\geq\zeta_2.$$

Thus,

$$-\phi(\zeta)(\Delta x(\zeta))^{\nu}>\delta\frac{x^{\nu}(\zeta)}{\theta^{\nu}(\zeta)}\tag{2.11}$$

or

$$-\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta)>\sqrt[\nu]{\delta}x(\zeta)=\varepsilon_0\delta_0x(\zeta),$$

where $\varepsilon_0=\frac{\sqrt[\nu]{\delta}}{\delta_0}$ is an arbitrary constant from $(0,1)$. Therefore,

$$\begin{aligned}\Delta\left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)&=\frac{\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta^{\sqrt[\nu]{\delta}}(\zeta)+\sqrt[\nu]{\delta}\theta^{\sqrt[\nu]{\delta}-1}(\zeta)x(\zeta)}{\phi^{\frac{1}{\nu}}(\zeta)\theta^{\sqrt[\nu]{\delta}}(\zeta)\theta^{\sqrt[\nu]{\delta}}(\zeta+1)}\\&=\frac{\theta^{\sqrt[\nu]{\delta}-1}(\zeta)(\sqrt[\nu]{\delta}x(\zeta)+\theta(\zeta)\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta))}{\phi^{\frac{1}{\nu}}(\zeta)\theta^{\sqrt[\nu]{\delta}}(\zeta)\theta^{\sqrt[\nu]{\delta}}(\zeta+1)}\leq 0,\quad \zeta\geq\zeta_2.\end{aligned}\tag{2.12}$$

Summing (1.1) from ζ_2 to $\zeta-1$ and using that $\{\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\}$ is decreasing, we have

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^\nu &\geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu + \left(\frac{x(\zeta-\eta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta-\eta)}\right)^\nu \\ &\times \sum_{s=\zeta_2}^{\zeta-1}\rho(s)\theta^{\sqrt[\nu]{\delta}}(s-\eta) \\ &\geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu + \left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu \\ &\times \sum_{s=\zeta_2}^{\zeta-1}\rho(s)\left(\frac{\theta(s-\eta)}{\theta(s)}\right)^{\sqrt[\nu]{\delta}}\theta^{\sqrt[\nu]{\delta}}(s).\end{aligned}$$

By virtue of (2.3), we see that

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^\nu \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu + \delta\left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu \\ \times \sum_{s=\zeta_2}^{\zeta-1}\frac{\nu\left(\frac{\theta(s-\eta)}{\theta(s)}\right)^{\sqrt[\nu]{\delta}}}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1-\nu\sqrt[\nu]{\delta}}(s+1)}\end{aligned}$$

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^\nu \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu \\ + \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu \\ \times \sum_{s=\zeta_2}^{\zeta-1}\frac{\nu(1-\sqrt[\nu]{\delta})}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1-\nu\sqrt[\nu]{\delta}}(s+1)}\end{aligned} \tag{2.13}$$

$$\begin{aligned}-\phi(\zeta)(\Delta x(\zeta))^\nu \geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu \\ + \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta^{\nu\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu \\ \times \left(\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\zeta)} - \frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\zeta_2)}\right).\end{aligned} \tag{2.14}$$

Now, we claim that $\lim_{\zeta\to\infty}\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)} = 0$. It sufficies to show that there is $\varepsilon > 0$ such that $\left\{\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}+\varepsilon}(\zeta)}\right\}$ is eventually decreasing sequence. Since $\{\theta(\zeta)\}$ tends to zero, there exists is a constant.

$$\xi \in \left(\frac{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}}{\mu^{\sqrt[\nu]{\delta}}}, 1\right)$$

and a $\zeta_3 \geq \zeta_2$ such that

$$\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\zeta)} - \frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\zeta_2)} > \xi^\nu\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\zeta)}, \quad \zeta \geq \zeta_3.$$

Using the above inequality in (2.14) yields

$$-\phi(\zeta)(\Delta x(\zeta))^\nu \geq \frac{\xi^\nu\delta}{1-\sqrt[\nu]{\delta})}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta(\zeta)}\right)^\nu,$$

i.e.,

$$-\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta) \geq \left(\sqrt[\nu]{\delta}+\varepsilon\right)\frac{x(\zeta)}{\theta(\zeta)}, \tag{2.15}$$

where

$$\varepsilon = \sqrt[\nu]{\delta}\left(\frac{\xi\mu^{\sqrt[\nu]{\delta}}}{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}} - 1\right) > 0.$$

Then, from (2.15),

$$\Delta\left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}+\in}(\zeta)}\right) \leq 0, \quad \zeta \geq \zeta_3,$$

and hence the claim holds. Therefore, for $\zeta_4 \geq \zeta_3$,

$$\begin{aligned}-\phi(\zeta_2)(\Delta x(\zeta_2))^\nu - \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}} \\ \times \left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu\frac{1}{\theta^{\nu-\nu\sqrt[\nu]{\delta}}(\zeta_2)} > 0, \quad \zeta \geq \zeta_4.\end{aligned}$$

Returning to (2.14) and applying the above inequality,

$$\begin{aligned}&-\phi(\zeta)(\Delta x(\zeta))^\nu \\ &\geq -\phi(\zeta_2)(\Delta x(\zeta_2))^\nu \\ &+ \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta(\zeta)}\right)^\nu \\ &- \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right)^\nu\frac{1}{\theta^{\nu-\nu\sqrt[\nu]{\delta}}(\zeta_2)} \\ &> \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\zeta)}{\theta(\zeta)}\right)^\nu,\end{aligned}$$

or

$$\begin{aligned}-\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta) &> \frac{\sqrt[\nu]{\delta}}{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}}\mu^{\nu\sqrt[\nu]{\delta}}x(\zeta) \\ &= \varepsilon_1\delta_1 x(\zeta), \quad \zeta \geq \zeta_4,\end{aligned}$$

where

$$\varepsilon_1 = \sqrt[\nu]{\frac{\delta(1-\sqrt[\nu]{\delta_*})}{\delta_*(1-\sqrt[\nu]{\delta})}}\frac{\mu^{\sqrt[\nu]{\delta}}}{\mu_*^{\sqrt[\nu]{\delta_*}}}$$

is an arbitrary constant from $(0,1)$ tends to 1 if $\delta \to \delta_*$ and $\mu \to \mu_*$. Hence,

$$\Delta\left(\frac{x(\zeta)}{\theta^{\varepsilon_1\delta_1}(\zeta)}\right) < 0, \quad \zeta \geq \zeta_4.$$

By induction, one can show that for any $k \in \mathbb{N}_0$ and ζ large enough,

$$\Delta\left(\frac{x(\zeta)}{\theta^{\varepsilon_k\delta_k}(\zeta)}\right) < 0,$$

where ε_k given by $\varepsilon_0 = \sqrt[\nu]{\frac{\delta}{\delta_*}}$

$$\varepsilon_{k+1} = \varepsilon_0 \sqrt[\nu]{\frac{1-\delta_k}{1-\varepsilon_k\delta_k}}\frac{\mu^{\varepsilon_k\delta_k}}{\mu_*^{\delta_k}}, \quad k \in \mathbb{N}_0$$

is an arbitrary constant from $(0,1)$ tends to 1 if $\delta \to \delta_*$ and $\mu \to \mu_*$. Now, we assert that from any $k \in \mathbb{N}_0$, $\left\{\frac{x(\zeta)}{\theta^{\varepsilon_{k+1}\delta_{k+1}}(\zeta)}\right\}$ decreasing implies that $\{\frac{x(\zeta)}{\theta^{\delta_k}}\}$ is a decreasing sequence as well. Using (2.6) and the fact that ε_{k+1} is arbitrarly closed to 1, we see that

$$\varepsilon_{k+1}\delta_{k+1} > \delta_k.$$

Then, for ζ sufficiently large enough,

$$-\phi^{\frac{1}{\nu}}(\zeta)\Delta x(\zeta)\theta(\zeta) > \varepsilon_{k+1}\delta_{k+1}x(\zeta) > \delta_k x(\zeta)$$

and so for any $\zeta \in \mathbb{N}_0$ and ζ large enough,

$$\Delta\left(\frac{x(\zeta)}{\theta^{\delta_k}(\zeta)}\right) < 0.$$

The proof is complete. □

3. Main Results

Theorem 3.1. *Let*

$$\mu_* := \liminf_{\zeta\to\infty}\frac{\theta(\zeta-\eta)}{\theta(\zeta)} < \infty. \tag{3.1}$$

If

$$\liminf_{\zeta\to\infty}\phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\rho(\zeta) > max\{c(\omega) : \nu\omega^{\nu}(1-\omega)\mu_*^{-\nu\omega} : 0<\omega<1\}, \tag{3.2}$$

then (1.1) *is oscillatory.*

Proof. Assume that $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Lemma 2.2 and 2.3 ensure that $\Delta\{\frac{x(\zeta)}{\theta(\zeta)}\} \geq 0$ and $\Delta\{\frac{x(\zeta)}{\theta^{\delta_k}(\zeta)}\} < 0$ for any $k \in \mathbb{N}_0$ and ζ sufficiently large enough. This case occurs when $\delta_k < 1$ for any $k \in \mathbb{N}_0$.

Thus, the sequence $\{\delta_k\}$ given by (2.5) is increasing and bounded sequence from above which implies that there exists a finite limit $\liminf_{k\to\infty}\delta_k = \omega$, where ω is the smallest positive root of the equation

$$c(\omega) = \liminf_{\zeta\to\infty}\phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\rho(\zeta). \tag{3.3}$$

Because of (3.2), equation (3.3) cannot have a positive solution.

This contradiction completes the proof. □

Corollary 3.2. *By simple computations, we obtain*

$$max\{c(\omega) : 0<\omega<1\} = c(max),$$

where

$$\omega_{max} = \begin{cases} \frac{\nu}{\nu+1}, for\ \mu_* = 1 \\ \frac{-\sqrt{(\nu\phi+\nu+1)^2-4\nu^2\phi}+\nu\phi+\nu+1}{2\nu\phi}, \\ for\ \mu_* \neq 1\ and\ \phi = \ln\mu_*, \end{cases}$$

and $c(\omega)$ *is defined by* (3.2).

We get the following result when (3.1) *is failed.*

Theorem 3.3. *Let*

$$\lim_{\zeta\to\infty}\frac{\theta(\zeta-\eta)}{\theta(\zeta)} = \infty. \tag{3.4}$$

If

$$\liminf_{\zeta\to\infty}\phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\rho(\zeta) > 0 \tag{3.5}$$

then (1.1) *is oscillatory.*

Proof. Let $\{x(\zeta)\}$ be an eventually positive solution of (1.1). Then there exists a $\zeta_1 \geq \zeta_0$ such that $x(\zeta-\eta) > 0$ for $\zeta \geq \zeta_1$. By virtue of (3.4), we see that for any $M > 0$ there exists a ζ sufficiently large enough such that

$$\frac{\theta(\zeta-\eta)}{\theta(\zeta)} \geq \left(\frac{M}{\sqrt[\nu]{\delta}}\right)^{\frac{1}{\sqrt[\nu]{\delta}}}. \tag{3.6}$$

As in the proof of Lemma 2.3, we can show that $\left\{\frac{x(\zeta)}{\theta^{\sqrt[\nu]{\delta}}(\zeta)}\right\}$ is decreasing eventually, say for $\zeta \geq \zeta_2 \geq \zeta_1$. Using this monotonicity in (2.9), we have

$$\begin{aligned} &-\phi(\zeta)(\Delta x(\zeta))^{\nu} \\ &= -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + \sum_{s=\zeta_2}^{\zeta-1}\rho(s)x^{\nu}(s-\eta) \\ &\geq -\phi(\zeta_2)(\Delta x(\zeta_2))^{\nu} + M^{\nu}x^{\nu}(\zeta) \\ &\times \sum_{s=\zeta_2}^{\zeta-1}\frac{\nu}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)} \end{aligned}$$

$$-\phi(\zeta)(\Delta x(\zeta))^{\nu} > M^{\nu}\left(\frac{x(\zeta)}{\theta(\zeta)}\right)^{\nu},$$

from which we derive that $\{\frac{x(\zeta)}{\theta^M(\zeta)}\}$ is decreasing sequence. From the fact that M is a arbitrary, we have $\{\frac{x(\zeta)}{\theta(\zeta)}\}$ is nondecreasing sequence.This is a contradiction with (ii)-part of Lemma 2.2 and this completes the proof. □

Now, we convert the oscillation behavior from (1.1) to the canonical equations in linear case $\nu = 1$

$$\Delta(\widetilde{\phi}(\zeta)\Delta u(\zeta)) + \widetilde{\rho}(\zeta)u(\zeta-\eta) = 0, \quad \zeta \geq \zeta_0, \quad (3.7)$$

where $\{\widetilde{\phi}(\zeta)\}$ is a positive real sequence and $\{\widetilde{\rho}(\zeta)\}$ is a nonnegative real sequence with $\rho(\zeta) \not\equiv 0$ for infinitely many values of ζ, and

$$R(\zeta) = \sum_{s=\zeta_0}^{\zeta-1} \frac{1}{\widetilde{\phi}(s)} \to \infty \quad as \quad \zeta \to \infty.$$

Theorem 3.4. *Let*

$$\delta_* := \liminf_{\zeta\to\infty} \frac{R(\zeta)}{R(\zeta-\eta)} < \infty.$$

If

$$\liminf_{\zeta\to\infty} \left(\widetilde{\phi}(\zeta)\widetilde{\rho}(\zeta)R(\zeta)R(\zeta-\eta)\right) > max\{\omega(1-\omega)\delta_*^{-\omega} : 0 < \omega < 1\},$$

then (3.7) *is oscillatory.*

Proof. we can readily check that the canonical equation (3.7) is equivalent to a noncanonical equation (1.1) with $\nu = 1$,

$$\phi(\zeta) = \widetilde{\phi}(\zeta)R(\zeta)R(\zeta+1)$$
$$\rho(\zeta) = \widetilde{\rho}(\zeta)R(\zeta+1)R(\zeta-\eta)$$

and

$$x(\zeta) = \frac{u(\zeta)}{R(\zeta)} > 0.$$

Now,

$$\theta(\zeta) = \sum_{s=\zeta}^{\infty} \frac{1}{\phi(s)} = \sum_{s=\zeta}^{\infty} \frac{\Delta R(s)}{R(s)R(s+1)} = \frac{1}{R(\zeta)}.$$

The result derives from Theorem 3.1 immediately. □

Theorem 3.5. *Let*

$$\lim_{\zeta\to\infty} \frac{R(\zeta)}{R(\zeta-\eta)} = \infty.$$

If

$$\liminf_{\zeta\to\infty} \{\widetilde{\phi}(\zeta)\widetilde{\rho}(\zeta)R(\zeta)R(\zeta-\eta)\} > 0,$$

then (3.7) *is oscillatory.*

Proof. By applying the equivalent noncanonical representation of (3.7) as in the proof of Theorem 3.4, the claim follows from Theorem 3.3. □

4. Examples

Example 4.1. *Let us discuss the second-order difference equation*

$$\Delta((\zeta(\zeta+1))^{\frac{1}{3}}(\Delta x(\zeta))^{\frac{1}{3}}) + \lambda_0 \frac{(\zeta+1)^{\frac{1}{3}}}{\zeta} x^{\frac{1}{3}}(\zeta-1) = 0; \quad \zeta = 1,2,3,\ldots \quad (4.1)$$

Here, we have $\phi(\zeta) = (\zeta(\zeta+1))^{\frac{1}{3}}$, $\rho(\zeta) = \lambda_0 \frac{(\zeta+1)^{\frac{1}{3}}}{\zeta}$, $\nu = \frac{1}{3}$ *and* $\eta = 1$.

By simple computation, we obtain

$$\theta(\zeta) = \frac{1}{\zeta},$$

$$\lambda_* = \liminf_{\zeta\to\infty} \frac{\theta(\zeta-1)}{\theta(\zeta)} = 1,$$

$$\liminf_{\zeta\to\infty} \phi^{\frac{1}{\nu}}(\zeta)\theta^{\nu+1}(\zeta+1)\rho(\zeta) = \lambda_0,$$

and

$$max\{c(\omega) : \nu\omega^{\nu}(1-\omega) : 0 < \omega < 1\} = \frac{1}{4\sqrt[3]{4}}.$$

Thus, by Theorem 3.1, every solution of (4.1) *is oscillatory if* $\lambda_0 > \frac{1}{4\sqrt[3]{4}}$

Example 4.2. *Let us investigate the oscillatory behavior of the second-order linear difference equation*

$$\Delta\left(\frac{1}{\zeta}\Delta x(\zeta)\right) + \frac{4\lambda_0}{(\zeta-2)(\zeta-1)^2} x(\zeta-1) = 0; \quad \zeta = 1,2,3,\ldots \quad (4.2)$$

We have $\widetilde{\phi}(\zeta) = \frac{1}{\zeta}$, $\widetilde{\rho}(\zeta) = \frac{4\lambda_0}{(\zeta-2)(\zeta-1)^2}$ *and* $\eta = 1$. *We can easily show that*

$$R(\zeta) = \frac{\zeta(\zeta-1)}{2}, \quad \delta_* = 1,$$

$$\liminf_{\zeta\to\infty} \widetilde{\phi}(\zeta)\widetilde{\rho}(\zeta)R(\zeta)R(\zeta-1) = \lambda_0,$$

and

$$max\{\omega(1-\omega) : 0 < \omega < 1\} = \frac{1}{4}.$$

Hence, by Theorem 3.4, the equation (4.2) *is oscillatory for* $\lambda_0 > \frac{1}{4}$.

Acknowledgment

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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