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Oscillatory and asymptotic behavior of solutions to second-order non-linear neutral difference equations of advanced type

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Abstract

In this paper, necessary and sufficient conditions are obtained for oscillatory and asymptotic behavior of solutions to second-order non-linear neutral advanced difference equations of the form

$$\Delta[\varphi(\zeta)(\Delta y(\zeta))^{\lambda}]+\sum_{i=1}^{m}\mu_i(\zeta)f(x(\zeta+\eta_i))=0; \quad \zeta\geq\zeta_0,$$

where $y(\zeta)=x(\zeta)+p(\zeta)x(\zeta-\kappa)$, under the assumption $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\xi}}(\zeta)}=\infty$. Our main tool is Lebesgue's dominated convergence theorem. Further, some illustrate examples showing the applicability of the new results are included.

Keywords

Oscillation, non-oscillation, neutral, second-order, non-linear, advanced, difference equations.

AMS Subject Classification

39A12, 39A13, 39A21.

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1. Introduction

We consider the second-order non-linear neutral advanced difference equations of the form

$$\Delta(\varphi(\zeta)(\Delta z(\zeta))^{\lambda})+\sum_{i=1}^{m}\mu_i(\zeta)x^{\lambda}f(x(\zeta+\eta_i))=0; \quad \zeta\geq\zeta_0. \tag{1.1}$$

where $y(\zeta)=x(\zeta)+p(\zeta)x(\zeta-\kappa)$ and Δ is the forward difference operator defined by $\Delta x(\zeta)=x(\zeta+1)-x(\zeta)$. Throughout the paper, we use the following assumptions:

(C_1) λ is a quotient of odd positive integers;

(C_2) η_i (i=1,2,...,m) are positive integers;

(C_3) κ is a positive integer;

(C_4) $\{\varphi(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is sequence of positive real numbers;

(C_5) for $i=1,2,\ldots,m$ $\{\mu_i(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of non-negative real numbers and $\mu_i(\zeta)\not\equiv 0$ for sufficiently large values of ζ;

(C_6) $f:R\to R$ is a continuous and strictly increasing function such that $uf(u)>0$ for $u\in 0$;

(C_7) $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\lambda}}(\zeta)}=\infty$; Letting

$$R(\zeta,\zeta_0)=\sum_{s=\zeta_0}^{\zeta-1}\frac{1}{\varphi^{\frac{1}{\lambda}}(s)}$$

and hence $R(\zeta,\zeta_0)\to\infty$ as $\zeta\to\infty$;

(C_8) $\{p(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of real numbers such that $-1\leq p(\zeta)\leq 0$.

By a solution of (1.1), we mean a real sequence $\{x(\zeta)\}$ which is defined for all $\zeta \geq \zeta_0 - \kappa$ and satisfies equation (1.1) for all $\zeta \in N(\zeta_0) = \{\zeta_0, \zeta_0+1, \zeta_0+2, \ldots\}$. A nontrivial solution $\{x(\zeta)\}$ of equation (1.1) is said to be oscillatory if it a neither eventually positive nor eventually negative and it is non-oscillatory otherwise.

In recent years, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions of nonlinear difference equations, see for example [1, 8, 11, 13, 16, 17] and the references cited therein. Neutral difference and differential equations arise in many areas of applied mathematics, such as population dynamics [6], stability theory [14, 15], circuit theory[4], bifurcation analysis [3], dynamical behavior of delayed network system [18], and so on. Therefore, these equations have attracted a great interest during the last few decades. For the general background of difference equations, one can refer to [1, 2, 7, 10]

Murugesan et al. [12] derived sufficient conditions for oscillation of all solutions of the second-order half-linear advanced difference equation

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\lambda}) + \mu(\zeta)x^{\lambda}(\zeta+\eta) = 0, \quad \zeta \geq \zeta_0.$$

Under the condition that $\sum_{\zeta=\zeta_0}^{\infty} \frac{1}{\varphi^{\frac{1}{\lambda}}(\zeta)} < \infty$.

Gopalakrishnan et al. [5], we derived oscillatory conditions for the second-order noncanonical difference equation of the delay and advanced type

$$\Delta(\varphi(\zeta)\Delta x(\zeta)) + \mu(\zeta)x(\zeta+\eta) = 0; \quad \zeta \geq \zeta_0.$$

Jayakumar et al. [9] investigated oscillatory and asymptotic behavior of a class of second-order non-linear delay difference equations

$$\Delta(r(\zeta)(\Delta x(\zeta))^{\alpha}) + q(\zeta)x^{\alpha}(\zeta-\sigma) = 0, \quad \zeta \geq \zeta_o,$$

and derived sufficient conditions under which every solution is either oscillatory or converges to zero.

2. Preliminary Results

Lemma 2.1. *If $\{x(\zeta)\}$ is an eventually positive solution of (1.1), then $\{y(\zeta)\}$ satisfies one of the following two posible cases;*

- (C_1) *$y(\zeta) < 0$, $\Delta y(\zeta) > 0$ and $\Delta(\varphi(\zeta)(\Delta y(\zeta)))^{\lambda} < 0$ for all large ζ;*
- (C_2) *$y(\zeta) > 0$, $\Delta y(\zeta) > 0$ and $\Delta(\varphi(\zeta)(\Delta y(\zeta)))^{\lambda} < 0$ for all large ζ.*

Proof. Let $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Then there exists a $\zeta_1 \geq \zeta_0$ such that $x(\zeta) > 0$ and $x(\zeta-\kappa) > 0$ for all $\zeta \geq \zeta_1$. Then from (1.1), we have

$$\Delta(\varphi(\zeta)(\Delta y(\zeta))^{\lambda}) = -\sum_{i=1}^{m} \mu_i(\zeta) f(x(\zeta+\eta_i)) < 0, \text{ for all } \zeta \geq \zeta_1 \tag{2.1}$$

Consequently, $\{\varphi(\zeta)(\Delta y(\zeta))^{\lambda}\}$ is nonincreasing for $\zeta \geq \zeta_1$. Since $\varphi(\zeta) > 0$, and thus either $\Delta y(\zeta) < 0$ or $\Delta y(\zeta) > 0$ for $\zeta \geq \zeta_2 \geq \zeta_1$.

If $\Delta y(\zeta) > 0$ for $\zeta \geq \zeta_2$, then we have (C_1) or (C_2). We have prove that $\Delta y(\zeta) < 0$ cannot occur.

If $\Delta y(\zeta) < 0$ for $\zeta \geq \zeta_2$, then there exists $\upsilon > 0$ such that $\varphi(\zeta)(\Delta y(\zeta))^{\lambda} \leq -\upsilon$ for $\zeta \geq \zeta_2$.

$$\Delta y(\zeta) \leq \frac{-\upsilon^{\frac{1}{\lambda}}}{\varphi^{\frac{1}{\lambda}}(\zeta)}.$$

Sum the last inequality from ζ_2 to $\zeta - 1$, we have

$$y(\zeta) \leq y(\zeta_2) - \upsilon^{\frac{1}{\lambda}} \sum_{s=\zeta_2}^{\zeta-1} \frac{1}{\varphi^{\frac{1}{\lambda}}(s)} \to \infty \text{ as } \zeta \to \infty.$$

We consider now the following possibilities seperately.

If $\{x(\zeta)\}$ is unbounded, then there exist a sequence $\{\zeta_k\}$ of positive integers such that $\zeta_k \to \infty$ as $k \to \infty$ and $x(\zeta_k) \to \infty$ as $k \to \infty$, where

$$x(\zeta_k) = max\{x(s) : \zeta_0 \leq s \leq \zeta_k\}.$$

Clearly, $x(\zeta_k - \kappa) \leq x(\zeta_k)$. Therefore, for all k,

$$\begin{aligned} y(\zeta_k) &= x(\zeta_k) + p(\zeta_k)x(\zeta_k - \kappa) \\ &\geq x(\zeta_k)(1 + p(\zeta_k)) > 0, \end{aligned}$$

which contradicts the fact that $\lim_{\zeta\to\infty} y(\zeta) = \infty$. If $\{x(\zeta)\}$ is bounded, then $\{y(\zeta)\}$ is also bounded, which contradicts $\lim_{\zeta\to\infty} y(\zeta) = -\infty$. Hence, $\{y(\zeta)\}$ satifies one of the (C_1) and (C_2). This completes the proof. □

Remark 2.2. *If follows from (C_2) of Lemma 2.1 that there exists $\theta > 0$ such that $y(\zeta) \geq \theta$ for all large ζ.*

We assume that there exists a constant ν such that $0 < \nu < \lambda$ and

$$\frac{f(u)}{u^{\nu}} \geq \frac{f(v)}{v^{\nu}}, \text{ for } 0 < u < v. \tag{2.2}$$

A typical example of a nonlinear function satisfying (2.2) is $f(y) = |y|^{\xi} sgn(y)$ with $0 < \xi < \nu$.

Remark 2.3. *The condition (2.2) implies that $f(u)/u^{\nu}$ is nonincreasing. We assume that there exists $\nu < \lambda > 0$ such that*

$$\frac{f(u)}{u^{\nu}} \leq \frac{f(v)}{v^{\nu}}, \text{ for } 0 < u < v. \tag{2.3}$$

A typical example of a nonlinear function satisfying (2.3) is $f(y) = |y|^{\xi} sgn(y)$ with $\nu < \xi$.

Remark 2.4. *The condition (2.3) implies that $f(u)/u^{\nu}$ is nondecreasing*

3. Main Results

Theorem 3.1. *Assume that $\Delta\varphi(\zeta) \geq 0$ and (2.2) hold. If there exists a constant p such that $-1+(2/3)^{\frac{1}{\lambda}} \leq -p \leq p(\zeta) \leq 0$, then every unbounded solution of (1.1) oscillates if and only if*

$$\sum_{\zeta=\zeta_0}^{\infty}\sum_{i=1}^{m}\mu_i(\zeta) f(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i)) = +\infty, \; \forall\, \theta > 0. \quad (3.1)$$

Proof. To prove sufficiency by contradiction, assume that $\{x(\zeta)\}$ is an unbounded non-oscillatory solution of (1.1). Without loss of generality, we may suppose that $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Then there exists $\zeta_1 \geq \zeta_0$ such that $x(\zeta) > 0$ and $x(\zeta-\kappa) > 0$. Then we have (2.1). From Lemma 2.1, $\{y(\zeta)\}$ satisfies one of the cases (C_1) and (C_2) for $\zeta \geq \zeta_2$, where $\zeta_2 \geq \zeta_1$. We consider each of two cases seperately.

Case 1. Let $\{y(\zeta)\}$ satisfies (C_1) for $\zeta \geq \zeta_2$. As $\{x(\zeta)\}$ is unbounded, there exists $N \geq \zeta_2$ such that $x(N) = max\{x(s) : \zeta_2 \leq s \leq N\}$. Then, $x(N) \leq y(\zeta) + (1-(2/3)^{\frac{1}{\lambda}})x(N-\kappa) < x(N)$, which is a contradiction.

Case 2. Let $\{y(\zeta)\}$ satisfies (C_2) for $\zeta \geq \zeta_2$. Since $\{\varphi(\zeta)(\Delta y(\zeta))^{\lambda}\}$ is a positive, non-increasing, and

$$\Delta y(\zeta) \leq \left(\frac{\varphi(\zeta_3)}{\varphi(\zeta)}\right)^{\frac{1}{\lambda}} \Delta y(\zeta_3) \text{ for } \zeta \geq \zeta_3, \text{ where } \zeta_3 \geq \zeta_2.$$

Summing this inequality from ζ_3 to $\zeta-1$, we have

$$y(\zeta) \leq y(\zeta_3) + \varphi^{\frac{1}{\lambda}}(\zeta_3)\Delta y(\zeta_3) R(\zeta,\zeta_3).$$

Since $R(\zeta) \to \infty$ as $\zeta \to \infty$ there exists $\theta > 0$ and $\zeta_4 \geq \zeta_3$ such that

$$y(\zeta) \leq \theta^{\frac{1}{\lambda}} R(\zeta) \text{ for } \zeta \geq \zeta_4. \quad (3.2)$$

Upon using $y(\zeta) \leq x(\zeta)$; (3.2) and by the assumption (2.2), we have

$$\begin{aligned} f(x(\zeta+\eta_i)) &\geq f(y(\zeta+\eta_i)) \\ &= \frac{f(y(\zeta+\eta_i))}{y^{\nu}(\zeta+\eta_i)} y^{\nu}(\zeta+\eta_i) \\ &\geq \frac{f(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i))^{\nu}} y^{\nu}(\zeta+\eta_i). \end{aligned}$$

Summing (1.1) from ζ to ∞, we have

$$\begin{aligned} \sum_{s=\zeta-\eta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta^{\frac{1}{\lambda}} R(s+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(s+\eta_i))^{\nu}} y^{\nu}(s+\eta_i) \\ \leq \varphi(\zeta)(\Delta y(\zeta))^{\lambda}, \quad \zeta \geq \zeta_4 \quad (3.3) \end{aligned}$$

Since $\Delta\varphi(\zeta) \geq 0$ and from (3.3), we obtain,

$$\Delta y(\zeta) \geq \left[\frac{1}{\varphi(\zeta+\eta)}\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta^{\frac{1}{\lambda}} R(s+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(s+\eta_i))^{\nu}} y^{\nu}(s+\eta_i)\right]^{\frac{1}{\lambda}}.$$

Summing the above inequality from ζ_4 to $\zeta-\eta-1$, we obtain

$$\begin{aligned} &y(\zeta-\eta) - y(\zeta_4) \\ &\geq \sum_{u=\zeta_4}^{\zeta-\eta-1}\left[\frac{1}{\varphi(u+\eta)}\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta^{\frac{1}{\lambda}} R(s+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(s+\eta_i))^{\nu}} \right. \\ &\left. \times y^{\nu}(s+\eta_i)\right]^{\frac{1}{\lambda}}. \end{aligned}$$

Using the increasing nature of $\{y(\zeta)\}$ in the above inequality, we have

$$y(\zeta) \geq \sum_{u=\zeta_4}^{\zeta-\eta-1}\left[\frac{1}{\varphi(u+\eta)}\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta^{\frac{1}{\lambda}} R(s+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(s+\eta_i))^{\nu}} y^{\nu}(s+\eta_i)\right]^{\frac{1}{\lambda}}. \quad (3.4)$$

Set

$$w(\zeta) = \sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta^{\frac{1}{\lambda}} R(s+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(s+\eta_i))^{\nu}} y^{\nu}(s+\eta_i). \quad (3.5)$$

From (3.4) and (3.5), and since $y(\zeta) > 0$, we obtain $y(\zeta) > R(\zeta,\zeta_4+\eta) w^{\frac{1}{\lambda}}(\zeta-\eta)$. Since $\lim_{\zeta\to\infty} R(\zeta) = \infty$, there exists $\zeta_5 \geq \zeta_4$ such that

$$R(\zeta,\zeta_4+\eta) \geq \frac{1}{2} R(\zeta) \text{ for } \zeta \geq \zeta_5. \quad (3.6)$$

Then

$$y(\zeta) > \frac{1}{2} R(\zeta) w^{\frac{1}{\lambda}}(\zeta-\eta) \text{ for } \zeta \geq \zeta_5, \quad (3.7)$$

and

$$\frac{y^{\nu}(\zeta)}{(\theta^{\frac{1}{\lambda}} R(\zeta))^{\nu}} \geq \frac{w^{\frac{\nu}{\lambda}}(\zeta-\eta)}{(2\theta^{\frac{1}{\lambda}})^{\nu}}.$$

Now,

$$\begin{aligned} &\Delta w(\zeta) \\ &= -\sum_{i=1}^{\zeta}\mu_i(\zeta)\frac{f(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i))}{(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i))^{\nu}} y^{\nu}(\zeta+\eta_i) \\ &\leq -\sum_{i=1}^{\zeta}\mu_i(\zeta)\frac{f(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i))}{(2\theta^{\frac{1}{\lambda}})^{\nu}} w^{\frac{\nu}{\lambda}}(\zeta-\eta+\eta_i) \leq 0. \end{aligned}$$

Thus, $\{w(\zeta)\}$ is non-increasing and hence $w^{\frac{\nu}{\lambda}}(\zeta-\eta+\eta_i)/w^{\frac{\nu}{\lambda}}(\zeta) \geq 1$, and

$$\begin{aligned} \Delta w^{1-\frac{\nu}{\lambda}}(\zeta) &\leq \left(1-\frac{\nu}{\lambda}\right) w^{-\frac{\nu}{\lambda}}(\zeta)\Delta w(\zeta) \\ &\leq -\frac{(1-\frac{\nu}{\lambda})}{(2\theta^{\frac{1}{\lambda}})^{\nu}}\sum_{i=1}^{m}\mu_i f(\theta^{\frac{1}{\lambda}} R(\zeta+\eta_i)). \end{aligned}$$

Summing this inequality from ζ_5 to $\zeta-1$, we get

$$\left[w^{1-\frac{\nu}{\lambda}}(s)\right]_{s=\zeta_5}^{\zeta-1} \leq -\frac{(1-\frac{\nu}{\lambda})}{(2\theta^{\frac{1}{\lambda}})^{\nu}}\sum_{s=\zeta_5}^{\zeta-1}\sum_{i=1}^{m}\mu_i(s) f(\theta^{\frac{1}{\lambda}} R(s+\eta_i)).$$

Since $\frac{\nu}{\lambda} < 1$ and $\{w(\zeta)\}$ is positive and non-increasing, we have

$$\sum_{s=\zeta_5}^{\zeta-1}\sum_{i=1}^{m}\mu_i f(\theta^{\frac{1}{\lambda}}R(s+\eta_i)) \leq \frac{(2\theta^{\frac{1}{\lambda}})^{\nu}}{(1-\frac{\nu}{\lambda})}w^{1-\frac{\nu}{\lambda}}(\zeta_5) < \infty.$$

This contradicts (3.1).

If $\{x(\zeta)\}$ is eventually negative, then $x(\zeta) < 0$ for $\zeta \geq \zeta_1$. Then we set $y(\zeta) := -x(\zeta)$ for $\zeta \geq \zeta_1$ in (1.1).

Using (C_6), we find

$$\Delta\Big[\varphi(\zeta)\left[\Delta(y(\zeta)+p(\zeta)y(\zeta-\kappa))\right]^{\lambda}\Big] + \sum_{i=1}^{m}\mu_i(\zeta)g(y(\zeta+\eta_i)) = 0 \text{ for } \zeta \geq \zeta_1,$$

where $g(u) = -f(-u)$ and g is also satifies (C_6). Then, proceeding as above, we find the same contradiction. This proves the oscillation of all solutions.

Next, we show that (3.1) is necessary. Suppose that (3.1) does not hold; so for some $\theta > 0$ the sum in (3.1) is finite. Then there exists $N \geq \eta$ such that

$$\sum_{\zeta=N}^{\infty}\sum_{i=1}^{m}\mu_i(\zeta)f(\theta^{\frac{1}{\lambda}}R(\zeta+\eta_i)) \leq \frac{\theta}{3} \tag{3.8}$$

we define the operator ϕ as follows:

For the sequence $\{x(\zeta)\}_{\zeta=N-\eta}^{\infty}$,

$$(\phi x)(\zeta) = \begin{cases} (\phi x)(N); & N-\eta \leq \zeta \leq N \\ -p(\zeta)x(\zeta-\kappa) & \\ +\sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\left[\frac{\theta}{3}\right.\right. & \\ \left.\left.+\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)f(x(s+\eta_i))\right]\right]^{\frac{1}{\lambda}}, & \zeta \geq N+1 \end{cases}$$

Now, consider the sequence $\{v^{(k)}(\zeta)\}$ of successive approximations defined by

$$v^{(1)}(\zeta) = \begin{cases} 0, & N-\eta \leq \zeta \leq N \\ (\frac{\theta}{3})^{\frac{1}{\lambda}}R(\zeta,N), & \zeta \geq N+1 \end{cases}$$

and for $k = 2,3,\ldots$.

$$v^{(k)}(\zeta) = (\phi v^{(k-1)})(\zeta)$$

Clearly for $\zeta \geq N+1$,

$$\left(\frac{\theta}{3}\right)^{\frac{1}{\lambda}}R(\zeta,N) \leq v^{(1)}(\zeta) \leq \theta^{\frac{1}{\lambda}}R(\zeta,N).$$

For, $\zeta \geq N+1$,

$$\begin{aligned} v^{(2)}(\zeta) =& (\phi v^{(1)})(\zeta) \\ =& -p(\zeta)v^{(1)}(\zeta-\kappa) + \sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\left[\frac{\theta}{3}\right.\right. \\ &\left.\left.+\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)f(v^{(1)}(s+\eta_i))\right]\right]^{\frac{1}{\lambda}} \end{aligned}$$

For $\zeta \geq N+1$, we have $v^{(1)}(\zeta) \leq \theta^{\frac{1}{\lambda}}R(\zeta,N)$ and using the increasing nature of f in the above inequality we obtain

$$\begin{aligned} v^{(2)}(\zeta) &\leq p\theta^{\frac{1}{\lambda}}R(\zeta-\kappa,N) + \sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\left[\frac{\theta}{3}+\frac{\theta}{3}\right]\right]^{\frac{1}{\lambda}} \\ &\leq p\theta^{\frac{1}{\lambda}}R(\zeta,N) + \left(\frac{2\theta}{3}\right)^{\frac{1}{\lambda}}R(\zeta,N) \\ &= \left(p+\left(\frac{2}{3}\right)^{\frac{1}{\lambda}}\right)\theta^{\frac{1}{\lambda}}R(\zeta,N) \\ &\leq p\theta^{\frac{1}{\lambda}}R(\zeta,N) \end{aligned}$$

Thus, for $\zeta \geq N+1$, we have

$$\left(\frac{\theta}{3}\right)^{\lambda}R(\zeta,N) \leq v^{(1)}(\zeta) \leq \theta^{\frac{1}{3}}R(\zeta,N).$$

Now, for $\zeta \geq N+1$

$$\begin{aligned} v^{(2)}(\zeta) &= (\phi v^{(1)})(\zeta) \\ &= \sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\left[\frac{\theta}{3}+\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)f(x(s+\eta_i))\right]\right]^{\frac{1}{\lambda}} \\ &\geq \sum_{u=N}^{\zeta-1}\left(\frac{1}{\varphi(u)}\frac{\theta}{3}\right)^{\frac{1}{\lambda}} \\ &= \left(\frac{\theta}{3}\right)^{\frac{1}{\lambda}}R(\zeta,N) \end{aligned}$$

Thus, we have, for $\zeta \geq N+1$,

$$\left(\frac{\theta}{3}\right)^{\frac{1}{\lambda}}R(\zeta,N) = v^{(1)}(\zeta) \leq v^{(2)}(\zeta) \leq \theta^{\frac{1}{\lambda}}R(\zeta,N).$$

By mathematics induction, we can easily prove that for $k > 1$.

$$\left(\frac{\theta}{3}\right)^{\frac{1}{\lambda}}R(\zeta,N) \leq v^{(k-1)}(\zeta) \leq v^{(k)}(\zeta) \leq \theta^{\frac{1}{\lambda}}R(\zeta,N)$$

for $\zeta \geq N+1$. Thus, $\{v^{(k)}(\zeta)\}$ is a pointwise convergent to some sequence $v^* = \{v^*(\zeta)\}$. By means of the Lebesgue dominated convergence theorem, we obtain

$(\phi v^*)(\zeta) = v^*(\zeta)$. We can easily show that $\{v^*(\zeta)\}$ is an eventually positive solution of the equation (1.1) for $\zeta \geq N-\eta$. This contradiction shows that (3.1) is necessary condition. This completes the proof. □

Theorem 3.2. *Assume that $\Delta\varphi(\zeta) \geq 0$ and (2.2). If there exists a constant p such that $-1+\left(\frac{2}{3}\right)^{\frac{1}{\lambda}} \leq -p \leq p(\zeta) \leq 0$, then every solution of (1.1) oscillates or converges to zero if and only if (3.1) holds for every $\theta > 0$.*

Proof. To prove sufficiency by contradiction, we assume that $\{x(\zeta)\}$ is an eventually positive solution of (1.1) which does not converges to zero. Then, there exist $\zeta_1 \geq \zeta_0$ such that $x(\zeta) > 0$, $x(\zeta-\kappa) > 0$ and $x(\zeta-\eta_i) > 0$ for $\zeta \geq \zeta_1$ and $i = 1,2,\dots,m$. Then we have (2.1). From Lemma 2.1, $\{y(\zeta)\}$ satisfies one of the cases (C_1) and (C_2) for $\zeta \geq \zeta_2$, where $\zeta_2 \geq \zeta_1$. We consider each of two cases separately.

Case 1. Let $\{y(\zeta)\}$ satisfies (C_1) for $\zeta \geq \zeta_2$. Therefore,

$$\begin{aligned}
0 \geq \lim_{\zeta\to\infty} y(\zeta) &= \limsup_{\zeta\to\infty} y(\zeta)\\
&\geq \limsup_{\zeta\to\infty} (x(\zeta) - px(\zeta-\kappa))\\
&\geq \limsup_{\zeta\to\infty} x(\zeta) + \liminf_{\zeta\to\infty}(-px(\zeta-\kappa))\\
&= (1-p)\limsup_{\zeta\to\infty} x(\zeta)
\end{aligned}$$

implies that $\limsup_{\zeta\to\infty} y(\zeta) = 0$ and hence $\lim_{\zeta\to\infty} x(\zeta) = 0$, which contradicts the assumption that $\{x(\zeta)\}$ does not converges to zero.

Case 2. Let $\{y(\zeta)\}$ satisfies (C_2) for $\zeta \geq \zeta_2$. The case follows from Theorem 3.1. Hence (3.1) is a sufficient condition.

The case where $\{x(\zeta)\}$ is an eventually negative solution is similar and we omit it here.

The necessary part is the same as in the Theorem 3.1. Thus, the proof of the theorem is complete. □

Theorem 3.3. *Assume that* (2.3) *hold and exists a constant p such that* $-1 \leq -p \leq p(\zeta) \leq 0$, *hold. Suppose that*

$$\sum_{\zeta=\zeta_0}^{\infty}\left[\frac{1}{\varphi(\zeta)}\sum_{s=\zeta}^{\infty}\mu_j(s)\right]^{\frac{1}{\lambda}} = +\infty \tag{3.9}$$

for some j, then every solution of (1.1) *is either oscillatory or converges to zero.*

Proof. To prove it by contradiction, suppose that $\{x(\zeta)\}$ is an eventually positive solution of (1.1) which does not converges to zero and we use same type of argument as in the proof of Theorem 3.2 for the case (C_1). Let us consider $\{y(\zeta)\}$ satisfies (C_2) for $\zeta \geq \zeta_2$. By Remark 2.2, there exist a constant $\theta > 0$ and $\zeta_3 \geq \zeta_2$ such that $y(\zeta+\eta_i) \geq \theta$ for $\zeta \geq \zeta_3$ and $i = 1,2,3,\dots,m$.

Upon using $y(\zeta) \leq x(\zeta)$ and by assumption (2.3), we have

$$\begin{aligned}
f(x(\zeta+\eta_i)) &\geq f(y(\zeta+\eta_i))\\
&= \frac{f(y(\zeta+\eta_i))}{y^{\nu}(\zeta+\eta_i)} y^{\nu}(\zeta+\eta_i)\\
&\geq \frac{f(\theta)}{\theta^{\nu}} y^{\nu}(\zeta+\eta_i).
\end{aligned}$$

Summing (1.1) from ζ to ∞, we have

$$\lim_{A\to\infty}\left[\varphi(\zeta)(\Delta y(\zeta))^{\lambda}\right]_{\zeta}^{\infty} + \sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta)}{\theta^{\nu}}y^{\nu}(s+\eta_i) \leq 0. \tag{3.10}$$

Using that $\varphi(\zeta)(\Delta y(\zeta))^{\lambda}$ is positive and non-increasing, we have

$$\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta)}{\theta^{\nu}}y^{\nu}(s+\eta_i) \leq \varphi(\zeta)(\Delta y(\zeta))^{\lambda}$$

for all $\zeta \geq \zeta_3$ and all $j \in \{1,2,\dots,m\}$. Therefore,

$$\left[\frac{1}{\varphi(\zeta)}\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{f(\theta)}{\theta^{\nu}}y^{\nu}(s+\eta_i)\right]^{\frac{1}{\lambda}} \leq \Delta y(\zeta). \tag{3.11}$$

Dividing by $y^{\frac{\nu}{\lambda}}(\zeta-\eta_j)$ and then summing from ζ_3 to ∞, we have

$$\begin{aligned}
&\left(\frac{f(\theta)}{\theta^{\nu}}\right)^{\frac{1}{\lambda}}\sum_{u=\zeta_3}^{\infty}\left[\frac{1}{\varphi(\zeta)}\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)\frac{y^{\nu}(s+\eta_i)}{y^{\nu}(u+\eta_j)}\right]^{\frac{1}{\lambda}}\\
&\leq \sum_{u=\zeta_3}^{\infty}\frac{\Delta y(u)}{y^{\frac{\nu}{\lambda}}(u+\eta_j)}\\
&\leq \sum_{u=\zeta_3}^{\infty}\frac{\Delta y(u)}{y^{\frac{\nu}{\lambda}}(u+1)}\\
&\leq \frac{y^{1-\frac{\nu}{\lambda}}(\zeta_3)}{\frac{\nu}{\lambda}-1}.
\end{aligned}$$

Since $\{y(\zeta)\}$ is increasing sequence, for $s \geq u$ we have $y^{\nu}(s+\eta_i) \geq y^{\nu}(u+\eta_i)$. Note that the summands $y^{\nu}(s+\eta_i)/y^{\nu}(u+\eta_j)$ are positive for all i, j and equal to 1 when $i = j$. Hence, the above inequality becomes

$$\left(\frac{f(\theta)}{\theta^{\nu}}\right)^{\frac{1}{\lambda}}\left[\sum_{u=\zeta_3}^{\infty}\frac{1}{\varphi(u)}\sum_{i=1}^{m}\mu_i(s)\right]^{\frac{1}{\lambda}} \leq \frac{y^{1-\frac{\nu}{\lambda}}(\zeta_3)}{\frac{\nu}{\lambda}-1} < \infty.$$

This contradicts (3.9). The case where $\{x(\zeta)\}$ is eventually negative solution is omitted since it can be dealt similarly. This proves the oscillation of all solutions. □

Theorem 3.4. *Assume that there exists a constant p such that* $-1 < -p \leq p(\zeta) \leq 0$. *If*

$$\sum_{\zeta=\zeta_0}^{\infty}\left[\frac{1}{\varphi(u)}\sum_{s=\zeta}^{\infty}\sum_{i=1}^{m}\mu_i(s)\right]^{\frac{1}{\lambda}} < \infty \tag{3.12}$$

holds, then (1.1) *admits a positive bounded solution.*

Proof. Due to (3.12), it is possible to find $N \geq \eta$ such that

$$\sum_{u=N}^{\infty}\left[\frac{1}{\varphi(u)}\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)\right]^{\frac{1}{\lambda}} \leq \frac{1-p}{(5f(1))^{\frac{1}{\lambda}}}, \quad \theta > 0 \tag{3.13}$$

We define the operator ϕ as follows: For the sequence $\{x(\zeta)\}_{\zeta=N-}^{\infty}$

$$(\phi x)(\zeta) = \begin{cases} (\phi x)(N); & \zeta_0 \le \zeta \le N \\ -p(\zeta)x(\zeta-\kappa) & \\ +\frac{1-p}{5}\sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s) \right. & \\ \left. \times f(x(s+\eta_i))\right]^{\frac{1}{\lambda}}, & \zeta \ge N+1 \end{cases}$$

Now, consider the sequence $\{v^{(k)}(\zeta)\}$ of successive approximations defined by

$$v^{(1)}(\zeta)\begin{cases} 0, & \zeta_0 \le \zeta \le N \\ \frac{1-p}{5}, & \zeta \ge N+1 \end{cases}$$

and for $k = 2, 3, \ldots$

$$v^{(k)}(\zeta) = (\phi v^{(k-1)})(\zeta).$$

Clearly,

$$\frac{1-p}{5} \le v^{(1)}(\zeta) \le 1, \text{ for } \zeta \ge N+1.$$

Now $v^{(2)}(\zeta) = (\phi v^{(1)})(\zeta) \ge \frac{1-p}{5}$, $\zeta \ge N+1$. Also for $\zeta \ge N+1$,

$$\begin{aligned} v^{(2)}(\zeta) =& (\phi v^{(1)})(\zeta) \\ =& -p(\zeta)v^{(1)}(\zeta-\kappa) + \frac{1-p}{5} \\ &+ \sum_{u=N}^{\zeta-1}\left[\frac{1}{\varphi(u)}\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)f(v^{(1)}(s+\eta_i))\right]^{\frac{1}{\lambda}} \\ \le& p + \frac{1-p}{5} + (f(1))^{\frac{1}{\lambda}}\sum_{u=N}^{\zeta}\left[\frac{1}{\varphi(u)}\sum_{s=u}^{\infty}\sum_{i=1}^{m}\mu_i(s)\right]^{\frac{1}{\lambda}} \\ =& p + \frac{1-p}{5} + \frac{1-p}{5}. \\ =& \frac{3p+2}{5} < 1. \end{aligned}$$

Hence

$$\frac{1-p}{5} \le v^{(1)}(\zeta) \le v^{(2)}(\zeta) \le 1.$$

By mathematical induction, we can easily show that for $k > 1$.

$$\frac{1-p}{5} \le v^{(k-1)}(\zeta) \le v^{(k)}(\zeta) \le 1, \quad \zeta \ge N+1.$$

The rest of the proof follows from Theorem 3.1. This completes the proof of the theorem. □

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