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$r-$Regular hypergraphs

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Abstract

In this paper, we have obtained the values of r for which $r-$regular simple hypergraphs exist, given the vertex set as $\{x_1, x_2, \cdots, x_n\}$. We have also discussed the procedure of constructing such a $r-$regular hypergraph. Further we have investigated the least value of the order of H for which $r-$regular simple hypergraph H exists, for a given value of r.

Keywords

Simple hypergraph, $r-$regular, $r-$uniform.

AMS Subject Classification

05C65.

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Contents

1. Introduction

Hypegraphs are a generalisation of graphs and hence many of the definitions of graphs carry verbatim to hypergraphs. Hypergraphs have important applications in science and engineering. In this paper we have discussed some properties of hypergraphs. We have obtained the values of r for which $r-$regular simple hypergraph exist, given the vertex set $X = \{x_1, x_2, \cdots, x_n\}$. We have also obtained bounds of n, i.e., the order of H for which a $r-$regular simple hypergraph can be constructed, given the value of r. Finally we have deduced the values of r for which $r-$regular $r-$uniform simple hypergraphs exist. The notations and terminology used in this paper are the same as in [1] and [2]. We recall that a hypergraph is called $r-$regular if $d_H(x) = r$ for all $x \in H$. We prove the existence of $r-$regular simple hypergraphs for all values of $r \leq 2^{n-1} - 1$. We have also given the procedure for constructing such a $r-$regular hypergraph.

2. Main results

Definition 2.1. *[1] A hyper graph $H = (X, \xi) = (E_1, E_2, \cdots, E_m) = (E_i : i \in M)$ is a family ξ of subsets E_i of a set $X = \{x_j : j \in M\}$ of vertices. The sets E_i are called edges or hyper edges. The elements of X are called the vertices of H.*

The order of a hypergraph $H = (X, \xi)$ is the cardinality of X and its size is the cardinality of ξ. If $|E_i| = r$ for all $i \in M$, the hyper graph is said to be $r-$uniform, where $|E_i|$ denotes the cardinality of E_i.
The degree of a vertex in a hypergraph is defined as follows

Definition 2.2. *[1] Given a hypergraph $H = (X, \xi)$, the degree of a vertex x is defined to be the maximum number of edges different from $\{x\}$ that form a partial family $\{E_j : j \in J\}$ with $E_i \cap E_j = \{x\}$, where $i, j \in J$ and $i \neq j$.*

Let the degree of x be denoted by $d_H(x)$. Clearly, $d_H(x) = 0$ if and only if the only edge that contains x is $\{x\}$. If $d_H(x) = r$ for all $x \in H$, the hypergraph is said to be $r-$regular.

A simple hypergraph is defined as follows

Definition 2.3. *[1] Let $H = (X, \xi)$ be a hypergraph. If the edges E_i are all distinct, the hypergraph is said to be simple and ξ is a set of non-empty subsets of X.*

We have proved the following result.

Theorem 2.4. *Given $X = \{x_1, x_2, \cdots, x_n\}$, there exist a $r-$regular simple hypergraph for all values of $r = 1, 2, \cdots, 2^{n-1} - 1$.*

Proof. The given vertex set is $X = \{x_1, x_2, \cdots, x_n\}$.
When $r = 1$, consider the hypergraph (X, ξ_1^1), where $\xi_1^1 = \{E_1\}$ and $E_1 = \{x_1, x_2, \cdots, x_n\}$.

Then (X,ξ_1^1) is the $1-$regular hypergraph.
When $r=2$ consider the hypergraph (X,ξ_1^2), where $\xi_1^2 = \{E_{11}^2, E_{21}^2, \cdots, E_{n1}^2\}$ and $E_{i1}^2 = \{x_i, x_{i+1}\}$ where the suffixes involving i are taken over modulo n.
Then (X,ξ_1^2), is a $2-$regular simple hypergraph.
Again consider (X,ξ_2^2), where $\xi_2^2 = \{E_{12}^2, E_{22}^2, \cdots, E_{n2}^2\}$ and $E_{i2}^2 = \{x_i, x_{i+2}\}$ where the suffixes involving i are taken over modulo n. Then (X,ξ_2^2) is a $2-$regular hypergraph.
Proceeding in this way we construct $2-$regular hypergraphs of the form $(X,\xi_1^2),(X,\xi_2^2),(X,\xi_3^2),\cdots,(X,\xi_{t-1}^2)$ where $t-1 < \lceil \frac{n}{2} \rceil$, where $\lceil x \rceil$ is the smallest integer $\geq x$.
It may be observed that these are all mutually edge disjoint $2-$regular hypergraphs as the families of edges $\xi_1^2, \xi_2^2, \cdots \xi_{t-1}^2$ are mutually disjoint where $t-1 < \lceil \frac{n}{2} \rceil$.
Thus we have $(t-1)$ $2-$regular hypergraphs containing n edges and these edges are formed by pairs of vertices from the set X.
Altogether we have ${}^nC_2$ distinct pairs and the number of edges that have been considered in these $(t-1)$ $2-$regular hypergraphs will be $n(t-1)$.
If n is odd, the edges that have been formed with pairs of vertices will be exactly partitioned into $(t-1)$ disjoint pairs. i.e.,$\frac{n-1}{2}$ disjoint pairs, which form the $\frac{n-1}{2}$ $2-$regular hypergraphs.
If n is even besides these $\frac{n-1}{2}$ $2-$regular hypergraphs, there will be exactly $\frac{n}{2}$ edges leftover which do not figure in any of these families $\xi_1^2, \xi_2^2, \cdots, \xi_{\frac{n-1}{2}}^2$ and these leftover edges are $\xi_{\frac{n}{2}}^2 = \{E_1, E_2, \cdots, E_{\frac{n}{2}}\}$ where $E_i = \{x_i, x_{\frac{n}{2}+i}\}$, $1 \leq i \leq \frac{n}{2}$.
It can be easily seen that $(X,\xi_{\frac{n}{2}}^2)$ forms a $1-$regular hypergraph.
If $r=3$, we form first the edges by taking 3 vertices at a time and altogether there are ${}^nC_3$ edges.
From among these edges we construct several $3-$regular hypergraphs as follows.
$\xi_1^3 = \{E_{11}^3, E_{21}^3, \cdots, E_{n1}^3\}$, where $E_{i1}^3 = \{x_i, x_{i+1}, x_{i+2}\}$
$\xi_2^3 = \{E_{12}^3, E_{22}^3, \cdots, E_{n2}^3\}$, where $E_{i2}^3 = \{x_i, x_{i+1}, x_{i+3}\}$
$\xi_3^3 = \{E_{13}^3, E_{23}^3, \cdots, E_{n3}^3\}$, where $E_{i3}^3 = \{x_i, x_{i+1}, x_{i+4}\}$
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$\xi_{t-1}^3 = \{E_{1(t-1)}^3, E_{2(t-1)}^3, \cdots, E_{n(t-1)}^3\}$, where
$E_{i(t-1)}^3 = \{x_i, x_{i+1}, x_{i+t}\}$ where the suffixes involving i are taken over modulo n and $t = n-2$.
we cannot proceed further for,
$\xi_t^3 = \xi_{n-2}^3 = \{(x_1,x_2,x_n),(x_2,x_3,x_{n+1}),\cdots\}$ but $\{x_1,x_2,x_n\}$ is already appearing in ξ_1^3, namely $E_{n1}^3 = \{x_n, x_{n+1}, x_{n+2}\}$.
Thus there are exactly $n-3$ mutually edge disjoint $3-$regular hypergraphs formed out of these ${}^nC_3$ edges.
There are still some more edges that are left and they are of the form $\xi_3^1 = \{E_i^3\}$ where $E_i^3 = \{x_i, x_{i+2}, x_{i+4}\}$ or $\{x_i, x_{i+3}, x_{i+6}\}$ etc.,.
The number of such edges will be ${}^nC_3 - n(n-3) = \frac{n}{6}[n^2 - 9n + 20]$.
There can be $3-$regular hypergraphs from the remaining edges if and only if $n \geq 7$.
Proceeding in this way in general we can construct an $r-$regular hypergraph as follows:
Form the edges by taking r vertices at a time where $\xi_1^r = \{E_{11}^r, E_{21}^r, \cdots E_{n1}^r\}$ where $E_{i1}^r = \{x_i, x_{i+1}, \cdots, x_{i+r-1}\}$.
clearly (X,ξ_1^r) is a $r-$regular hypergraph.
In this way we construct all $r-$regular simple hypergraphs for all values of $r < n$.
For $r \geq n$ the construction of $r-$regular hypergraph is as follows.
If $r = n$ then $(X, \xi_1^1 \cup \xi_1^{n-1})$ is a $n-$regular hypergraph.
Similarly $(X, \xi_1^2 \cup \xi_1^{n-1})$ is a $(n+1)-$regular hypergraph.
Proceeding in this way we can construct regular hypergraphs up to the stage $(X, \xi_1^1 \cup \xi_1^2 \cup \cdots \cup \xi_1^{n-1})$ giving a $r-$regular simple hypergraph, where $r = 1+2+3+\cdots+(n-1) = \frac{(n-1)n}{2} = {}^nC_2$.
For $r > {}^nC_2$ we adopt the following procedure.
Choose the appropriate $s-$regular hypergraphs where $1 \leq s \leq {}^nC_2$ Such that the sum of superscripts of the families is equal to r.
For example if $r = {}^nC_2 + 1$ we obtain the hypergraph $(X, \xi_1^2 \cup \xi_2^2 \cup \xi_1^2 \cup \cdots \cup \xi_1^{n-1})$.
Now consider the set $X = \{x_1, x_2, \cdots, x_n\}$.
Construct all possible edges from these vertices and they are nothing but all possible subsets of X except the empty set and singleton sets.
Each vertex x_i appears in exactly ${}^{(n-1)}C_1 + {}^{(n-1)}C_2 + \cdots + {}^{(n-1)}C_{(n-1)}) = 2^{(n-1)} - 1$ edges.
Thus the degree of each vertex is $2^{n-1} - 1$.
Hence the maximum value of r for which the $r-$regular simple hypergraph can be constructed with X as the vertex set is $2^{n-1} - 1$. □

Next we consider the converse of the above problem, i.e., for a given r, what is the least value of n (order of H) for which we have a $r-$regular simple hypergraph?
We have obtained the following interesting result.

Theorem 2.5. *If r is a given positive integer such that $2^t - 1 < r < 2^{t+1} - 1$. Then the smallest set in which the $r-$regular simple hypergraph can be constructed is $X = \{x_1, x_2, \cdots, x_{t+2}\}$. If $r = 2^t - 1$ then the smallest set in which the $r-$regular simple hypergraph can be constructed is $X = \{x_1, x_2, \cdots, x_{t+1}\}$.*

Proof. From the above Theorem-2.4, if $X = \{x_1, x_2, \cdots, x_n\}$ the maximum value of r for which a $r-$regular hypergraph exists with its vertex set X is $2^{n-1} - 1$. From this theorem the result follows at once. □

The following is an immediate consequence of Theorem-2.4.

Theorem 2.6. *If $X = \{x_1, x_2, \cdots, x_n\}$, then there exist $r-$regular $r-$uniform hypergraphs for all values of r, $2 \leq r \leq n-1$.*

References

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