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# Necessary and sufficient conditions for oscillation of solutions to second-order non-linear difference equations with delay argument

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**Abstract**
In this paper, we established necessary and sufficient conditions for the oscillation of all solutions of second-order half-linear delay difference equation of the form

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\xi})+\mu(\zeta)x^{\nu}(\eta(\zeta))=0;\quad \zeta\geq\zeta_0,$$

Under the assumption $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\xi}}(\zeta)}=\infty$, we consider the cases when $\xi>\nu$ and $\xi<\nu$. Further, some illustrate examples showing applicability of the new results are included.

**Keywords**
Oscillation, non-oscillation, second-order, non-linear, delay, difference equations.

**AMS Subject Classification**
39A12, 39A13, 39A21.

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## Contents



## 1. Introduction

The main feature of this paper is having an oscillation condition that is necessary and sufficient at the same time. We mainly consider the following second-order half-linear delay difference equation

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\xi})+\mu(\zeta)x^{\nu}(\eta(\zeta))=0;\quad \zeta\geq\zeta_0,\quad (1.1)$$

by considering two cases $\xi>\nu$ and $\xi<\nu$, where $\Delta$ is the forward difference operator defined by $\Delta x(\zeta)=x(\zeta+1)-x(\zeta)$. We suppose that the following assumptions hold:

$(A_1)$ $\xi$ and $\nu$ are quotient of odd positive integers ;

$(A_2)$ $\{\varphi(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of positive real numbers;

$(A_3)$ $\{\mu(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of non-negative real numbers and $\mu(\zeta)$ is not identically zero for sufficiently large values of $\zeta$;

$(A_4)$ $\{\eta(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of positive integers such that $\Delta\eta(\zeta)\geq 0$ and $\eta(\zeta)\leq\zeta$ for $\zeta\geq\zeta_0$ and $\lim_{\zeta\to\infty}\eta(\zeta)=\infty$;

$(A_5)$ $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\xi}}(\zeta)}=\infty$. Letting $R(\zeta)=\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\xi}}(\zeta)}$, we have $R(\zeta)\to\infty$ as $\zeta\to\infty$.

By a solution of (1.1), we mean a real sequence $\{x(\zeta)\}$ satisfying equation (1.1) for all $\zeta\geq\zeta_0-M$ where $M=\inf_{i\geq\zeta_0}\{\eta(i)\}$. A nontrival solution $\{x(\zeta)\}$ is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise. We will say that an equation is oscillatory if all its solution are oscillatory.

The problem of determining the oscillation and nonoscillation of solution of difference equations has been a very active area of research in the last ten years, and for surveys of recent results, we refer the reader to the monographs by Agarwal [2] and Agarwal and Wong [4]. Half-linear equations derive their name from the fact that if $\{x(\zeta)\}$ is a solution, then so is $\{(x(\zeta))\}$ for any constant c. Half-linear equations and their generalizations have received a good bit of attention in the literature in the last few years, and we cite as recent contributions the papers of Chen et al. [6, 7], Li and Yeh [11], Thandapani et al. [15–20], and Wong and Agarwal [22]. The oscillation theory of discrete analogues of delay differential equation has also attracted growing attention in the recent few years. The reader is referred to [1, 5, 10, 12, 14, 21] and the reference cited there in. For the general theory of second-order difference equation, the reader is referred to [2–4].

In [8], Dinakar et al. established sufficient conditions for oscillation of all solutions of second-order half-linear advanced difference equation

$$\Delta(a(\zeta)(\Delta x(\zeta))^{\nu})+\mu(\zeta)x^{\nu}(\eta(\zeta))=0, \quad \zeta\geq\zeta_0$$

under the condition that $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{a^{\frac{1}{\nu}}(\zeta)}<\infty$.

Murugesan and Jayakumar [13] determined sufficient condition for the oscillation of all solution of second-order half-linear advanced difference equations of the form

$$\Delta(\varphi(\zeta)\Delta x(\zeta)^{\nu})+\mu(\zeta)x^{\nu}(\zeta+\eta)=0, \quad \zeta\geq\zeta_0$$

under the condition that $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\nu}}(\zeta)}<\infty$.

Gopalakrishan et al. [9] derived new oscillatory conditions for the second-order noncanonical difference equations of the type

$$\Delta(\varphi(\zeta)\Delta x(\zeta))+\mu(\zeta)x(\zeta+\eta)=0, \quad \zeta\geq\zeta_0,$$

where $\eta$ is an integer.

In this paper, we derive necessary and sufficient condition for oscillation of all solutions to (1.1). Under the assumption $\sum_{\zeta=\zeta_0}^{\infty}\frac{1}{\varphi^{\frac{1}{\xi}}(\zeta)}=\infty$, we consider the cases when $\xi>\nu$ and $\xi<\nu$.

In the following sections, we presume that all functional inequalities are satisfied, eventually, that is, for all $\zeta$ large enough.

## 2. Main Results

**Lemma 2.1.** *Assume that $(A_1)-(A_5)$ hold. If $\{x(\zeta)\}$ is an eventually positive solution of (1.1), then $\{x(\zeta)\}$ satisfies $\Delta x(\zeta)>0$ and*

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\nu})<0 \quad \textit{for all large } \zeta. \tag{2.1}$$

*Proof.* Since $\{x(\zeta)\}$ is an eventually positive solution of (1.1), then there exists an integer $\zeta_1\geq\zeta_0$ such that $x(\zeta)>0$ and $x(\eta(\zeta))>0$ for $\zeta\geq\zeta_1$. From (1.1), it follows that

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\xi})=-\mu(\zeta)x^{\nu}(\eta(\zeta))\leq 0, \quad \zeta\geq\zeta_1. \tag{2.2}$$

This shows that $\{\varphi(\zeta)(\Delta x(\zeta))^{\xi}\}$ is non-increasing sequence. We claim that $\varphi(\zeta)(\Delta x(\zeta))^{\xi}>0$ for $\zeta\geq\zeta_1$. On the contrary, assume that $\varphi(\zeta)(\Delta x(\zeta))^{\xi}\leq 0$ for some $\zeta\geq\zeta_1$, then we can find $\zeta^*\geq\zeta_1$ and $k_1>0$ such that $\varphi(\zeta)(\Delta x(\zeta))\leq -k_1$ for all $\zeta\geq\zeta^*$. Summing the inequality $\Delta x(\zeta)\geq\left(\frac{-k}{\varphi(\zeta)}\right)^{\frac{1}{\xi}}$ from $\zeta^*$ to $\zeta-1$ $(\zeta>\zeta^*)$, we obtain

$$x(\zeta)\leq x(\zeta^*)-k_1^{\frac{1}{\xi}}\sum_{s=\zeta^*}^{\zeta-1}\frac{1}{\varphi^{\frac{1}{\xi}}(s)}\to\infty, \text{as } \zeta\to\infty.$$

This contradicts $\{x(\zeta)\}$ being a positive solution. So, $\varphi(\zeta)(\Delta x(\zeta))^{\xi}>0$ for $\zeta\geq\zeta_1$. Since, $\varphi(\zeta)>0$ then $\Delta x(\zeta)>0$ for $\zeta\geq\zeta_1$. □

### 2.1 The Case $\xi>\nu$ .

In this subsection we assume that there exists a constant $\beta$ such that $0<\nu<\beta<\xi$ and

$$u^{\nu-\beta}\geq v^{\nu-\beta}, \quad for \quad o<u\leq v. \tag{2.3}$$

**Lemma 2.2.** *Assume that $(A_1)-(A_5)$ hold. If $\{x(\zeta)\}$ is an eventually positive solution of (1.1), then there exists $\zeta_1\geq\zeta_0$ and $k>0$ such that for $\zeta\geq\zeta_1$, the following holds*

$$x(\zeta)\leq k^{\frac{1}{\xi}}R(\zeta) \tag{2.4}$$

$$(R(\zeta)-R(\zeta_1))\left[\sum_{s=\zeta}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(s))\right)^{\nu-\beta}x^{\beta}(\eta(s))\right]^{\frac{1}{\xi}}\leq x(\zeta) \tag{2.5}$$

*Proof.* By Lemma 2.1, $\{\varphi(\zeta)(\Delta x(\zeta))^{\xi}\}$ is positive and non-increasing sequence. Then there exists $k>0$ and $\zeta_1\geq\zeta_0$ such that

$$\varphi(\zeta)(\Delta x(\zeta))^{\xi}\leq k.$$

Summing the above inequality from $\zeta_1$ to $\zeta-1$, we have

$$x(\zeta)\leq x(\zeta_1)+k^{\frac{1}{\xi}}\left(R(\zeta)-R(\zeta_1)\right).$$

Since $\lim_{\zeta\to\infty}R(\zeta)=\infty$, then the last inequality becomes that $x(\zeta)\leq k^{\frac{1}{\xi}}R(\zeta)$ for $\zeta\geq\zeta_1$, which is (2.4).

By (2.4) and the assumption (2.3), we have

$$x^{\nu}(\eta(\zeta)) = x^{\nu-\beta}(\eta^{\zeta})x^{\beta}(\eta(\zeta)) \geq \left(k^{\frac{1}{\xi}}R(\eta(\zeta))\right)^{\nu-\beta} x^{\beta}(\eta(\zeta)).$$

Summing (1.1) from $\zeta$ to $\infty$, we have

$$\lim_{l\to\infty}\left[\varphi(\zeta)(\Delta x(\zeta))^{\xi}\right]_t^l + \sum_{s=\zeta}^{\infty}\mu(\zeta)\left(k^{\frac{1}{\xi}}R(\eta(\zeta))\right)^{\nu-\beta}x^{\beta}(\eta(\zeta)) \leq 0.$$

Using that $\{\varphi(\zeta)(\Delta x(\zeta))^{\xi}\}$ is positive and non-increasing, we have

$$\sum_{s=\zeta}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(s))\right)^{\nu-\beta} x^{\beta}(\eta(s)) \leq \varphi(\zeta)(\Delta x(\zeta))^{\xi}, \quad for \quad \zeta \geq \zeta_1.$$

Therefore,

$$\Delta x(\zeta) \geq \left[\frac{1}{\varphi(\zeta)}\sum_{s=\zeta}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(\zeta))\right)^{\nu-\beta}x^{\beta}(\eta(s))\right]^{\frac{1}{\xi}}. \tag{2.6}$$

Summing (2.6) from $\zeta_1$ to $\zeta-1$, we obtain

$$\begin{aligned} x(\zeta) &\geq \sum_{u=\zeta_1}^{\zeta-1}\left[\frac{1}{\varphi(u)}\sum_{s=u}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(s))\right)^{\nu-\beta}x^{\beta}(\eta(s))\right]^{\frac{1}{\xi}} \\ &\geq (R(\zeta)-R(\zeta_1))\left[\sum_{s=\zeta}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(s))\right)^{\nu-\beta} \times x^{\beta}(\eta(s))\right]^{\frac{1}{\xi}}, \end{aligned}$$

which is (2.5). □

**Theorem 2.3.** *Assume that the $(A_1)-(A_5)$ hold. The even solution of (1.1) is oscillatory if and only if*

$$\sum_{s=\zeta_0}^{\infty}\mu(s)R^{\nu}(\eta(\zeta)) = +\infty. \tag{2.7}$$

*Proof.* To prove sufficiency by contradiction, assume that $\{x(\zeta)\}$ is a non-oscillatory solution of (1.1). Without loss of generality, we may assume that $\{x(\zeta)\}$ is an eventually positive. Then Lemma 2.1 and 2.2 hold for $\zeta \geq \zeta_1$. So,

$$x(\zeta) \geq (R(\zeta)-R(\zeta_1))w^{\frac{1}{\xi}}(\zeta) \quad forall \quad \zeta \geq \zeta_1,$$

where

$$w(\zeta) = \sum_{s=\zeta}^{\infty}\mu(s)\left(k^{\frac{1}{\xi}}R(\eta(s))\right)^{\nu-\beta}x^{\beta}(\eta(s)) \geq 0.$$

Since $\lim_{\zeta\to\infty}R(\zeta)=\infty$, there exists $\zeta_2 \geq \zeta_1$, such that $R(\zeta)-R(\zeta_1) \geq \frac{1}{2}R(\zeta)$ for $\zeta \geq \zeta_2$. Then

$$x(\zeta) > \frac{1}{2}R(\zeta)w^{\frac{1}{\xi}}(\zeta) \quad for \quad \zeta \geq \zeta_2,$$

and

$$x^{\beta}/\left(k^{\frac{1}{\xi}}R(\zeta)\right)^{\beta} \geq w^{\frac{\beta}{\xi}}(\zeta)/(2k^{\frac{1}{\xi}})^{\beta}.$$

Taking the difference of $w(\zeta)$, we have

$$\begin{aligned} \Delta w(\zeta) &= -\mu(\zeta)\left(k^{\frac{1}{\xi}}R(\eta(\zeta))^{\nu-\beta}x^{\beta}(\eta(\zeta))\right) \\ &\leq -\mu(\zeta)\left(k^{\frac{1}{\xi}}R(\eta(\zeta))\right)^{\nu}w^{\frac{\beta}{\xi}}(\eta(\zeta))(2k^{\frac{1}{\xi}})^{-\beta} \leq 0. \end{aligned}$$

Therefore, $\{w(\zeta)\}$ is non-increasing so $w^{\frac{\beta}{\xi}}(\eta(\zeta))/w^{\frac{\beta}{\xi}}(\zeta) \geq 1$, and

$$\begin{aligned} \Delta\left(w^{1-\frac{\beta}{\xi}}(\zeta)\right) &\leq \left(1-\frac{\beta}{\xi}\right)w^{-\frac{\beta}{\xi}}(\zeta)\Delta w(\zeta) \\ &\leq -\left(1-\frac{\beta}{\xi}\right)2^{-\beta}k^{\frac{\nu-\beta}{\xi}}\mu(\zeta)R^{\nu}(\eta(\zeta)). \end{aligned}$$

Summing this inequality from $\zeta_2$ to $\zeta-1$, we have

$$w^{1-\frac{\beta}{\xi}}(\zeta) - w^{1-\frac{\beta}{\xi}}(\zeta_2) \leq -\left(1-\frac{\beta}{\xi}\right)2^{-\beta}k^{\frac{\nu-\beta}{\xi}}\sum_{s=\zeta_2}^{\zeta-1}\mu(s)R^{\nu}(\eta(s)).$$

Since $\frac{\beta}{\xi} < 1$ and $\{w(\zeta)\}$ is a positive and non-increasing sequence, we have

$$\sum_{s=\zeta_2}^{\zeta-1}\mu(s)R^{\nu}(\eta(s)) \leq \frac{2^{\beta}k^{\frac{\beta-\nu}{\xi}}w^{1-\frac{\beta}{\xi}}(\zeta_2)}{\left(1-\frac{\beta}{\xi}\right)}.$$

This contradicts (2.7) and proves the oscillation of all solutions.

Next, we show that (2.7) is necessary. Suppose that (2.7) does not hold; so for given $\alpha > 0$ there exists an integer $N \geq \zeta_0$ such that

$$\sum_{s=N}^{\infty}\mu(s)R^{\nu}(\eta(s)) \leq \frac{\alpha^{1-\frac{\nu}{\xi}}}{2}. \tag{2.8}$$

We define the operator $\phi$ as follows: For the sequence $\{x(\zeta)\}_{\zeta=\zeta_0}^{\infty}$,

$$(\phi x)(\zeta) = \begin{cases} 0, & \zeta_0 \le \zeta \le N-1, \\ \sum_{u=N}^{\zeta-1} \left[ \frac{1}{\varphi(u)} \left[ \frac{\alpha}{2} \right.\right. & \\ \left.\left. + \sum_{s=u}^{\infty} \mu(s) x^{\nu}(\eta(s)) \right]\right]^{\frac{1}{\xi}}, & \zeta \ge N+1. \end{cases}$$

Now, consider the sequence $\{\nu^{(k)}(\zeta)\}$ of successive approxiamately defined by

$$\nu^{(1)}(\zeta) = \begin{cases} 0, & \zeta_0 \le \zeta \le N-1, \\ \left(\frac{\alpha}{2}\right)^{\frac{1}{\xi}} [R(\zeta) - R(N)], & \zeta \ge N. \end{cases}$$

and for $k = 2, 3, \dots$

$$\nu^{(k)}(\zeta) = (\phi \nu^{k-1})(\zeta).$$

Clearly, for $\zeta \ge N$

$$\left(\frac{\alpha}{2}\right)^{\frac{1}{\xi}} [R(\zeta) - R(N)] \le \nu^{(1)}(\zeta) \le \alpha^{\frac{1}{\xi}} [R(\zeta) - R(N)].$$

For $\zeta \ge N$,

$$\begin{aligned} \nu^{(2)}(\zeta) &= (\phi \nu^{(1)})(\zeta) \\ &= \sum_{u=N}^{\zeta-1} \left[ \frac{1}{\varphi(u)} \left[ \frac{\alpha}{2} + \sum_{s=u}^{\infty} \mu(s) (\nu^{(1)}(\eta(s))) \right]^{\nu} \right]^{\frac{1}{\xi}} \\ &\ge \sum_{u=N}^{\zeta-1} \left( \frac{1}{\varphi(u)} \frac{\alpha}{2} \right)^{\frac{1}{\xi}} \\ &= \left(\frac{\alpha}{2}\right)^{\frac{1}{\xi}} [R(\zeta) - R(N)] \\ &= \nu^{(1)}(\zeta) \end{aligned}$$

Also, for $\zeta \ge N$, we have $\nu^{(1)}(\zeta) \le \alpha^{\frac{1}{\xi}} R(\zeta)$ and $(\nu^{(1)}(\eta(\zeta)))^{\nu} \le (\alpha^{\frac{1}{\xi}} R(\eta(\zeta)))^{\nu}$. Then using (2.8), for $\zeta \ge N$

$$\begin{aligned} \nu^{(2)}(\zeta) &= (\phi \nu^{(1)})(\zeta) \\ &\le \sum_{u=N}^{\zeta-1} \left[ \frac{1}{\varphi(\zeta)} \left[ \frac{\alpha}{2} + \frac{\alpha}{2} \right] \right]^{\frac{1}{\xi}} \\ &= \alpha^{\frac{1}{\xi}} [R(\zeta) - R(N)] \end{aligned}$$

Thus, we have

$$\left(\frac{\alpha}{2}\right)^{\frac{1}{\xi}} [R(\zeta) - R(N)] \le \nu^{(1)}(\zeta) \le \nu^{(2)}(\zeta) \le \alpha^{\frac{1}{\xi}} [R(\zeta) - R(N)]$$

By induction principle, we can easily show that for $k = 2, 3, \dots$

$$\begin{aligned} \left(\frac{\alpha}{2}\right)^{\frac{1}{\xi}} [R(\zeta) - R(N)] &\le \nu^{(k-1)}(\zeta) \\ &\le \nu^{(k)}(\zeta) \le \alpha^{\frac{1}{\xi}} [R(\zeta) - R(N)]. \end{aligned}$$

Thus $\{\nu^{(k)}(\zeta)\}$ is a pointwise convergent to some sequence $\nu^* = \{\nu^*(\zeta)\}$. By means of the Lebesgue dominated convergence theorem, are obtain $(\phi \nu^*)(\zeta) = \nu^*(\zeta)$. We can easily show that $\{\nu^*(\zeta)\}$ is an eventually positive solution of the equation (1.1). This contradiction shows that (2.7) is a necessary condition. This completes the proof. □

### 2.2 For the case $\xi < \nu$

In this subsection, we assume that there exists $\nu > \beta > \xi > 0$ such that

$$u^{\nu-\beta} \le \nu^{\nu-\beta}, \quad for \quad 0 < u \le \nu. \tag{2.9}$$

**Lemma 2.4.** *Assume that $(A_1) - (A_5)$ hold. If $\{x(\zeta)\}$ is an eventually positive solution of (1.1) then there exists an integer $\zeta_1 \ge \zeta_0$ such that for $\zeta \ge \zeta_1$, the following holds:*

$$x^{\nu}(\eta(\zeta)) \ge \kappa^{\nu-\beta} x^{\beta}(\eta(\zeta)). \tag{2.10}$$

*Proof.* By Lemma 2.1, if follows that $\Delta x(\zeta) > 0$, so $\{x(\zeta)\}$ is increasing and $x(\zeta) \ge x(\zeta_0)$ for $\zeta \ge \zeta_0$. Thus

$$x(\eta(\zeta)) \ge x(\eta(\zeta_0)) := \kappa > 0 \quad for \quad \zeta \ge \zeta_1.$$

From (2.9), we have

$$\begin{aligned} x^{\nu}(\eta(\zeta)) &= x^{\nu-\beta}(\eta(\zeta)).x^{\beta}(\eta(\zeta)) \\ &\ge \kappa^{\nu-\beta} x^{\beta}(\eta(\zeta)) \text{ for } \zeta \ge \zeta_1, \end{aligned}$$

which is (2.10) □

**Theorem 2.5.** *Assume that $(A_1) - (A_5)$ hold and $\Delta\varphi(\zeta) \ge 0$. Then every solution of (1.1) is oscillatory if and only if*

$$\sum_{u=N}^{\infty} \left[ \frac{1}{\varphi(u+1)} \sum_{s=u+1}^{\infty} \mu(s) \right]^{\frac{1}{\xi}} = +\infty, \textit{for all } N > \zeta_0. \tag{2.11}$$

*Proof.* To prove sufficiency by contradiction, assume that $\{x(\zeta)\}$ is a non-oscillatory solution of (1.1). Without loss of generality we may assume that $\{x(\zeta)\}$ is eventually positive. Then by Lemma 2.1 and 2.4 hold for $\zeta \ge \zeta_1$. Using (2.10) in (1.1) and then summing the fixed inequality from $\zeta + 1$ to $\infty$, we have

$$\lim_{l \to \infty} \left[ \Delta\varphi(\zeta)(\Delta x(\zeta))^{\xi} \right]_{\zeta+1}^{l} + \kappa^{\nu-\beta} \sum_{s=\zeta+1}^{\infty} \mu(s) x^{\beta}(\eta(s)) \le 0.$$

Using that $\{\varphi(\zeta)(\Delta x(\zeta))^{\xi}\}$ is positive and non-increasing, and $\Delta\varphi(\zeta) \ge 0$, we have

$$\begin{aligned} &\kappa^{\nu-\beta} \sum_{s=\zeta+1}^{\infty} \mu(s) x^{\beta}(\eta(\zeta)) \le \varphi(n+1)(\Delta x(\zeta))^{\xi} \\ &\le \varphi(\zeta)(\Delta x(\zeta))^{\xi} \le \varphi(\eta(\zeta))(\Delta x(\eta(\zeta)))^{\xi} \\ &\qquad \le \varphi(\zeta)(\Delta x(\eta(\zeta)))^{\xi} \le \varphi(\zeta+1)(\Delta x(\eta(\zeta)))^{\xi} \end{aligned}$$

for all $\zeta \geq \zeta_1$. Therefore,

$$\kappa^{\frac{v-\beta}{\xi}}\left[\frac{1}{\varphi(\zeta+1)}\sum_{s=\zeta+1}^{\infty}\mu(s)x^{\beta}(\eta(\zeta))\right]^{\frac{1}{\xi}} \leq \Delta x(\eta(\zeta))$$

implies that

$$\kappa^{\frac{v-\beta}{\xi}}\left[\frac{1}{\varphi(\zeta+1)}\sum_{s=\zeta+1}^{\infty}\mu(s)\right]^{\frac{1}{\xi}} \leq \frac{\Delta x(\eta(\zeta))}{x^{\frac{\beta}{\xi}}(\eta(\zeta+1))}. \quad (2.12)$$

Summing (2.12) from $\zeta_1$ to $\infty$, we have

$$\kappa^{\frac{v-\beta}{\xi}}\sum_{u=\zeta_1}^{\infty}\left[\frac{1}{\varphi(\zeta+1)}\sum_{s=u+1}^{\infty}\mu(s)\right]^{\frac{1}{\xi}} \leq \sum_{u=\zeta_1}^{\infty}\frac{\Delta x(\eta(u))}{x^{\frac{\beta}{\xi}}(\eta(u+1))}. \quad (2.13)$$

Let $y(t)=x(\eta(u))+(t-u)\Delta x(\eta(u))$, $u \leq t \leq u+1$. Then $y(u)=x(\eta(u))$, $y(u+1)=x(\eta(u+1))$ and $y'(t)=\Delta x(\eta(u))$, $u<t<u+1$. Thus $y(t)$ is continuous and increasing for $u \geq \zeta_1$. Now,

$$\begin{aligned}\frac{\Delta x(\eta(u))}{x^{\frac{\beta}{\xi}}(\eta(u+1))} &= \int_u^{u+1}\frac{\Delta x(\eta(u))}{x^{\frac{\beta}{\xi}}(\eta(u+1))\mathrm{d}t} \\ &= \int_u^{u+1}\frac{y'(t)\mathrm{d}t}{y^{\frac{\beta}{\xi}}(u+1)} \\ &\leq \int_u^{u+1}\frac{y'(t)\mathrm{d}t}{y^{\frac{\beta}{\xi}}(t)} \\ &= \frac{1}{1-\frac{\beta}{\xi}}\left[x^{1-\frac{\beta}{\xi}}(\eta(u+1))-x^{1-\frac{\beta}{\xi}}(\eta(u))\right]\end{aligned}$$

This implies that

$$\sum_{u=\zeta_1}^{k-1}\frac{\Delta x(\eta(u))}{x^{\frac{\beta}{\xi}}(\eta(u+1))} \leq \frac{1}{1-\frac{\beta}{\xi}}\left[x^{1-\frac{\beta}{\xi}}(\eta(k+1))-x^{1-\frac{\beta}{\xi}}(\eta(\zeta_1))\right]$$

Taking $k \to \infty$, we have

$$\sum_{u=\zeta_1}^{\infty}\frac{\Delta x(\eta(u))}{x^{\frac{\beta}{\xi}}(\eta(u+1))} \leq \frac{x^{1-\frac{\beta}{\xi}}(\eta(\zeta_1))}{\frac{\beta}{\xi}-1} < \infty. \quad (2.14)$$

Using (2.14) in (2.13), we have

$$\sum_{u=\zeta_1}^{\infty}\left[\frac{1}{\varphi(u+1)}\sum_{s=u+1}^{\infty}\mu(s)\right]^{\frac{1}{\xi}} < \infty,$$

which contradicts (2.11) and proves the oscillation of all solutions.

Next, we show that (2.11) is necessary. Suppose that (2.11) does not hold; So for each $\alpha > 0$, there exists $N \geq \zeta_0$ such that

$$\sum_{u=N}^{\infty}\left[\frac{1}{\varphi(u+1)}\sum_{s=u+1}^{\infty}\mu(s)\right]^{\frac{1}{\xi}} \leq \frac{\alpha^{1-\frac{v}{\xi}}}{2}. \quad (2.15)$$

We define the operator $\phi$ as follows:

For the sequence $\{x(\zeta)\}_{\zeta=\zeta_0}^{\infty}$,

$$(\phi x)(\zeta) = \begin{cases}\frac{\alpha}{2}, & \zeta_0 \leq \zeta \leq N+1, \\ \frac{\alpha}{2}+\sum_{u=N}^{\zeta-2}\left[\frac{1}{\varphi(u+1)}\right. & \\ \left.\times\sum_{s=u+1}^{\infty}\mu(s)x^{v}(\eta(s))\right]^{\frac{1}{\xi}}, & \zeta \geq N+2.\end{cases}$$

Now, consider the sequence $\{v^{(k)}(\zeta)\}$ of successive approximations defined by

$$\{v^{(1)}(\zeta)=\frac{\alpha}{2}; \quad \zeta \geq \zeta_0.$$

and for $k=2,3\ldots$

$$\{v^{(k)}(\zeta)=(\phi v^{k-1})(\zeta).$$

Clearly $\frac{\alpha}{2} \leq v^{(1)}(\zeta) \leq \alpha, \quad \zeta \geq \zeta_0.$
Also,

$$v^{(2)}(\zeta)=(\phi v^{1})(\zeta)$$

For $\zeta_0 \leq \zeta \leq \zeta \leq N+1$, we have

$$v^{(2)}(\zeta)=\frac{\alpha}{2}=v^{(1)}(\zeta)$$

and for $\zeta \geq N+2$,

$$\begin{aligned}v^{(2)}(\zeta) &= \frac{\alpha}{2}+\sum_{u=N}^{\zeta-2}\left[\frac{1}{\varphi(u+1)}\sum_{s=u+1}^{\infty}\mu(s)v^{(1)}(\eta(s))^{v}\right]^{\frac{1}{\xi}} \\ &\leq \frac{\alpha}{2}+\alpha^{\frac{v}{\xi}}\sum_{u=N}^{\zeta-2}\left(\frac{1}{\varphi(u+1)}\sum_{s=u+1}^{\infty}\mu(s)\right)^{\frac{1}{\xi}}.\end{aligned} \quad (2.16)$$

Using (2.15) in (2.16), we obtain

$$v^{(2)}(\zeta) \leq \alpha.$$

Hence, we have

$$\frac{\alpha}{2} \leq v^{1}(\zeta) \leq v^{(2)}(\zeta) \leq \alpha.$$

By induction, we can easily prove that

$$\frac{\alpha}{2} \leq v^{k-1}(\zeta) \leq v^{(k)}(\zeta) \leq \alpha, \quad k=2,3,\ldots$$

Thus, $\{v^{(k)}(\zeta)\}$ is a pointwise convergent to some sequence $v^* = \{v^*(\zeta)\}$.

By means of the Lebesgue dominated convergence theorem, we obtain $(\phi v^*)(\zeta) = v^*(\zeta)$. We can easily show that $\{v^*(\zeta)\}$ is an eventually positive solution of the equation (1.1) for $\zeta \geq N+1$. This contradiction shows that (2.11) is necessary condition. This completes the proof. □

**Remark 2.6.** *The result of this paper also hold for equation of the form*

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\xi}) + \sum_{j=1}^{m} \mu_j(\zeta) x^{\nu_j}(\eta_j(\zeta)) = 0; \quad \zeta \geq \zeta_0 \tag{2.17}$$

*where* $\{\varphi(\zeta)\}$, $\{\mu_j(\zeta)\}$, $\nu_j$ *and* $\eta_j(\zeta)$ $(j = 1,2,3,\ldots,m)$ *satify the assumptions in* $(A_1)$ *to* $(A_5)$*,* (2.3) *and* (2.5)*. In order to extend Theorem 2.3 and Theorem 2.5, there exists an index j such that* $\{\mu_j(\zeta)\}$, $\nu_j$ *and* $\{\eta_j(\zeta)\}$ *fulfiles* (2.7) *and* (2.11)*.*

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