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σ-algebra of complex fuzzy subgroups associated with a finite group and the measure on the σ-algebra

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Abstract

Complex fuzzy subgroups of a finite group are fuzzy subgroups in which the membership values are in the unit disc. In this paper we introduce the term σ-algebra of complex fuzzy subgroups associated with a finite group by using the discussion in [3]. We prove that the collection of complex fuzzy subgroups associated with a finite group will form a σ-algebra of complex fuzzy sets. Also, we define measures on this newly introduced σ-algebra.

Keywords

Complex fuzzy sets, Fuzzy subgroups, Complex fuzzy subgroups, Similarity measure, Coherence Measure.

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Contents

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [1] in 1965. A fuzzy set in the universe of discourse X is defined by the grade of membership $\mu_A : X \longrightarrow [0,1], \mu_A(x)$ denotes the degree of membership of x in the fuzzy set A. Many authors noticed the theory of fuzzy sets has many applications in several branches of Mathematics. Rosenfeld introduced the concept of fuzzy subgroups in 1971. He also discussed several properties of fuzzy subgroups.

Ramot et. al. [6] introduced the most applicable concept of complex fuzzy sets in 2002, instead of taking the grade of membership in $[0,1]$, the complex fuzzy sets assigns grade of membership in the unit disc in the complex plane. Also, he introduced several operations and relations of the complex fuzzy sets like union, intersection complements etc. David H Glass [4]studied the coherence measure and similarity measure of fuzzy sets in 2006. Mikhled Okleh Alsarahead and Abd Ghafur Ahmad [3] introduced the concept of complex fuzzy subgroups in 2017.

In this paper, we consider the collection of complex fuzzy subgroups of a finite group G and we prove that the collection will form a σ - algebra of complex fuzzy sets. Also, we investigate the characteristics of the algebra and finally we define some measures on this algebra and also, we define some measure on the σ- algebra of complex fuzzy sets.

2. PRELIMINARIES

2.1 Fuzzy Group

Let G be a group and (A,μ) be a fuzzy set. Then (A,μ) is said to be a fuzzy group if the following holds.

(i) $\mu(xy) \geq min\{\mu(x),\mu(y)\}$, for all $x,y\varepsilon G$;

(ii) $\mu(x^{-1}) = \mu(x)$;
(iii) $\mu(e) = 1, e\varepsilon G$.

Example 2.1.

(1) Let

$$A_4 = \{e, (12)(34), (13)(24), (123), (132), (142), (124), (234), (243), (134), (143)\}.$$

Define $\mu(e) = 1$, $\mu((12)(34)) = \frac{1}{2}$, $\mu\{(14)(23), (13)(24)\} = \frac{1}{3}$, $\mu(ijk) = 0$, then μ is a fuzzy subgroup of G.
(2) Let $G = \{e, a, b, b^2, ab, ab^2\}$
define $\mu(a) = \mu(e) = 1$, $\mu(b) = \mu(b^2) = \mu(ab) = \mu(ab^2) = 0$, then μ is a fuzzy subgroup of G.

2.2 Complex Fuzzy Set

A complex fuzzy set defined on the universe of discourse U, is characterized by a membership function $\mu_A(x)$, that assign any element $a\varepsilon A$ a complex valued grade of membership in A.

By definition, the value $\mu_A(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_A(x)e^{i\omega_A(x)}$, $i = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ both real valued $r_A(x)\varepsilon[0,1]$ and $\omega_A(x)\varepsilon[0,2\pi)$ the complex fuzzy set may be represented as the set of ordered pairs. $A = \{(x, \mu_A(x)) : x\varepsilon U\}$.

2.3 Definition

Let $A = \{(x, \mu_A(x)) : x\varepsilon X\}$ and $B = \{(x, \mu_B(x)) : x\varepsilon X\}$ be two complex fuzzy sets of the same universe of discourseX with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ respectively.

Then

$$(i)\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = \min\{r_A(x), r_B(x)\}\, e^{i\min\{\omega_A(x),\omega_B(x)\}}.$$

$$(ii)\mu_{A\cup B}(x) = r_{A\cup B}(x)e^{i\omega_{A\cup B}(x)} = \max\{r_A(x), r_B(x)\}\, e^{i\max\{\omega_A(x),\omega_B(x)\}}.$$

$$(iii)\mu_{A^c}(x) = [1 - r_A(x)]\, e^{i[2\pi - \omega_A(x)]}.$$

2.4 Complex Fuzzy Subgroups

Let G be a finite group. A be a complex fuzzy set of G. Then A is said to be a complex fuzzy subgroup if,

(i) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$

(ii) $\mu_A(x) = \mu_A(x^{-1})$.
In other words,

$$(i)\mu_A(xy) = r_A(xy)e^{i\omega_A(xy)} \geq min\{r_A(x), r_A(y)\}\, e^{i min\{\omega_A(x),\omega_A(y)\}}$$

if and only if
$r_A(xy) \geq min\{r_A(x), r_A(y)\}$ and

$\omega_A(xy) \geq min\{\omega_A(x), \omega_A(y)\}$.

2.5 Algebra of Sets

A collection X of sets is said to be an algebra if,
(i) $A\varepsilon X \Longrightarrow A^c\varepsilon X$;
(ii) $A, B\varepsilon X \Longrightarrow A\cap B$ or $A\cup B\varepsilon X$.

2.6 σ- Algebra

A collection X of sets is said to be a σ algebra if,
(i) $A\varepsilon X \Longrightarrow A^c\varepsilon X$;
(ii) $\{A_i : i\varepsilon I\} \subset X \Longrightarrow \bigcap_{i\varepsilon I} A_i$ or $\bigcup_{i\varepsilon I} A_i\varepsilon X$.

2.7 Coherence Measure of Fuzzy Sets

Let X be any set, $P^f(X)$ denote the set of all fuzzy subsets of X. A mapping $S : P^f(X) \times P^f(X) \longrightarrow [0,1]$ is said to be a similarity measure if,

(i) $S(A,B) = S(B,A)$;

(ii) $S(A,B)$ is maximal iff A and B are identical.

(iii) $S(A,B)$ is minimal iff $|A\cap B| = 0$.

Most commonly used similarity measure is, $S(A,B) = |\frac{A\cap B}{A\cup B}|$, where the cardinality $|A|$ of a fuzzy set A is the sum of all the membership values in A.

Define $S_{new} : P^f(X) \times P^f(X) \longrightarrow [-1,1]$ by

$$S_{new}(A,B) = \frac{1}{2}\{(S(A,B) + S(A^c,B^c)) - (S(A,B^c) + S(A^c,B))\}.$$

The coherence measure on the collection $P^f(x) \times P^f(X)$ is a real valued function $cohe : P^f(X) \times P^f(X) \longrightarrow [0,1]$ defined by,

$$Cohe(A,B) = \tfrac{1}{2}\{S_{new}(A,B) + 1\}.$$

3. σ-Algebra of Complex Fuzzy Subgroups Associated with a Finite Group and the Measure on the σ- Algebra

3.1 Theorem

Let G be a finite group. $P^{(cf)}(G)$ be the collection of all complex fuzzy subgroups of G. Then,
(I) $A\varepsilon P^{(cf)}(G) \Longrightarrow A^c\varepsilon P^{(cf)}(G)$;
(II) $\{A_i : i\varepsilon I\} \subset P^{(cf)}(G)$
$\Longrightarrow \bigcap_{i\varepsilon I} A_i\varepsilon P^{(cf)}(G)$ or $\bigcup_{i\varepsilon I} A_i\varepsilon P^{(cf)}(G)$.

In other words, the collection $P^{(cf)}(G)$ form a σ - algebra of complex fuzzy sets, known as σ- algebra of complex fuzzy subgroups associated with a finite group G.

Proof: - (I) Let $A \varepsilon P^{(cf)}(G) \Longrightarrow$ A is a complex fuzzy subgroup of G.

$$\Longrightarrow (i)\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}, (ii)\mu_A(x) = \mu_A(x^{-1})$$

Claim: - A^c is a complex fuzzy subgroup. For that, consider

$$\begin{aligned}
&\mu_{A^c}(xy) \\
&= [1 - r_A(x)]e^{i[2\pi - \omega_A(x)]} \\
&\geq (1 - min\{r_A(x), r_A(y)\})e^{i(2\pi - min\{\omega_A(x), \omega_A(y)\})} \\
&= \max\{(1 - r_A(x)), (1 - r_A(y))\} e^{imax\{(2\pi - \omega_A(x)), 2\pi - \omega_A(y)\}} \\
&\geq \min\{(1 - r_A(x)), (1 - r_A(y))\} e^{imin\{(2\pi - \omega_A(x)), 2\pi - \omega_A(y)\}} \\
&\geq \min\left\{(1 - r_A(x))e^{i(2\pi - \omega_A(x))}, (1 - r_A(y))e^{i(2\pi - \omega_A(y))}\right\} \\
&= \min\{\mu_{A^c}(x), \mu_{A^c}(y)\}
\end{aligned}$$

So, $\mu_{A^c}(xy) \geq min\{\mu_{A^c}(x), \mu_{A^c}(y)\}$

On the other hand,

$$\mu_{A^c}(x^{-1}) = [1 - r_A(x^{-1})]e^{i[2\pi - \omega_A(x^{-1})]}$$

$= [1 - r_A(x)]e^{i[2\pi - \omega_A(x)]}$ (Since A is a complex fuzzy subgroup)

$$= \mu_{A^c}(x)$$

So, $\mu_{A^c}(x^{-1}) = \mu_{A^c}(x)$

Therefore A^c is a complex fuzzy subgroup of G , whenever A is.

(II) Let $\{A_i : i\varepsilon I\}$ be the collection of complex fuzzy subgroups.

Now let $x, y\varepsilon G$

$$\begin{aligned}
&\mu_{\cap_{i\varepsilon I} A_i}(xy) \\
&= r_{\cap_{i\varepsilon I} A_i}(xy)e^{i\omega_{\cap_{i\varepsilon I} A_i}(xy)} \\
&= \min_{i\varepsilon I}\{r_{A_i}(xy)\} e^{imin_{i\varepsilon I}\{\omega_{A_i}(xy)\}} \\
&\geq min_{i\varepsilon I}(min\{r_{A_i}(x), r_{A_i}(y)\})e^{imin_{i\varepsilon I}(min\{\omega_{A_i}(x), \omega_{A_i}(y)\})}
\end{aligned}$$

Since

$$\begin{aligned}
&\mu_{A\cap B}(x) \\
&= \min\left\{min_{i\varepsilon I}(r_{A_i}(x))e^{imin(\omega_{A_i}(x))}, min_{i\varepsilon I}(r_{A_i}(y))e^{imin(\omega_{A_i}(y))}\right\} \\
&= \min\left\{r_{\cap_{i\varepsilon I} A_i}(x)e^{i\omega_{\cap_{i\varepsilon I} A_i}(x)}, r_{\cap_{i\varepsilon I} A_i}(y)e^{i\omega_{\cap_{i\varepsilon I} A_i}(y)}\right\} \\
&= \min_{i\varepsilon I}\left\{\mu_{\cap_{i\varepsilon I} A_i(x)}, \mu_{\cap_{i\varepsilon I} A_i(y)}\right\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mu_{\cap_{i\varepsilon I} A_i}(x^{-1}) &= r_{\cap_{i\varepsilon I} A_i}(x^{-1})e^{i\omega_{\cap_{i\varepsilon I} A_i}(x^{-1})} \\
&= min_{i\varepsilon I}\left\{r_{A_i}(x^{-1})\right\} e^{imin_{i\varepsilon I}\left\{\omega_{A_i}(x^{-1})\right\}} \\
&= min_{i\varepsilon I}\{r_{A_i}(x)\} e^{imin_{i\varepsilon I}\{\omega_{A_i}(x)\}} \\
&= r_{\cap_{i\varepsilon I} A_i}(x)e^{i\omega_{\cap_{i\varepsilon I} A_i}(x)} \\
&= \mu_{\cap_{i\varepsilon I} A_i}(x)
\end{aligned}$$

So, $\mu_{\cap_{i\varepsilon I} A_i}(x^{-1}) = \mu_{\cap_{i\varepsilon I} A_i}(x)$

Therefore from (I) and (II) ,

(I) $A\varepsilon P^{(cf)}(G) \Longrightarrow A^c \varepsilon P^{(cf)}(G)$

(II) $\{A_i : i\varepsilon I\} \subset P^{(cf)}(G) \Longrightarrow \bigcap_{i\varepsilon I} A_i \varepsilon P^{(cf)}(G)$.

So we can say that, the collection $P^{(cf)}(G)$ form a σ - algebra of complex fuzzy sets, known as σ - algebra of complex fuzzy subgroups associated with a finite group G.

3.2 Measure on the σ- Algebra

3.2.1 Similarity measure on complex fuzzy subgroup σ-algebra $P^{(cf)}(G)$

Consider the complex fuzzy subgroup σ- algebra $P^{(cf)}(G)$ associated with a finite group G. Consider two complex fuzzy subgroups A and B, the similarity measure S is a function from $P^{(cf)}(G) \times P^{(cf)}(G)$ into $[0,1]$ and the similarity measure S of A and B is defined as:

$S(A,B) = |\frac{A\cap B}{A\cup B}|$

where, $|A \cap B| = |\sum \mu_{A\cap B}(x)| = |Min\{\mu_{A\cap B}(x)\}|$

and, $|A \cup B| = |\sum \mu_{A\cup B}(x)| = |Min\{\mu_{A\cup B}(x)\}|$.

4. Illustration

Let $G = \{a, b, ab, e\}$

Let
$$A = \left\{0.3e^{i\pi}, 0.5e^{i\frac{\pi}{2}}, 0.3e^{i\pi}, 1e^{i0}\right\}$$
and
$$B = \left\{0.2e^{i0}, 0.45e^{i\frac{\pi}{4}}, 0.3e^{i\frac{\pi}{3}}, 1e^{i\pi}\right\}.$$

$\mu_{A\cap B}(a) = min\left\{0.3e^{i\pi}, 0.2e^{i0}\right\} = min\{0.3, 0.2\} e^{imin\{\pi,0\}} = 0.2e^{i0}$

$$\mu_{A\cap B}(b) = min\left\{0.5e^{i\frac{\pi}{2}}, 0.45e^{i\frac{\pi}{4}}\right\} = 0.45e^{i\frac{\pi}{4}}$$

$$\mu_{A\cap B}(ab) = min\left\{0.3e^{i\pi}, 0.3e^{i\frac{\pi}{3}}\right\} = 0.3e^{i\frac{\pi}{3}}$$

$\mu_{A\cap B}(e) = min\left\{1e^{i0}, 1e^{i\pi}\right\} = 1e^{i0}$

$|A\cap B = |\sum \mu_{A\cap B}(x)| = |Min\{\mu_{A\cap B}(x)\}|$

$= |Min\left\{0.2e^{i0}, 0.45e^{i\frac{\pi}{4}}, 0.3e^{i\frac{\pi}{3}}, 1e^{i0}\right\}|$

$= |Min\{0.2, 0.45, 0.3, 1\}\, e^{imin\left\{0,\frac{\pi}{4},\frac{\pi}{3},0\right\}}|$

$= |0.2e^{i0}| = 0.2$

Similarly,

$\mu_{A\cup B}(a) = max\left\{0.3e^{i\pi}, 0.2e^{i0}\right\} = max\{0.3, 0.2\}\, e^{imax\{\pi,0\}} = 0.3e^{i\pi}$

$\mu_{A\cup B}(b) = max\left\{0.5e^{i\frac{\pi}{2}}, 0.45e^{i\frac{\pi}{4}}\right\} = 0.5e^{i\frac{\pi}{2}}$

$\mu_{A\cup B}(ab) = max\left\{0.3e^{i\pi}, 0.3e^{i\frac{\pi}{3}}\right\} = 0.3e^{i\pi}$

$\mu_{A\cup B}(e) = max\left\{1e^{i0}, 1e^{i\pi}\right\} = 1e^{i\pi}$

$|A\cup B = |\sum \mu_{A\cup B}(x)| = |Min\{\mu_{A\cup B}(x)\}|$

$= |Min\left\{0.3e^{i\pi}, 0.5e^{i\frac{\pi}{2}}, 0.3e^{i\pi}, 1e^{i\pi}\right\}|$

$= |Min\{0.3, 0.5, 0.3, 1\}\, e^{imin\left\{\pi,\frac{\pi}{2},\pi,\pi\right\}}|$

$= |0.3e^{i\frac{\pi}{2}}| = 0.3$

$S(A,B) = |\frac{A\cap B}{A\cup B}| = \frac{0.2}{0.3} = 0.667.$

4.0.1 S_{New}- Measure

S_{New} - measure is a function from $P^{(cf)}(G)\times P^{cf}(G)$ into $[-1,1]$ and is defined by,

$$S_{new}(A,B) = \frac{1}{2}\left\{(S(A,B)+S(A^c,B^c)) - (S(A,B^c)+S(A^c,B))\right\}$$

$S(A,B) = |\frac{A\cap B}{A\cup B}| = \frac{0.2}{0.3} = 0.667$

$$A^c = \left\{(1-r_A(a))e^{i(2\pi-\omega_A(a))}, (1-r_A(b))e^{i(2\pi-\omega_A(b))}, (1-r_A(ab))e^{i(2\pi-\omega_A(ab))}, (1-r_A(e))e^{i(2\pi-\omega_A(e))}\right.$$

$$= \left\{0.7e^{i\pi}, 0.5e^{\frac{3\pi}{2}}, 0.7e^{i\pi}, 0e^{i2\pi}\right\}.$$

Similarly,

$$B^c = \left\{(1-r_B(a))e^{i(2\pi-\omega_B(a))}, (1-r_B(b))e^{i(2\pi-\omega_B(b))}, (1-r_B(ab))e^{i(2\pi-\omega_B(ab))}, (1-r_B(e))e^{i(2\pi-\omega_B(e))}\right.$$

$$= \left\{0.8e^{i2\pi}, 0.55e^{\frac{7\pi}{4}}, 0.7e^{i\frac{5\pi}{3}}, 0e^{i\pi}\right\}$$

Using similar method, we get,

$S(A^c,B^c) = |\frac{A^c\cap B^c}{A^c\cup B^c}| = 0.$

$S(A,B^c) = |\frac{A\cap B^c}{A\cup B^c}| = 0.$

$S(A^c,B) = |\frac{A^c\cap B}{A^c\cup B}| = 0.$

So,

$$S_{new}(A,B) = \frac{1}{2}\left\{(S(A,B)+S(A^c,B^c)) - (S(A,B^c)+S(A^c,B))\right\}.$$

$$= \frac{1}{2}\{0.667+0+0+0\} = \frac{0.667}{2} = 0.3335.$$

4.0.2 The Coherence Measure

The coherence measure is a function $Cohe: P^{(cf)}(G)\times P^{(cf)}(G) \longrightarrow [0,1]$ and is defined as $Cohe(A,B) = \frac{1}{2}\{S_{New}(A,B)+1\}$, where A and B are complex fuzzy subgroups of G.

So here,
$Cohe(A,B) = \frac{1}{2}\{0.3335+1\} = \frac{1.3335}{2} = 0.66675.$

4.0.3 The Ambiguity Measure

Let G be a finite group and let $P^{(cf)}(G)$ be the σ - algebra of complex fuzzy subgroups associated with G. Then the Ambiguity Measure of $P^{(cf)}(G)$ is a function $\alpha: P^{(cf)}(G) \longrightarrow [0,1]$ and is defined as
$\alpha(A) = 1 - Cohe(A,A).$

5. Conclusion

In this paper, we introduce the concept of σ-algebra of complex fuzzy subgroups associated with a finite group. The concept of complex fuzzy set has undergone several evolutionary processes since they first introduced and it has many applications in different fields. So, different algebraic structures of complex fuzzy sets can be used to develope the applications of the concept. In this paper, we also introduce several measures that can be defined on the mentioned σ-algebra of complex fuzzy subgroups associated with a finite group. This concept will leads to further developments of complex fuzzy sets and helps to embedding some important notions of different branches of Mathematics especially, the area of theory of measure and integration to the theory of complex fuzzy sets.

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