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Solution of non-linear integro-differential equations by using modified Laplace transform Adomian decomposition method

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Abstract
In last few decades, integro-differential equations are used in various fields of sciences and engineering. Recently most of researchers have taken considerable effort to the study of exact and numerical solutions of the linear, nonlinear ordinary, or partial differential equations. In this paper, we have discussed the Modified Laplace Transform Adomian Decomposition Method (MLTADM) which is the combination of Laplace transform and Adomian decomposition method to solve the second and third-order nonlinear integro-differential equations.The main advantage of this method is that it gives an analytical solution. The method overcomes the difficulties arising in calculating the Adomian polynomials. The efficiency of the method was tested on some numerical examples, and the results show that the method is easier than many other numerical techniques. It is also observed that (MLTADM) is a reliable tool for the solution of linear and nonlinear integro-differential equations.

Keywords
Laplace Transform, Adomian decomposition method, Volterra-Fredholm integro-differential equation, Non-Linear Volterra integral equation.

AMS Subject Classification
34G20.

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Contents

1. Introduction

In the recent decade, the use of integro-differential equations has become a powerful tool to solve problems in science and technology, integro-differential equation which gives us to provides efficiency for the description of many practical problems arising in engineering and scientific disciplines such as, physics, biology, economics, chemistry, electromagnetic, control theory and viscoelasticity. In recent years, most of researchers have developed numerical and analytical techniques for fractional integro-differential equations. In this paper, we have studied the nonlinear Volterra-Fredholm integro-differential equations of the second kind.

The nonlinear Fredholm integro-differential equations are given by

$$u^n(x) = f(x) + \int_a^x k(x,t)[Ru(t) + Nu(t)]dt$$
$$u^k(x) = \alpha_k, 0 \leq k \leq n-1, n \geq 0, \tag{1.1}$$

and the nonlinear Volterra integrodifferential equations are given by

$$u^n(x) = f(x) + \int_a^b k(x,t)[Ru(t) + Nu(t)]dt$$
$$u^k(x) = \alpha_k, 0 \leq k \leq n-1, n \geq 0, \tag{1.2}$$

where $u^n(x)$ is the n^th derivative of the unknown function $u(x)$ that has be determined, $k(x,t)$ is the kernel of the integral equation, $f(x)$ is an analytic function, $R(u)$ and $N(u)$ are linear and nonlinear functions of u, respectively. For $n=0$, Equations (1.1) and (1.2) are the nonlinear Fredholm integral equations and the nonlinear Volterra integral equations respectively.

M. Dehghan et. al. [1] had studied the solution of the linear fractional partial differential equations using the homotopy analysis method. M. Hussain and M. Khan [2] have applied modified Laplace decomposition method. S. Shahmorad [3] have investigated numerical solution of the general form linear Fredholm-Volterra integro- differential equations by the Tau method with an error estimation. Later, N Bildik et.al. [4] had work on a comparison between adomian decomposition and Tau methods. Then, S. Alkan et.al. [5] have studied approximate solutions of Volterra-Fredholm integro-differential equations of fractional order; M. Al-Mazmumy et.al. [6] Some modifications of adomian decomposition methods for nonlinear partial differential equations; S. Agarwal et al. [7] had studied application of Mahgoub transform for solving linear Volterra integral equations of first kind; R.B. Thete et. al. [8] have investigated temperature distribution of an inverse steady state thermo elastic problem of thin rectangular plate by numerical method; Tarik M. Elzaki et.al. [9] had studied solution of volterra integro differential equation by triple Laplace transform; R.B. Thete et.al. [10] had investigated estimation of temperature distribution and thermal stress analysis of composite circular rod by finite element method. Recently, R.B. Thete and Arihant Jain [11] had studied analytical solution of linear volterra integro-differential equation of first and second kind by Laplace transform. For additional detail, see for instance [12, 13].

In this paper, we have discussed the modified Laplace transform Adomian decomposition method (MLTADM) which is the combination of Laplace transform and Adomian decomposition method to solve the second and third-order nonlinear integro-differential equations. The main advantage of this method is the fact that it gives an analytical solution. The method overcomes the difficulties arising in calculating the Adomian polynomials. The efficiency of the method was tested on some numerical examples, and the results show that the method is easier than many other numerical techniques. It is also observed that (MLTADM) is a reliable tool for the solution of linear and nonlinear integro-differential equations.

2. Modified Laplace Adomian Decomposition Method

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the integro differential equations. Let us consider the most general form of second order nonlinear partial differential equations with initial conditions is of the form

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=h(x,t),$$
$$u(x,0)=g(x), \tag{1.3}$$

where L is the second order differential operator $L_{xx}=\frac{d^2}{dx^2}$, R is the linear operator, N represents a general non-linear differential operator and $h(x,t)$ represent the source term.

Taking Laplace transform on both sides of Equation (1.3) we have

$$L[Lu(x,t)]+L[Ru(x,t)]+L[Nu(x,t)]=L[h(x,t)]$$

and by using the derivative property of Laplace transform we get,

$$\{s^2L[u(x,t)]-sf(x)-g(x)\}+L[Ru(x,t)]$$
$$+L[Nu(x,t)]=L[h(x,t)]$$
$$L[u(x,t)]=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)-\frac{1}{s^2}L[Ru(x,t)]$$
$$-\frac{1}{s^2}L[Nu(x,t)]+\frac{1}{s^2}L[h(x,t)]. \tag{1.4}$$

The next step in Laplace decomposition method is representing the solution as an infinite series given below

$$[u(x,t)]=\sum_{n=0}^{\infty}u_n(x,t). \tag{1.5}$$

The nonlinear operator is decomposed as

$$[Nu(x,t)]=\sum_{n=0}^{\infty}A_n(x,t), \tag{1.6}$$

where for every $n\in N$, the Adomian polynomial A_n is given by

$$A_n=\frac{1}{n!}\frac{d^n}{dx^n}\left[N\sum_{n=0}^{\infty}\tau^i u_i\right]_{\tau=0}.$$

Using (1.4), (1.5) and (1.6) we get

$$\sum_{n=0}^{\infty}L[u_n(x,t)]=\frac{f(x)}{s}+\frac{g(x)}{s^2}-\frac{1}{s^2}L[Ru(x,t)]$$
$$-\frac{1}{s^2}L\left[\sum_{n=0}^{\infty}A_n(x,t)\right]+\frac{1}{s^2}L[h(x,t)]. \tag{1.7}$$

Comparing equation (1.7) to both side we get,

$$L[u_0(x,t)]=K_1(x,s) \tag{1.8}$$
$$L[u_1(x,t)]=K_2(x,s)-\frac{1}{s^2}L[R_0u(x,t)]-\frac{1}{s^2}L[A_0(x,t)] \tag{1.9}$$
$$L[u_{n+1}(x,t)]=-\frac{1}{s^2}L[R_nu(x,t)]-\frac{1}{s^2}L[A_n(x,t)]\ n\geq 1, \tag{1.10}$$

where $K_1(x,s)$ and $K_2(x,s)$ are Laplace transform of $K_1(x,t)$ and $K_2(x,t)$ respectively. Applying the inverse Laplace transform to equations (1.8)-(1.10) gives us recursive relation is as follows

$$u_0(x,t) = K_1(x,t), \tag{1.11}$$

$$[u_1(x,t)] = K_2(x,t) - L^{-1}\left[\frac{1}{s^2}L[R_0u(x,t)] - \frac{1}{s^2}L[A_0(x,t)]\right], \tag{1.12}$$

$$[u_{n+1}(x,t)] = -L^{-1}\left[\frac{1}{s^2}L[R_nu(x,t)] + \frac{1}{s^2}L[A_n(x,t)]\right], \quad n \geq 1. \tag{1.13}$$

The solution through the modified Adomian decomposition method highly depends upon the choice of $K_0(x,t)$ and $K_1(x,t)$, where $K_0(x,t)$ and $K_1(x,t)$ represent the terms arising from the source term and prescribed initial conditions.

3. Application of the Modified Adomian Decomposition Method

Example 1. Let us consider the nonlinear integro differential equation $u'(x) = -1 + \int_0^x u^2(t)dt, \quad u(0) = 0.$

Applying the Laplace transform and by using the initial condition, we have

$$sU(s) = -\frac{1}{s} + L\left[\int_0^x u^2(t)dt\right]$$

$$U(s) = -\frac{1}{s^2} + \frac{1}{s}L\left[\int_0^x u^2(t)dt\right]$$

Applying Inverse Laplace transform to both side

$$u(x) = -x + L^{-1}\left\{\frac{1}{s}L\left[\int_0^x u^2(t)dt\right]\right\}. \tag{1.14}$$

Now, we decompose the solution in the form of an infinite sum given below

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{1.15}$$

Using (1.15) on (1.14), we get

$$\sum_{n=0}^{\infty} u_n(x) = -x + L^{-1}\left\{\frac{1}{s}L\left[\int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right]\right\},$$

in which $\sum_{n=0}^{\infty} u_j u_{n-j}$.

The recursive relation is given below

$$u_0(x) = -x;$$

$$u_1(x) = \frac{x^4}{12};$$

$$u_2(x) = -\frac{x^7}{252}.$$

By repeating the above procedure for $n \geq 3$, we get to the approximate solution.

Example 2. Consider the second order nonlinear integro-differential equation

$$u''(x) = \sinh(x) + x - \int_0^1 x\left[\cosh^2(t) + u^2(t)\right]dt,$$
$$u(0) = 0, \quad u'(0) = 1.$$

Applying the Laplace transform and by using the initial conditions, we obtain

$$s^2U(s) - 1 = \frac{1}{s^2-1} + \frac{1}{s^2} - L\left[\int_0^1 x\cosh^2(t) + u^2(t)dt\right]$$

$$U(s) = \frac{1}{s^2} + \frac{1}{s^2(s^2-1)} + \frac{1}{s^4} - \frac{1}{s^2}L\left[\int_0^1 x\cosh^2(t) + u^2(t)dt\right]$$

.

Applying the inverse Laplace transform we get

$$u(x) = \sinh x + \frac{x^3}{6} + L^{-1}\left\{\frac{1}{s^2}L\left[\int_0^1 x\cosh^2(t) + u^2(t)dt\right]\right\}$$

.

Now we get the modified recursive relation is as given below

$$u_0(x) = \sinh x,$$

$$u_1(x) = \frac{x^3}{6} + L^{-1}\left\{\frac{1}{s^2}L\left[\int_0^1 x\cosh^2(t) + u_0^2(t)dt\right]\right\} = 0$$

$$u_{n+1}(x) = \frac{x^3}{6} + L^{-1}\left\{\frac{1}{s^2}L\left[\int_0^1 x\cosh^2(t) + A_n(t)dt\right]\right\} = 0, n \geq 1,$$

in which $\sum_{n=0}^{\infty} u_j u_{n-j}$ where for every $n \geq 1, A_n = 0$.

Thus the exact solution is

$$u(x) = \sinh x.$$

Example 3.

Let us consider the nonlinear integro differential equation

$$u'''(x) = \sin x - x - \int_0^{\frac{\pi}{2}} xtu'(t)dt,$$
$$u(0) = 1, u'(0) = 0, u'(0) = 1.$$

Applying the Laplace transform and by using the initial condition, we have

$$s^3U(s) - s^2 + 1 = \frac{1}{s^2+1} - \frac{1}{s^2} - L\left[\int_0^{\frac{\pi}{2}} xtu'(t)dt\right]$$

$$U(s) = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5} - \frac{1}{s^3}L\left[\int_0^{\frac{\pi}{2}} xtu'(t)dt\right].$$

Applying Inverse Laplace transform to both side

$$u(x) = \cos x - \frac{1}{24x^4} - L^{-1}\left\{\frac{1}{s^3}L\left[\int_0^{\frac{\pi}{2}} xtu'(t)dt\right]\right\}.$$

Now we get the modified recursive relation is as given below

$$u_0(x) = \cos x$$
$$u_1(x) = -\frac{1}{24x^4} - L^{-1}\left\{\frac{1}{s^3}L\left[x\int_n^{\frac{\pi}{2}} tu_0'(t)dt\right]\right\} = 0$$
$$u_{n+1}(x) = -L^{-1}\left\{\frac{1}{s^3}L\left[\int_0^{\frac{\pi}{2}} xtu_n'(t)dt\right]\right\} = 0, n \geq 1.$$

So the exact solution of the above problem is given by

$$u(x) = \cos x.$$

Example 4. Consider the nonlinear integro-differential equation

$$u''(x) = \frac{1}{2}e^x + \frac{1}{2}\int_0^x e^{x-2t}u^2(t)dt,$$
$$u(0) = u'(0) = 1.$$

Taking the Laplace transform and by use the give initial condition we get

$$sU(s) = -\frac{1}{s} + L\left[\int_0^x u^2(t)dt\right]$$
$$U(s) = -\frac{1}{s^2} + \frac{1}{s}L\left[\int_0^x u^2(t)dt\right].$$

Applying inverse Laplace transform to both sides

$$u(x) = -x + L^{-1}\left\{\frac{1}{s}L\left[\int_0^x u^2(t)dt\right]\right\}. \tag{1.16}$$

Now we decompose the solution in the form of an infinite sum given below

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{1.17}$$

Using (1.17) on (1.16), we get

$$\sum_{n=0}^{\infty} u_n(x) = -x + L^{-1}\left\{\frac{1}{s}L\left[\int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right]\right\},$$

in which $\sum_{n=0}^{\infty} u_j u_{n-j}$ the recursive relation is given below

$$u_0(x) = -x,$$
$$u_1(x) = \frac{x^4}{12}$$
$$u_2(x) = -\frac{x^7}{252}.$$

By repeating the above procedure for $n \geq 3$, we get to an approximate solution.

4. Conclusion

In this paper, we have discussed new Modified Laplace Transform Adomian Decomposition Methods to find solutions of integro-differential equations (IDEs). We have successfully applied to find efficient numerical solutions of nonlinear Volterra-Fredholm integro-differential equations. In the above examples, we observed that the LADM with the initial approximation obtained from initial conditions gives us a good approximation to the exact solution only in a few iterations. The LADM gives approximate solutions with fewer computational steps in comparison with ADM. The results reveal that the proposed method is simple to execute and effective from a computational point of view.

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