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Remarks on the fractional abstract differential equation with nonlocal conditions

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Abstract

In this paper, we study the existence and uniqueness of a solution to an initial value problem for a class of nonlinear fractional involving Riemann-Liouville derivative with nonlocal initial conditions in Banach spaces. We prove our main result by introducing a regular measure of noncompactness in the weighted space of continuous functions and using fixed point theory. Our result improve and complement several earlier related works. An example is given to illustrate the applications of the abstract result.

Keywords

Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, nonlocal initial conditions, point fixed, measure of noncompactness.

AMS Subject Classification

26A33, 34A08, 34K30, 47H08, 47H10.

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1. Introduction

Recently, fractional differential equations have attracted considerable interest in both mathematics and applications, since they have been proved to be valuable tools in modeling many physical phenomena. There has been significant development in fractional differential equations in recent years, see the monographs of Samko et al.[27], Kilbas et al.[20], Miller and Ross [22], Podlubny [26], and the references therein. The definitions of Riemann-Liouville fractional derivatives or integrals initial conditions play an important role, in some practical problems. Heymans and Podlubny [19], have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions. In [14], Gaston et al. studied fractional order differential equations with Caputo derivative

$$D^q x(t) = f(t,x(t)); t \in [0,b]; 0<\alpha<1,$$

with nonlocal condition

$$x(0)+g(x)=x_0.$$

As indicated in Deng's pioneering paper [11], the nonlocal condition $x(0)+g(0)=x_0$ can be applied in physics with better effect than the classical Cauchy problem with initial condition $x(0)=x_0$. For instance the author used

$$g(x)=\Sigma_{i=1}^{p} c_i x(t_i),$$

where $c_i=1,2,\cdots,p$ are given constants and $0<t_1<t_2<\cdots<t_p\leq T$. To describe the diffusion phenomenon of a small amount in a transparent tube. In this case, the Cauchy problem allows the additional measurements at t_i, $i=1,2,\cdots,p$.
In this work we consider the following Cauchy problem for the nonlocal initial conditions fractional differential equation

$$^{L}D^{\alpha}_{0^+}x(t)=f(t,x(t)); \;\; t\in J':=(0,b], \tag{1.1}$$

$$(I^{1-\alpha}_{0^+}x)(0)+g(x)=x_0, \tag{1.2}$$

where ${}^{L}D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α, $I^{1-\alpha}_{0^+}$ is the Riemann-Liouville integral of order $1-\alpha$, $0<\alpha<1$.
This paper is organized in the following way. In Sect 2 we introduce the notations, definitions, and preliminary facts that will be used in remainder of this paper. In Sect 3 we prove the main results. Finally an illustrative example is given in Sect 4.

2. Preliminaries

Let $J := [0,b], b>0$ and $(E,\|\cdot\|)$ be a Banach space, $C(J,E)$ be the space of E-valued continuous functions on J endowed with the uniform norm topology

$$\|x\|_{\infty} = \sup\{\|x(t)\|, t\in J\}.$$

Let $L^1(J,E)$ the space of E-valued Bochner integrable functions on J with norm

$$\|f\|_{L^1} = \int_0^b \|f(t)\|dt.$$

We consider the Banach space of continuous functions

$$C_{1-\alpha}(J,E) = \{x\in C(J',E) : \lim_{t\to 0^+} t^{1-\alpha}x(t) \text{ exists }\}.$$

A norm in this space is given by

$$\|x\|_{\alpha} = \sup_{t\in J}\left\{t^{1-\alpha}\|x\|_E\right\},$$

it easy to see $(C_{1-\alpha}(J,E), \|x\|_{\alpha})$ is a Banach space. For Ω a subset of the space $C_{1-\alpha}(J,E)$, define Ω_{α} by

$$\Omega_{\alpha} = \{x_{\alpha}, x\in\Omega\},$$

where

$$x_{\alpha}(t) = \begin{cases} t^{1-\alpha}x(t), & \text{if } t\in(0,b], \\ \lim_{t\to 0^+} t^{1-\alpha}x(t), & \text{if } t=0. \end{cases}$$

It is clear that $x_{\alpha}\in C(J,E)$.

Lemma 2.1. *[19] A set $\Omega\subset C_{1-\alpha}(J,E)$ is relatively compact if and only if Ω_{α} is relatively compact in $C(J,E)$.*

Definition 2.2. *Let $0<\alpha<1$. A function $x : J\to E$ has a fractional integral if the following integral*

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha}x(s)ds,$$

is defined for $t\geq 0$, where $\Gamma(\cdot)$ is the Gamma function. The Reimann-Liouville derivative of x of order α is defined as

$${}^{L}D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}x(s)ds = \frac{d}{dt}I^{1-\alpha}x(t),$$

provided it is well defined for $t\geq 0$. The previous integral is taken in Bochner sense. Let $\phi_{\alpha}(t) : \mathbb{R}\to\mathbb{R}$ defined by

$$\phi_{\alpha} = \begin{cases} \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}, & \text{if } t>0, \\ 0, & \text{if } t\leq 0. \end{cases}$$

Then

$$I^{\alpha}x(t) = (\phi_{\alpha} * x)(t),$$

and

$${}^{L}D^{\alpha}x(t) = \frac{d}{dt}(\phi_{1-\alpha} * x)(t).$$

Lemma 2.3. *[12] Let $\alpha,\beta\in\mathbb{R}_+$. Then*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and hence

$$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}dt = x^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The integral in the first equation of Lemma 2.3 is known as Beta function $B(\alpha,\beta)$.
Next, we recall some definitions and properties of measure of noncompactness, for more details, we refer the reader to [[3],[5],[6],[13][17],[28]].

Definition 2.4. *Let E be a Banach space, $\mathscr{P}(E)$ denote the collection of all nonempty subsets of E, and $(\mathscr{A},\geq)$ a partially ordered set A map $\beta : \mathscr{P}(E)\longrightarrow\mathscr{A}$ is called a measure of noncompactness on E, MNC for short, if*

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega\in\mathscr{P}(E)$, where $\overline{co}\Omega$ is the closure of convex hull of Ω.

Definition 2.5. *A measure of noncompactness β is called*
(1) *monotone if for each $\Omega_0,\Omega_1\in\mathscr{P}(E)$,from $\Omega_0\subset\Omega_1$ follows $\beta(\Omega_0)\leq\beta(\Omega_1)$.*
(2) *nonsingular, if for each $a\in E$ and each $\Omega\in\mathscr{P}(E)$ we have $\beta(\{a\}\cup\Omega) = \beta(\Omega)$.*
If $\mathscr{A}$ is a cone in Banach space, the MNC β is called:
(3) *regular, if $\beta(\Omega)=0$ is equivalent to the relative compactness of $\Omega\in\mathscr{P}(E)$,*
(4) *real, if $\mathscr{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.*
As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon>0, \text{ for which } \Omega \text{ has a finite } \varepsilon\text{-net in } E\}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition, i.e.:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for each $\lambda\in\mathbb{R}$ and each $\Omega\in\mathscr{P}(E)$.

For any $W \subset C(J,E)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t x(s)ds : x \in W, \quad \text{for } t \in J = [0,b] \right\},$$

where $W(s) = \{x(s) \in E : x \in w\}$.

Lemma 2.6. *If $W \subset C(J,E)$ is bounded and equicontinuous then $\beta(W(t))$ is continuous on J and*

$$\beta\left(\int_0^t W(s)ds\right) \leq \int_0^t \beta(W(s))ds, \quad \textit{for } t \in [0,b].$$

Definition 2.7. *A continuous map $F : E \subset X \to X$ is said to be condensing with respect to a MNCβ (β-condensing) if for every bounded set $\Omega \subset E$ that is not relatively compact, we have*

$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Lemma 2.8. *[7] If $\{u_n\}_{n=1}^{\infty} \subset L^1(J,E)$ satisfies $\|u_n(t)\| \leq \kappa(t)$ a.e. on J for all $n \geq 1$ with some $\kappa \in L^1(J,\mathbb{R}_+)$. then the function $\chi(\{u_n\}_{n=1}^{\infty})$ be long to $L^1(J,\mathbb{R}_+)$ and*

$$\chi\left\{\left(\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \chi(u_n(s)ds : n \geq 1).$$

According to Lemma 2.4 in [25], we can obtain the results immediately. The map $F : B \subseteq Y \to Y$ is said to be an β-contraction if there exists a positive constant $k < 1$ such that $\beta(F(B_0)) \leq k\beta(B_0)$ for any bounded closed subset $B_0 \subseteq B$.

Theorem 2.9. *[6](Darbo-Sadovskii's fixed point theorem). If B is a bounded, closed and convex subset of a Banach space Y, and the continuous map $F : B \to B$ is an β-contraction, then the map F has at least one fixed point in B.*

3. Main Results

We investigate in our the Cauchy problem for the fractional differential equation above with the following assumptions.

(H_1) $f : [0,b] \times E \to E$ is continuous function.

(H_2) $\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall t \in [0,b], x,y \in E$.

(H_3) $g : C_{1-\alpha}([0,b],E) \to E$ is continuous and $\|g(x)-g(y)\| \leq L_g\|x-y\|_\alpha$.

Theorem 3.1. *Under assumptions (H_1-H_3), if $L_g < \dfrac{1}{2}$ and $L \leq \dfrac{\Gamma(2\alpha)}{2b^\alpha\Gamma(\alpha)}$.*
Then (1.1)-(1.2) has a unique solution.

Proof. Defined $T : C_{1-\alpha}(J,E) \to C_{1-\alpha}(J,E)$ by:

$$T(x)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds.$$

Let $B_r = \{x \in C_{1-\alpha}(J,E), \|x\|_\alpha \leq r\}$, where

$$r \geq 2\left[(\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha+1)}\right].$$

Then we can show that $T(B_r) \subset B_r$. So that $x \in B_r$ and set $g^* = \sup_{x \in C_{1-\alpha}(J,E)} \|g(x)\|$, $M = \sup_{t\in J} \|f(t,0)\|$ then we get

$$\begin{aligned}
t^{1-\alpha}\|T(x)(t)\| &\leq \|(x_0 - g(x))\| \\
&+ \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds \\
&\leq (\|x_0\| + g^*) + \frac{b^{1-\alpha}}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}(\|f(s,x(s)) - f(s,0)\| \right. \\
&\left. + \|f(s,0)\|)ds\right] \\
&\leq (\|x_0\| + g^*) + \frac{Mb^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&+ \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s,x(s)) - f(s,0)\|ds \\
&\leq (\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha+1)} \\
&+ \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|f(s,x(s)) - f(s,0)\|_\alpha ds \\
&\leq (\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha+1)} + \frac{Lb^\alpha B(\alpha,\alpha)}{\Gamma(\alpha)} r \leq r.
\end{aligned}$$

Now take $x, y \in C_{1-\alpha}(J,E)$, we get

$$\begin{aligned}
&t^{1-\alpha}\|T(x)(t) - T(y)(t)\| \\
&\leq \|g(x) - g(y)\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s)) - f(s,y(s))\|ds \\
&\leq L_g\|x-y\|_\alpha + \frac{Lb^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|x-y\|ds \\
&\leq L_g\|x-y\|_\alpha + \frac{Lb^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|x-y\|_\alpha ds \\
&\leq L_g\|x-y\|_\alpha + \frac{Lb^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\|x-y\|_\alpha \\
&\leq \left(L_g + \frac{Lb^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\right)\|x-y\|_\alpha \\
&\leq \Omega_{L,L_g,b,\alpha}\|x-y\|_\alpha,
\end{aligned}$$

where $\Omega_{L,L_g,b,\alpha} := \left(L_g + \dfrac{Lb^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\right)$, which depends only on the parameters involved in the problem. And since

$\Omega_{L,L_g,b,\alpha} < 1$, then T is contraction mapping. Therefor, for by Banach's contraction principle T has a unique fixed point. $\square$

Our next result is based on the following the now assume

(H_4) there exists a constant $c_1 > 0$ such that

$$\|f(t,x)\| \leq c_1(1+t^{1-\alpha}\|x(t)\|) \ \text{ for all } t\in[0,b], \text{and } x\in E.$$

(H_5) there exists a constant $\widehat{L} > 0$ such that for each nonempty, bounded set $\Omega \subset C_{1-\alpha}(J,E)$

$$\chi(f(t,\Omega)) \leq \widehat{L}\chi(\Omega(t)), \text{ for all } t\in J,$$

where χ is the Hausdorff measure of noncompactness in E.

For brevity, let

$$M_1 = \frac{c_1 b}{\Gamma(\alpha+1)},$$

$$M_2 = (\|x_0\| + g^*) + \frac{c_1 b}{\Gamma(\alpha+1)}.$$

Define an operator T on $C_{1-\alpha}(J,E)$ by

$$(Tx)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \ t\in(0,b],$$

for any $x\in C_{1-\alpha}(J,E)$, let $(Tx)(t) = (T_1x)(t) + (T_2x)(t)$, where

$$(T_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds,$$

$$(T_2x)(t) = t^{\alpha-1}(x_0 - g(x)).$$

Assume that $M_1 < 1$, and let $B_r = \{x\in C_{1-\alpha}(J,E) : \|x\|_\alpha \leq r\}$, where $r \geq \dfrac{M_2}{1-M_1}$.

Lemma 3.2. *If the assumptions (H_1),(H_4) are satisfied with $M_1 < 1$, and (H_5). Then $T_1(B_r)$ is relatively compact set in $C_{1-\alpha}(J,E)$.*

Proof. Using (H_4) we can easily prove that $T_1x\in C_{1-\alpha}(J,E)$ for any $x\in C_{1-\alpha}(J,E)$. Then T_1 is well defined on $C_{1-\alpha}(J,E)$. We divide the proof into a sequence of steps.

Step 1. T_1 is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}(J,E)$. Then

$$t^{1-\alpha}\|T_1(x_n)(t) - T_1(x)(t)\|$$

$$\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s)) - f(s,x(s))\|ds$$

$$\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s,x_n(s)) - f(s,x(s))\|ds$$

$$\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_\alpha ds.$$

Using hypothesis (H_4) we have

$$\|T_1(x_n) - T_1(x)\|_\alpha \leq \frac{b^\alpha}{\Gamma(\alpha)}B(\alpha,\alpha)\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_\alpha.$$

Hence

$$\|T_1(x_n) - T_1(x)\|_\alpha \to 0, \quad \text{as } \ n\to+\infty.$$

Step 2. T_1 maps bounded sets into bounded sets in $C_{1-\alpha}(J,E)$. Indeed, it is enough to show that there exists a positive constant l such that for each $x\in B_r = \{x\in C_{1-\alpha}(J,E) : \|x\|_\alpha \leq r\}$ one has $\|T_1(x)\|_\alpha \leq l$.

$$t^{1-\alpha}\|T_1(x)(t)\| \leq \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds$$

$$\leq \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}c_1(1+s^{1-\alpha}\|x(s)\|)ds$$

$$\leq \frac{c_1 b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+r)ds$$

$$\leq \frac{c_1 b}{\Gamma(\alpha+1)}(1+r) := l.$$

Step 3. T_1 maps bounded sets into equicontinuous sets.

$$\|t_2^{1-\alpha}T_1(x)(t_2) - t_1^{1-\alpha}T_1(x)(t_1)\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}\left\|\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]f(s,x(s))ds\right\|$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\right\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]c_1(1+s^{1-\alpha}\|x(s)\|)ds$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}c_1(1+s^{1-\alpha}\|x(s)\|)ds$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}c_1(1+r)\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]ds$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}c_1(1+r)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds.$$

Thus

$$\|t_2^{1-\alpha}T_1(x)(t_2) - t_1^{1-\alpha}T_1(x)(t_1)\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha+1)}c_1(1+r)[(t_2-t_1)^\alpha + (t_1^\alpha - t_2^\alpha)]$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha+1)}c_1(1+r)(t_2-t_1)^\alpha.$$

As $t_2 \to t_1$, the right-hand side of above expression tends to zero. Then $T_1(B_r)$ is equicontinuous.

Step 4. T_1 is ν-condensing. We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset C_{1-\alpha}(J,E)$.

$$\nu(\Omega) = \max_{\Omega \in \Delta(\Omega)} (\gamma(\Omega), \quad \mathrm{mod}\,_{C_{1-\alpha}}(\Omega)), \tag{3.1}$$

$\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the partial order in the cone $\mathbb{R}^2_+$, γ is the damped modules of fiber noncompactness

$$\gamma(\Omega) = \sup_{t\in J} e^{-\mu t}\chi(\Omega_\alpha(t)),\ \mu \geq 0, \tag{3.2}$$

where $\Omega_\alpha(t) = \{x_\alpha(t) : x(t) \in \Omega\}$ and $\mathrm{mod}\,_{C_{1-\alpha}}(\Omega)$ is the modulus of equicontinuity of the set of functions Ω given by formula

$$\mathrm{mod}\,_{C_{1-\alpha}}(\Omega) = \lim_{\delta\to 0}\sup_{x\in\Omega}\max_{|t_1-t_2|\leq\delta} \|x_\alpha(t_1) - x_\alpha(t_2)\|. \tag{3.3}$$

Let

$$\sigma(\mu) = \sup_{t\in J}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{-\mu(t-s)}ds. \tag{3.4}$$

It is clear that

$$\sup_{t\in[0,b]}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{-\mu(t-s)}ds \to 0 \ \text{ as } \ \mu \to +\infty.$$

We can choose μ such that

$$\overline{\sigma} = \frac{2\widehat{L}b^{1-\alpha}}{\Gamma(\alpha)}\sigma(\mu) < 1. \tag{3.5}$$

From Lemma , the measure ν is well defined and give a monotone, nonsingular and regular measure of noncompactness in $C_{1-\alpha}(J,E)$.

Let $\Omega \subset C_{1-\alpha}(J,E)$ be a bounded subset such that

$$\nu(T_1(\Omega)) \geq \nu(\Omega). \tag{3.6}$$

We will show that (3.6) implies that Ω is relatively compact. Let the maximum on the left-hand side of the inequality (3.6) be a chieved for the countable set $\{y^n\}_{n=1}^{+\infty}$ with

$$y^n(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_n(s)ds,\ \{x^n\}_{n=1}^{+\infty} \subset \Omega, \tag{3.7}$$

and $f_n(t) = f(t,x^n(t))$.

We give now an upper estimate for $\gamma(\{y^n\}_{n=1}^{+\infty}$. By using (H_5) we have

$$\begin{aligned}\chi(\{(t-s)^{\alpha-1}f_n(s)\}_{n=1}^{+\infty}) &\leq (t-s)^{\alpha-1}\widehat{L}\chi(\{x^n(s)\}_{n=1}^{+\infty})\\ &\leq \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\chi(\{x^n(s)\}_{n=1}^{+\infty})\\ &= \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}\chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty})\\ &\leq \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\sup_{0\leq s\leq t} e^{-\mu s}\chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty})\\ &= \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\gamma(\{x^n\}_{n=1}^{+\infty}),\end{aligned}$$

for all $t \in [0,b]$, $s \leq t$. Then applying Lemma 2.8, we obtain

$$\chi(\{y^n\}_{n=1}^{+\infty}) \leq \frac{2\widehat{L}b^{1-\alpha}}{\Gamma(\alpha)}\sup_{t\in[0,b]}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Taking (3.5) and (3.7) into account, we derive

$$\gamma(\{y^n\}_{n=1}^{+\infty}) \leq \overline{\sigma}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Combining the last inequality with (3.6), we have

$$\gamma(\{x^n\}_{n=1}^{+\infty}) \leq \overline{\sigma}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Therefore

$$\gamma(\{x^n\}_{n=1}^{+\infty}) = 0.$$

Furthermore, from step 3, we know that $\mathrm{mod}\,_{C_{1-\alpha}}(T_1(\Omega)) = 0$ and (3.6) yields $\mathrm{mod}\,_{C_{1-\alpha}}(\Omega) = 0$. Finally,

$$\nu(\Omega) = (0,0),$$

which prove the relative compactness of set Ω. □

Theorem 3.3. *Assume that (H_1), (H_3), (H_4), (H_5) hold, with $M_1 < 1$. Then (1.1)-(1.2) has least one solution.*

Proof. Using (H_1), (H_4) can be prove that $Tx \in C_{1-\alpha}(J,E)$ for any $x \in C_{1-\alpha}(J,E)$. Then T is well defined on $C_{1-\alpha}(J,E)$. We will show that T satisfies all conditions of Theorem 3.1, the proof will be given in several steps.

For any $x \in B_r$ and $t \in J$, taking into account the imposed assumptions, we obtain

$$\begin{aligned}&t^{1-\alpha}\|(Tx)(t)\| \leq (\|x_0\| + g^*)\\ &+\frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds\\ &\leq (\|x_0\| + g^*) + \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}c_1(1+s^{1-\alpha}\|x(s)\|)ds\\ &\leq (\|x_0\| + g^*) + \frac{c_1b^{1-\alpha}(1+r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\\ &\leq (\|x_0\| + g^*) + \frac{c_1b(1+r)}{\Gamma(\alpha+1)} \leq r.\end{aligned}$$

Then T is maps B_r into B_r.

Next, we will show that T is continuous in B_r.

By (H_3), for $L_g < 1$ it is clear that T_2 is a contraction mapping. This means that T is continuous in B_r.

According to Lemma 3.2, $T_1(B_r)$ is relatively compact in $C_{1-\alpha}(J,E)$, then $\chi(T_1(B_r)) = 0$. For any $x_1, x_2 \in B_r$, we have

$$t^{1-\alpha}\|T_2x_2(t) - T_2x_1(t)\| \leq \|g(x_2) - g(x_1)\|$$

which implies that

$$\|T_2x_2 - T_2x_1\|_\alpha \leq L_g\|x_2 - x_1\|_\alpha.$$

Hence

$$\beta(T_2(B_r)) \leq L_g \beta(B_r).$$

Therefore

$$\chi(T(B_r)) \leq \chi(T_1(B_r)) + \chi(T_2(B_r))$$

$$\leq L_g \chi(B_r).$$

Noting that $L_g < 1$, we find that the operator T is an χ-contraction in B_r. Then problem (1.1)-(1.2) has at least one solution in B_r. The proof is complet. □

4. An example

In section, we discuss an example to illustrate our results. Let us consider the fractional differential equation nonlocal

$$^{L}D^{\alpha}_{0^+}x(t) = \frac{1}{e^{t^2}+1}\left\{\ln(|x_k|+1) + \frac{1}{k^2}\right\}_{k=1}^{\infty}, \quad t \in J = [0,1], \tag{4.1}$$

$$(I^{1-\alpha}_{0^+}x)(0) + g(x) = x_0, \tag{4.2}$$

c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm

$$\|x\| = \sup_k |x_k|.$$

Let $t \in J$ and $x = \{x_k\}_k \in c_0$, we have

$$\|f(t,x)\|_\infty = \frac{1}{e^{t^2}+1}\|\ln(|x_k|+1) + \tfrac{1}{k^2}\|_\infty$$

$$\leq \frac{1}{e^{t^2}+1}\left(\sup_k |x_k| + 1\right)$$

$$\leq \frac{1}{e^{t^2}+1}(1 + \|x\|_\infty).$$

Hence conditions (H_1), (H_4) are satisfied with $p(t) = \dfrac{1}{e^{t^2}+1}$, for all $t \in [0,1]$.
We recall that the measure of noncompactness χ in space c_0 can be computed by means of the formula

$$\chi(\Omega) = \lim_{n\to+\infty} \sup_{x\in\Omega} \|(I - P_n)x\|_\infty.$$

Where Ω is a bounded subset in c_0 and P_n is the projection onto the linear span of n vectors, we get

$$\chi(f(t,\Omega)) \leq \eta(t)\chi(\Omega(t)) \quad \text{for all} \quad t \in [0,1],$$

with $\eta(t) = (e^{t^2}+1)^{-1}$. Hence (H_5) is satisfied.
Denote $g(x) = \sum_{i=1}^{m} c_i x(t_i)$, then for any $x = \{x_k\}_k, y = \{y_k\}_k \in c_0$, one has

$$\|g(x) - g(y)\|_\infty \leq \sum_{i=1}^{m} |c_i| \|x - y\|_\infty.$$

Clearly, $L_g = \sum_{i=1}^{m} |c_i|$ and choose c_i such that $L_g < 1$.
Assume that (H_1), (H_3), (H_4), (H_5) is satisfied and $M_1 < 1$. Then by Theorem 3.3 the fractional problem (4.1)-(4.2) has least one solution.

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References

[1] Abbas S.,Benchohra M.,N'Guérékata G.M., Topics in fractional differential equations. Springer, New York (2012)

[2] Agarwal R.P,Lupulescu V.,O'Regan D., ur RahmanG., Fractional calculus and fractional differential equations in nonreflexive Banach spaces, *Commun. Nonlinear Sci. Numer.Simulat.*, 2015, 20, 59–73.

[3] Aghajani A.,Pourhadi E.,Trujillo J.J., Application of measure of noncompactness to Cauchy problem for fractional differential equations in Banach spaces, *Frac. Calc. Appl. Annl.*, 2013, 16, 362–377.

[4] Aizicovici S., McKibben M., Existence results for a class of abstract nonlocal Cauchy problems, *Nonlinear Anal. TMA*, 2000, 39, 649–668.

[5] Akhmerov R.R.,Kamenskii M.I.,Potapov A.S.,Rodkina A.E.,Sadovskii B.N., Measures of noncompactness and condensing operators. Birkhauser, Boston, Basel, Berlin (1992)

[6] Banas.J ,Goebel.K , Measure of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Math.,Vol 60, Dekker, New York, 1980

[7] Bothe D., Multivalued perturbations of m-accretive differential inclusions, *Israel J. Math.*, 1998, 108, 109–138.

[8] Byszewski L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J.Math. Anal. Appl.*, 1991, 162, 494–505.

[9] Delbosco D., Rodino L., Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, 1996, 204, 609–625.

[10] Deimling K., Nonlinear functional analysis. Springer-Verlag, Berlin, 1985.

[11] Deng K., Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, 1993, 179, 630–637.

[12] Diethelm K., The analysis of fractional differential equations. Lecture Notes in Mathmatics 2004, Springer-verlag, Berlin, 2010.

[13] Dong Q.,Li G., Measure of noncompactness and semilinear nonlocal functional differential equations in Banach spaces, *Acta Mathematica Sinica, English Series.*, 2015, 31, No. 1, 140–150.

[14] Gaston M., N'Guérékata, A Cauchy problem for some fractional abstract differential equation with non local conditions, *ScienceDirect Nonlinear Analysis* ., 2009, 70, 1873–1876.

[15] Gaston M., N'Guérékata, Existence and uniquencess of an integral to some Cauchy problem with nonlocal conditions, *In: Differential and Difference Equations and Applications, Hindawi Publ.Cop,New York*, 2006, 843–849.

[16] Glockle W.G., Nounenmacher T.F, A fractional calculus approach of self-similar protein dynamics, *Biophys.J.*,1995, 68, 46–53.

[17] Heinz H.P., On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Anal. TMA.*, 1983, 7, 1351–1371.

[18] Hernández E., O'Regan D., On a new class of abstract impulsive differential equations, *Proc. Am. Math. Soc*, 2013, 141(5), 1641–1649.

[19] Heymans N., Podlubny I., physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, *Rheol. Acta.*, 2006, 45, 765–771.

[20] Kilbas A.A.,Srivastava H.M.,Trujillo J.J., Theory and applications of fractional differential equations. North-Holland Mathematics studies 204. Elsevier Science B.V., Amsterdam (2006)

[21] Mainandi F., Fractional calculus: Some basic problems in continuum and statistical mechanis. In: Carpinteri A.,Mainardi F.(eds.) Fractals and fractional calculus in continuum mechanics, *Springer-verlag, Wien*, 1997, 291–348.

[22] Miller K.S.,Ross B., An introdution to the fractional calculus and differential equations. John Wiley, New York (1993)

[23] Lakshmikantham V., Devi J.V., Theory of fractional differential equations in Banach space, *European J. Pur Appl. Math.*, 2008, 1(1), 38–45.

[24] Lasry J., Robert R., Analyse nonlineare multique, VER mathematiques de la decision, Numero 249. *Université de Paris-Dauphine.*

[25] Liu L., Guo, F., Wu C., Wu Y., Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces, *J. Math. Anal. Appl.*, 2005, 309, 638–649.

[26] Podlubny I., Fractional differential equations. Academic Press, SanDiego (1999)

[27] Samko S.G., Kilbas A.A., Marichev O.L., Fractional Integral and Derivatives, Theory and applications. Gordon and Breach, Yverdon (1993)

[28] Toledano J.M.A.,Benavides T.D.,Azedo D.L., Measures of noncompactness in metric fixed point theory. Birkhauser, Basel (1997)

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