



Unique isolate domination in graphs

Sivagnanam Mutharasu¹ and V. Nirmala^{2*}

Abstract

A dominating set S of a graph G is said to be an isolate dominating set of G if the induced subgraph $\langle S \rangle$ has at least one isolated vertex [6].

A dominating set S of a graph G is said to be an unique isolate dominating set (UIDS) of G if $\langle S \rangle$ has exactly one isolated vertex. An UIDS S is said to be minimal if no proper subset of S is an UIDS. The minimum cardinality of a minimal UIDS of G is called the UID number, denoted by $\gamma_0^U(G)$. This paper includes some properties of UIDS and gives the UID number of paths, complete k -partite graphs and disconnected graphs. Finally, the role played by UIDS in the domination chain has been discussed in detail.

Keywords

Isolate dominating set, unique isolate dominating set, unique isolate domination number.

AMS Subject Classification

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¹Department of Mathematics, C. B. M. College, Coimbatore - 641042, Tamil Nadu, India.

²Department of Science and Humanities (Mathematics), R.M.K. Engineering College, Chennai-601206, Tamil Nadu, India.

*Corresponding author: ² nirmalradha2001@yahoo.co.in

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1. Introduction

In this paper, we consider only finite, non-trivial and undirected graphs with no loops and no multiple edges. For graph theoretic terminology, we refer to [2].

Let $G = (V, E)$ be a simple connected graph. For $v \in V$, the open neighborhood $N(v)$ is the set of all vertices which are adjacent to v . The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is defined by $deg(v) = |N(v)|$. The minimum and maximum degree of G is defined by $\delta(G) = \min_{v \in V} \{deg(v)\}$ and $\Delta(G) = \max_{v \in V} \{deg(v)\}$ respectively.

A set $S \subseteq V$ is called a dominating set if every vertex in V is either an element of S or adjacent to an element of S . A dominating set S is minimal if no proper subset of S is a dominating set. The minimum and maximum cardinality of a minimal dominating set of G are called the domination num-

ber $\gamma(G)$ and the upper domination number $\Gamma(G)$ respectively.

In 2016, Hameed and Balamurugan [6] introduced the concept of isolate domination in graphs. Further, in [5], they characterized unicycle graphs on which the order equals the sum of the isolate domination number and its maximum degree. A dominating set S of a graph G is said to be an isolate dominating set if $\langle S \rangle$ has at least one isolated vertex [6]. An isolate dominating set S is said to be minimal if no proper subset of S is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of G are called the isolate domination number $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively. An isolate dominating set of cardinality γ_0 is called a γ_0 -set.

By using the above concept of isolate domination, we define a new concept called "Unique Isolate Domination (UID)". A dominating set S of G is said to be an UIDS of G if $\langle S \rangle$ has exactly one isolated vertex. An UIDS S is said to be minimal if no proper subset of S is an UIDS. The minimum and maximum cardinality of a minimal UIDS of G are called the UID number $\gamma_0^U(G)$ and the upper UID number $\Gamma_0^U(G)$ respectively. An UIDS of cardinality γ_0^U is called a γ_0^U -set. Note that the cycle C_4 does not admit UIDS but it admits isolate dominating sets. So many differences between these

two domination parameters that we have discussed in the next section. This paper includes some basic properties of UIDS and the role played by UIDS in the domination chain has been discussed.

2. Extended domination chain

A set $S \subseteq V$ of vertices is called an independent dominating set if S is a dominating set and the induced subgraph $\langle S \rangle$ is totally disconnected. The independent domination number $i(G)$ of a graph G equals the minimum cardinality of an independent dominating set in G . The independence number $\beta_0(G)$ of a graph G is the maximum cardinality of an independent set of G .

For a set S of vertices, a vertex v is said to be a private neighbor of a vertex $u \in S$ with respect to S if $N[v] \cap S = \{u\}$. A set S of vertices is irredundant if every vertex in S has at least one private neighbor with respect to S . The minimum and maximum cardinality of a maximal irredundant set are called the irredundance number $ir(G)$ and the upper irredundance number $IR(G)$ respectively. An inequality chain connecting these parameters was established in [3] as given below:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G). \quad (2.1)$$

A detailed study of this domination chain can be found in [4]. Extending this chain by new parameters whose values lie between any two parameters in the chain is one direction of research.

In [6], Hameed extend this domination chain and obtained the following:

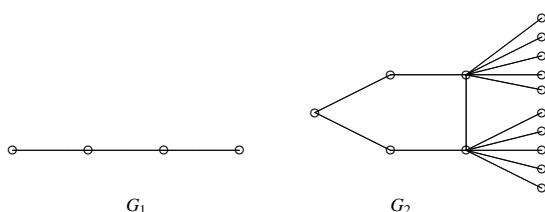
$$ir(G) \leq \gamma(G) \leq \gamma_0(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_0(G) \leq \Gamma(G) \leq IR(G). \quad (2.2)$$

In this section, we study the position of γ_0^U in the domination chain.

Theorem 2.1. For any graph G , we have $\gamma_0(G) \leq \gamma_0^U(G)$.

Proof. Since every UIDS of G is also an isolate dominating set of G , we have $\gamma_0(G) \leq \gamma_0^U(G)$. \square

Remark 2.2. Consider the following graphs G_1 and G_2 .



From the following table, it is concluded that the parameter $\gamma_0^U(G)$ is non-comparable with all these parameters $i(G), \beta_0(G), \Gamma(G), \Gamma_0(G)$ and $IR(G)$.

Parameter	Graphs	
	G_1	G_2
γ_0^U	3	3
i	2	7
β_0	2	12
Γ	2	12
Γ_0	2	12
IR	2	12

From Theorem 2.1 and Remark 2.2, we arrived a new domination chain as given below. For any graph G , we have

$$ir(G) \leq \gamma(G) \leq \gamma_0(G) \leq \gamma_0^U(G) \leq \Gamma_0^U(G). \quad (2.3)$$

3. Basics of unique isolate domination

In this section, the UID number of paths and complete k -partite graphs have been obtained. Also some properties of UIDS are given.

Remark 3.1. If S is an UIDS of a graph G , then the induced subgraph $\langle S \rangle$ has exactly one isolated vertex and all other vertices of S has a neighbor in S .

Lemma 3.2. Let S be any UIDS of a graph G such that every non-isolated vertex of $\langle S \rangle$ has a private neighbor with respect to S . Then S is minimal.

Proof. Let $v \in S$. If v is the isolated vertex in $\langle S \rangle$, then $S - \{v\}$ will not dominate the vertex v . If v is non isolated vertex of $\langle S \rangle$, then there exist $w \in S$ such that w is the private neighbor of v in S . In this case, $S - \{v\}$ will not dominate the vertex w . Thus S is minimal. \square

The corona of two graphs G and H , denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H , and then joining the i -th vertex of G to every vertex in the i -th copy of H [1]. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are joined or attached to the vertex v .

Remark 3.3. The converse of Lemma 3.2 is not true. For example, let $G = C_4 \circ K_1$, where $V(C_4) = \{a_i | 1 \leq i \leq 4\}$ and for $1 \leq i \leq 4$, let b_i be the newly added vertex adjacent to a_i . Then $S = \{a_1, a_2, a_3, b_4\}$ is a minimal UIDS but the vertex a_1 has no private neighbor in S .

Remark 3.4. (a). If a graph G has a full vertex, say x , then $\{x\}$ is an UIDS and $\gamma_0^U(G) = 1$. Thus the complete graphs, wheels and stars admit UIDS.

(b). Since any UIDS S of a graph G contains exactly one isolated vertex in $\langle S \rangle$, $\gamma_0^U(G) \neq 2$.



Lemma 3.5. *Let P_n be a path of n vertices for $n \geq 1$. Then*
 (a). $\gamma_0^U(P_n) = 2t + 1$ if $n = 4t + 1$ or $4t + 2$ or $4t + 3$ for some integer $t \geq 0$.
 (b). $\gamma_0^U(P_4) = 3$.
 (c). $\gamma_0^U(P_n) = 2t$ if $n = 4t$ for some integer $t \geq 2$.

Proof. (a). Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Suppose $n = 4t + 1$ or $4t + 2$ or $4t + 3$ for some integer $t \geq 0$. When $t = 0$, each of the graphs P_1, P_2 and P_3 has a full vertex and hence by Remark 3.4(a), $\gamma_0^U(P_1) = \gamma_0^U(P_2) = \gamma_0^U(P_3) = 1 = 2t + 1$.

Suppose $n = 4t + 1$ for some integer $t \geq 1$. Let S be any UIDS of P_n and x be the isolated vertex in $\langle S \rangle$. Note that the vertex x can dominate a maximum of 3 vertices. Thus the remaining $n - 3$ vertices of P_n must be dominated by $S - \{x\}$. Also every vertex of $S - \{x\}$ is adjacent to at least one other vertex of $S - \{x\}$ and any two adjacent vertices of $S - \{x\}$ can dominate a maximum of 4 vertices. Since $n = 4t + 1 = 3 + 4(t - 1) + 2$, to dominate the $3 + 4(t - 1)$ vertices, S must have $1 + 2(t - 1) = 2t - 1$ vertices. Let v_a and v_b be two vertices of P_n which are not dominated by these $2t - 1$ vertices.

Case 1: If v_a and v_b are adjacent, then there exists a path P with at least 3 vertices such that $v_a, v_b \in P$ and $P \cap S = \emptyset$. If we choose one vertex of this path to dominate all the vertices of P , then it must be isolated in $\langle S \rangle$, a contradiction to S is uniquely isolated. Thus we need at least two more vertices in S to dominate all the vertices of P .

Case 2: Suppose v_a and v_b are not adjacent. Suppose there exists a vertex v_c such that v_c is adjacent with both v_a and v_b . Then $v_c \notin S$ and there exists a path P with at least 3 vertices such that $v_a, v_b \in P$ and $P \cap S = \emptyset$. In this case, as discussed above, we need at least two more vertices in S to dominate all the vertices of P_n . In the other case, to dominate v_a and v_b , we need at least two more vertices in S .

Thus in all the cases, S must have $1 + 2(t - 1) + 2 = 2t + 1$ vertices and hence $\gamma_0^U(P_n) \geq 2t + 1$. Also the set $\{v_1\} \cup \{v_{3+4i}, v_{4(i+1)} : i = 0, 1, 2, \dots, t - 1\}$ is an UIDS with $2t + 1$ vertices and so $\gamma_0^U(P_n) \leq 2t + 1$.

By using the facts $4t + 2 = 3 + 4(t - 1) + 3$ and $4t + 3 = 3 + 4t$, we can prove that $\gamma_0^U(P_n) = 2t + 1$ if $n = 4t + 2$ and $n = 4t + 3$ respectively.

(b). When $n = 4$, $\{v_1, v_3, v_4\}$ is a γ_0^U -set of P_4 .
 (c). Let $n = 4t$ for some integer $t \geq 2$. Since $n = 4t = 3 + 4(t - 1) + 1$, as discussed in (a), S must have $1 + 2(t - 1) + 1 = 2t$ vertices. Thus $\gamma_0^U(P_n) \geq 2t$. Also the set $\{v_2, v_{4t-1}\} \cup \{v_{4i+1}, v_{4i+2}/i = 1, 2, \dots, t - 1\}$ is an UIDS with $2t$ vertices and so $\gamma_0^U(P_n) \leq 2t$. \square

Lemma 3.6. *Let $k \geq 2$ be an integer and $G = K_{m_1, m_2, \dots, m_k} = (M_1, M_2, \dots, M_k)$ be a complete k -partite graph. Then G admits an UIDS if, and only if, $m_i = 1$ for some integer i with $1 \leq i \leq k$.*

Proof. Suppose G admits an UIDS, say S . On the contrary, assume that $m_i \geq 2$ for all $1 \leq i \leq k$.

Let x be the isolated vertex of $\langle S \rangle$. Without loss of generality, assume that $x \in M_1$. Since $|M_1| \geq 2$, we can choose a vertex $y \in M_1$ such that $y \neq x$. Note that no vertex of $M_2 \cup M_3 \cup \dots \cup M_k$ will be in S (otherwise x will not be isolated in $\langle S \rangle$). Thus to dominate the vertex y , S must include y and hence $\langle S \rangle$ has more than one isolated vertex, namely y , a contradiction.

The converse part follows from Remark 3.4(a). \square

Lemma 3.7. *Let H be a graph of order $n (\geq 3)$ such that H has a maximum of one isolated vertex and $G = H \circ K_1$. Then $\gamma_0^U(G) = \gamma(G) = n$.*

Proof. Since every pendant vertex or its support should be included in any dominating set and all the n pendant vertices together forms a dominating set, we have $\gamma(G) = n$ and hence $\gamma_0^U(G) \geq n$.

If H has an isolated vertex, let it be v . Let v' be the newly added vertex which is adjacent to v . Then $V(H)$ is an UIDS of G with n elements and v is the only isolated vertex in $\langle V(H) \rangle$. If H does not have isolated vertices, then $(V(H) - \{v\}) \cup \{v'\}$ is an UIDS of G with n elements for each $v \in V(H)$ and so $\gamma_0^U(G) \leq n$. \square

In the above Lemma, if we let $n = 2$, then H is isomorphic to the complete graph K_2 . In this case, G is a path on four vertices with $\gamma_0^U(G) = 3$ and $\gamma(G) = 2$.

When $n = 1$, H is isomorphic to the complete graph K_1 and G is the path on two vertices and hence by Remark 3.4(a), $\gamma_0^U(G) = \gamma(G) = 1$.

Theorem 3.8. *Let $n \geq 2$ be an integer and let G be a disconnected graph with n components G_1, G_2, \dots, G_n such that the first r components G_1, G_2, \dots, G_r admit UIDS. Then $\gamma_0^U(G) = \min_{1 \leq i \leq r} \{t_i\}$, where $t_i = \gamma_0^U(G_i) + \sum_{j=1, j \neq i}^n \gamma(G_j)$ for $1 \leq i \leq r$.*

Proof. With out loss of generality, let $t_1 = \min_{1 \leq i \leq r} \{t_i\}$.

Let S be a γ_0^U -set of G_1 and D_i be a γ -set of G_i for each i with $2 \leq i \leq n$. Then $S \cup (\bigcup_{i=2}^n D_i)$ is an UIDS of G with cardinality

$\gamma_0^U(G_1) + \sum_{i=2}^n \gamma(G_i)$ and so $\gamma_0^U(G) \leq \gamma_0^U(G_1) + \sum_{i=2}^n \gamma(G_i) = t_1$.

Let S be a minimal UIDS of G . Then S must intersect $V(G_i)$ for each $1 \leq i \leq n$. Further, there exists an integer j such that $S \cap V(G_j)$ is a minimal UIDS of G_j and $1 \leq j \leq r$. Also for each $1 \leq i \leq n, i \neq j$, the set $S \cap V(G_i)$ is a minimal total dominating set of G_i .

Therefore $|S| \geq \gamma_0^U(G_j) + \sum_{i=1, i \neq j}^n \gamma(G_i) \geq t_1$ and hence $\gamma_0^U(G) =$

$\min_{1 \leq i \leq r} \{t_i\}$. \square

Theorem 3.9. *Let $k \geq 1$ be an integer such that $k \neq 2$. Then there exists a graph G such that $\gamma(G) = \gamma_0^U(G) = k$.*



Proof. Let C_{k+2} be a cycle of order $k + 2$ and $V(C_{k+2}) = \{u_1, u_2, \dots, u_{k+2}\}$. Let G be a graph obtained from C_{k+2} by adding one pendant edge at each vertex of $V(C_{k+2}) - \{u_1, u_2, u_3\}$. Let S be a dominating set of G . To dominate all the pendant vertices in G , S must have at least $k - 1$ vertices of $V(G) - \{u_1, u_2, u_3\}$. And S must include at least one more vertex to dominate the vertex u_2 . Thus $|S| \geq (k - 1) + 1 = k$ and so $\gamma(G) \geq k$.

Note that $S = \{u_2, u_4, u_5, \dots, u_{k+2}\}$ is an UIDS with $|S| = k$ and u_2 is the only isolated vertex of $\langle S \rangle$. Thus $\gamma_0^U(G) \leq k$. Since $\gamma(G) \leq \gamma_0^U(G)$, we have $\gamma(G) = \gamma_0^U(G) = k$. \square

Theorem 3.10. *Let a and b be two integers such that $b > a \geq 2$. Then there exists a graph G such that $\gamma(G) = a$ and $\gamma_0^U(G) = b$.*

Proof. Let C_a be a cycle of order a with $V(C_a) = \{u_1, u_2, \dots, u_a\}$. Let H be any graph which admits UIDS with $\gamma_0^U(H) = \gamma(H) = b - a + 1$ (which is possible by Theorem 3.9) and $G = C_a \circ H$. Let S be a dominating set of G . Then S must include u_i or at least one vertex of H^{u_i} for each $1 \leq i \leq a$ and so $\gamma(G) \geq a$. Since $V(C_a)$ is a dominating set of G , we have $\gamma(G) \leq a$ and so $\gamma(G) = a$.

Let S be any UIDS of G and u be the isolated vertex of $\langle S \rangle$. Suppose $u \in V(C_a)$, without loss of generality, let it be u_1 . Then $u_2 \notin S$. To dominate the vertices of $V(H^{u_2})$, S must include at least $b - a + 1$ vertices of $V(H^{u_2})$ (since $\gamma(H^{u_2}) = b - a + 1$). Now to dominate the vertices of $V(H^{u_i})$ for each $i \neq 2$, S must include at least one vertex of $V(H^{u_i})$ or u_i . Thus $|S| \geq b - a + 1 + a - 1 \geq b$.

Suppose $u \in V(H^{u_i})$ for some $1 \leq i \leq a$. Without loss of generality, assume that $u \in H^{u_1}$. Then $u_1 \notin S$. To dominate the vertices of $V(H^{u_1})$, S must include at least $b - a + 1$ vertices of $V(H^{u_1})$ (since $\gamma(H^{u_1}) = \gamma_0^U(H^{u_1}) = b - a + 1$). Now to dominate the vertices of $V(H^{u_i})$ for each $i \neq 1$, S must include at least one vertex of $V(H^{u_i})$ or u_i . Thus $|S| \geq b - a + 1 + a - 1 \geq b$.

Hence in both the cases, we have $|S| \geq b$ and so $\gamma_0^U(G) \geq b$. Let S be a γ_0^U -set of H^{u_1} . Then $|S| = b - a + 1$ and $D = \{u_2, u_3, \dots, u_a\} \cup S$ is an UIDS of G with $|D| = a - 1 + b - a + 1 = b$. Thus $\gamma_0^U(G) \leq b$ and so $\gamma_0^U(G) = b$. \square

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