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Common fixed point theorems for generalized contractive mappings in an $F-$cone metric space over a Banach algebra

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Abstract
This paper is dealt with some common fixed point theorems for generalized contractive mappings in an $F-$cone metric space over a Banach algebra. Examples have been cited in support of our theorems.

Keywords
Banach algebra; $F-$cone metric space; common fixed point.

AMS Subject Classification
47H10, 54H25.

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1. Introduction

The concept of $b-$metric spaces had been initiated by Bakhtin [1] in 1989 which generalizes metric spaces, usually we mean, and thereby he had been succeeded to generalize the Banach contraction principle theorem over it. With the introduction of partial metric spaces by Matthews [22] in 1994, it had been possible to show an application of program verification in case of data flow network by generalizing the Banach contraction principle. The speciality of such partial metric p on a partial metric space (X,p) is that $p(x,x)$ may not be zero, a convergent sequence in (X,p) need not be necessarily have a unique limit.

In light of the same spirit Huang and Zhang [16] in 2007 had been able to set up a cone metric space as a generalization of metric space in which the set of real numbers are replaced by the elements of ordered Banach space. With this concept he had been succeeded to define Cauchy sequences and convergence of sequences in it. Subsequently many researchers (See [2],[19],[5],[8],[9],[17]) had proved fixed point theorems in such spaces. As we go through in a survey of literatures, we note that many authors had pointed out that fixed point theorems so far obtained in a cone metric space have no importance as such, because the results directly follows from the usual metric spaces where the real valued metric function is defined by some equivalent functions (See [3],[17]). Due to the loss of significant facts researchers had loose their interest to work with fixed point theory on such a cone metric space. Surprisingly Liu and Xu [20] switched on the concept of cone metric spaces over a Banach algebra and proved some fixed point theorems by weakening the conditions of Lipschitz constant of spectral radius for generalized contractive mapping. In recent years some authors (See [4],[7],[10],[20]) had been able to establish some fixed point theorems in a setting of cone metric space over a Banach algebra.

Following these concepts and with the generalization of partial metric spaces and cone metric spaces over Banach algebra, Fernandez et. al. [9] introduced the notion of partial cone metric spaces over a Banach algebra. Subsequently Fernandez et. al. (See [11],[12],[13]) had been succeeded to introduce the structures of (i) N_p-cone metric space over Banach algebra and (ii) N_b-cone metric space over Banach algebra respectively, as a generalization of (a) $N-$cone metric space over Banach algebra and (b) $N-$cone metric space over Banach algebra together with $b-$metric space.

Motivated with the introduction of $F-$cone metric space by Fernandez et. al. [6] as a generalization of N_p-cone metric space and N_b-cone metric space over a Banach algebra, we have been able to prove some common fixed point theorems for a class of generalized contractive mappings over such an $F-$cone metric space over a Banach algebra. The illustrative examples are given in strengthening of the hypothesis of our theorems.

2. Preliminaries

In this section we recall some basic definitions and some basic results together with its consequences which are relevant to our findings.

Definition 2.1. *[20] A linear space A over a field K($\mathbb{R}$ or $\mathbb{C}$) is called an algebra if it is closed under multiplication (i.e. if for all $x,y \in A$, $xy \in A$) and satisfies the following properties:*
(i) $(xy)z = x(yz)$, $\forall\ x,y,z \in A$
(ii) $x(y+z) = xy + yz$ and $(x+y)z = xz + yz$, $\forall\ x,y,z \in A$
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$, $\forall\ x \in A$, $\forall \alpha \in K$.
A Banach space A over the field K ($\mathbb{R}$ or $\mathbb{C}$) is said to be a Banach algebra if
(i) A is an algebra and (ii) $\forall\ x,y \in A$, $||xy|| \leq ||x||\ ||y||$.

Throughout this paper, we shall assume that a Banach algebra is always unital, that is it has a unity element e such that $ex = xe = x\ \forall x \in A$. A non-zero element $x \in A$ is said to be invertible if its inverse exists. i.e. if there exists a non-zero element y such that $xy = yx = e$, we write $y = x^{-1}$ and say that y is the inverse of x. Note that the unity element of a Banach algebra A, if it exists, is unique. Also it can be shown that in a Banach algebra A, with the unity element e, the inverse of an element is unique. Note that in a Banach algebra with unity e we have $(xy)^{-1} = y^{-1}x^{-1}$ and $(x^{-1})^{-1} = x\ \forall x,y \in A$. For further details reader may take help of [24].

Proposition 2.2. *[24] Let A be a Banach algebra with a unit e, the spectral radius of an element $x \in A$ is denoted by $\rho(x)$ and defined by $\rho(x) = sup_{\lambda \in \sigma(x)}|\lambda| = lim_{n\to\infty}||x^n||^{\frac{1}{n}}$, where $\sigma(x)$ is the spectrum of $x \in A$. If $\rho(x) < 1$ then $e - x$ is invertible and $(e-x)^{-1} = e + \sum_{i=1}^{\infty} x^i$.*

Remark 2.3. *[24] The spectral radius $\rho(x)$ of $x \in A$ satisfies $\rho(x) \leq ||x||$, where A is a Banach algebra with a unity e.*

Remark 2.4. *[25] In Proposition 2.2 if we replace the condition $'\rho(x) < 1'$ by $||x|| \leq 1$ then we get the same conclusion.*

Remark 2.5. *[25] If $\rho(x) < 1$, then we get $||x||^n \to 0$ as $n \to \infty$.*

Definition 2.6. *[6] A subset P of a unital Banach algebra A is called a cone if*
1. P is non empty, closed and $\theta, e \in P$
2. If $x,y \in P$ and $\alpha,\beta \geq 0$ then $\alpha x + \beta y \in P$
3. $x,y \in P$ implies $xy \in P$
4. If $x, -x \in P$ for some $x \in A$ then $x = \theta$, where θ is the zero element of A.

A cone P is called a solid cone if $int(P) \neq \phi$. Each cone P induces a partial ordering $\preceq$ on A by $x \preceq y$ iff $y - x \in P$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. When the cone is solid $x \ll y$ will stand for $y - x \in int(P)$. The cone P is said to be normal if there exists a number $R > 0$ such that $\theta \preceq x \preceq y \Rightarrow ||x|| \leq R||y||$. The least positive number which satisfies the previous normality condition is called the normal constant of P (See [16]).

Lemma 2.7. *[23] If E be a real Banach space with a solid cone P and if $\theta \preceq a \ll c$ for all $c \gg \theta$, then $a = \theta$.*

Definition 2.8. *[18] Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is called a $c-$sequence if for each $\theta \ll c$ there exists a natural number N such that $u_n \ll c$ whenever $n \geq N$.*

Lemma 2.9. *[14] If E is a real Banach space with a solid cone P and $\{u_n\} \subset P$ is a sequence with $||u_n|| \to 0$ as $n \to \infty$, then u_n is a $c-$sequence.*

Lemma 2.10. *[25] Let A be a Banach algebra with a unity e. Let $x,y \in H$ such that x and y commute, then*
1. $\rho(xy) \leq \rho(x)\rho(y)$
2. $\rho(x+y) \leq \rho(x) + \rho(y)$
3. $|\rho(x) - \rho(y)| \leq \rho(x-y)$.

Lemma 2.11. *[14] If E be a real Banach space and P be a solid cone of E then for $a,b,c \in E$ with $a \preceq b \ll c$ implies $a \ll c$.*

Lemma 2.12. *[25] Let P be a solid cone of a Banach algebra A. Suppose that $k \in P$ is an arbitrary vector and $\{u_n\} \subset P$ is a $c-$sequence, then $\{ku_n\}$ is also a $c-$sequence.*

Lemma 2.13. *[14] Let A be a Banach algebra with unity e and $k \in A$. Let λ be a complex constant and $\rho(k) < |\lambda|$. Then we get $\rho((\lambda e - k)^{-1}) \leq \frac{1}{|\lambda| - \rho(k)}$.*

Lemma 2.14. *[14] Let A be a Banach space and P be a solid cone of A. Let $a,k,l \in P$, $l \preceq k$ and $a \preceq la$ with $\rho(k) < 1$, then $a = \theta$.*

Lemma 2.15. *[25] Let A be a Banach algebra with unity e and $\{x_n\} \subset A$. Suppose that $\{x_n\}$ converges $x \in A$ and that x_n and x commute for all n, then we have $\rho(x_n) \to \rho(x)$ as $n \to \infty$.*

Lemma 2.16. *[15] Let A be a Banach algebra with unity e and P be a solid cone of A. Let $h \in A$ and $u_n = h^n$ for all $n \in \mathbb{N}$. If $\rho(h) < 1$ then $\{u_n\}$ is a $c-$sequence.*

Lemma 2.17. *[18] The following conditions are equivalent for a cone K in the Banach space $(E, ||.||)$*
1. K is normal
2. for arbitrary sequence $\{x_n\},\{y_n\},\{z_n\}$ in E,
$x_n \preceq y_n \preceq z_n\ \forall n \in \mathbb{N}$ and $lim_{n\to\infty}x_n = lim_{n\to\infty}z_n = x$ imply $lim_{n\to\infty}y_n = x$
3. there exists a norm $||.||_1$ on H equivalent to $||.||$ such that the cone is monotone with respect to $||.||_1$.

In the following definitions we always assume that A is a Banach algebra with unity e, P is a solid cone of A and $\preceq$ is the partial order generated by the cone P.

Definition 2.18. *([16],[20]) Let X be a nonempty set. The mapping $d : X \times X \to A$ is said to be a cone metric on X if it satisfies the following conditions:*

1. $d(x,y) \succeq \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ iff $x = y$

2. $d(x,y) = d(y,x)$ for all $x,y \in X$

3. $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

The space (X,d) is called a cone metric space over the Banach algebra A.

Definition 2.19. *[1] Let X be a nonempty set and s be a real number satisfying $s \geq 1$. A function $d : X \times X \to \mathbb{R}^+$ is a $b-$metric on X if for all $x,y,z \in X$, the following conditions hold*

1. $d(x,y) = 0$ iff $x = y$

2. $d(x,y) = d(y,x)$ for all $x,y \in X$

3. $d(x,z) \preceq s[d(x,y) + d(y,z)]$ for all $x,y,z \in X$.

The space (X,d) is called a $b-$metric space.

Definition 2.20. *[22] Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x,y,z \in X$ p satisfies the following conditions:*

1. $p(x,x) = p(x,y) = p(y,y)$ iff $x = y$

2. $p(x,x) \leq p(x,y)$

3. $p(x,y) = p(y,x)$

4. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$

The space (X,p) is called a partial metric space.

Definition 2.21. *[21] Let X be a nonempty set. A function $N : X \times X \times X \to A$ is said to be $N-$cone metric on X if it satisfies the following conditions:*

1. $N(x,x,x) \succeq \theta$ for all $x \in X$

2. for any $x,y,z \in X$ $N(x,y,z) = \theta$ if and only if $x = y = z$

3. $N(x,y,z) \preceq N(x,x,a) + N(y,y,a) + N(z,z,a)$ for all $x,y,z,a \in X$.

Then (X,N) is called an $N-$cone metric space over the Banach algebra A.

Definition 2.22. *[12] Let X be a nonempty set. A function $N_b : X \times X \times X \to A$ is said to be N_b-cone metric on X if it satisfies the following conditions:*

1. $N_b(x,y,z) \succeq \theta$ for all $x,y,z \in X$

2. for any $x,y,z \in X$ $N_b(x,y,z) = \theta$ if and only if $x = y = z$

3. $N_b(x,y,z) \preceq s[N_b(x,x,a) + N_b(y,y,a) + N_b(z,z,a)]$ for all $x,y,z,a \in X$, where s is a real number greater than or equal to 1.

Then (X,N_b) is called an N_b-cone metric space over the Banach algebra A.

Definition 2.23. *[13] Let X be a nonempty set. A function $N_p : X \times X \times X \to A$ is said to be N_p-cone metric on X if it satisfies the following conditions:*

1. $N_p(x,x,x) = N_p(y,y,y) = N_p(z,z,z) = N_p(x,y,z)$ iff $x = y = z$

2. $\theta \preceq N_p(x,x,x) \preceq N_p(x,x,y) \preceq N_p(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$

3. $N_p(x,y,z) \preceq N_p(x,x,a) + N_p(y,y,a) + N_p(z,z,a) - N_p(a,a,a)$ for all $x,y,z,a \in X$.

The triplet (X,N_p) is called an N_p-cone metric space over the Banach algebra A.

3. Introduction to $F-$cone metric space over a Banach algebra

Keeping in view with the definitions given by [6], in this section we now redefine the following definitions. In the following definitions we always take A as an ordered Banach algebra.

Definition 3.1. *Let X be a nonempty set. A function $F : X \times X \times X \to A$ is said to be $F-$cone metric on X if it satisfies the following conditions:*

1. $F(x,x,x) = F(y,y,y) = F(z,z,z) = F(x,y,z)$ iff $x = y = z$,

2. $\theta \preceq F(x,x,x) \preceq F(x,x,y) \preceq F(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$,

3. $F(x,y,z) \preceq s[F(x,x,a) + F(y,y,a) + F(z,z,a) - F(a,a,a)]$ for some $s \geq 1$ and for all $x,y,z,a \in X$.

Then the triplet (X,A,F) is called an $F-$cone metric space over the Banach algebra A and the number $s \geq 1$ is called the coefficient of (X,A,F).

Definition 3.2. *Let (X,F) be an $F-$cone metric space over the Banach algebra A. A sequence $\{x_n\} \subset X$ is said to be convergent and converges to a point $x \in X$ if for each $c \gg \theta$ there is a natural number N such that $F(x_n,x_n,x) \ll c$ whenever $n \geq N$. We write it as $lim_{n\to\infty} x_n = x$.*

Definition 3.3. *Let (X,F) be an $F-$cone metric space over the Banach algebra A. A sequence $\{x_n\} \subset X$ is said to be $\theta-$Cauchy sequence if for each $c \gg \theta$ there is a natural number N_0 such that $F(x_n,x_n,x_m) \ll c$ whenever $n,m \geq N_0$.*

Definition 3.4. *Let (X,F) be an $F-$cone metric space over the Banach algebra A. Then X is called $\theta-$complete if every $\theta-$Cauchy sequence $\{x_n\} \subset X$ is convergent and converges to some $x \in X$ such that $F(x,x,x) = \theta$.*

Definition 3.5. *Let (X,F) and (X',F') be two $F-$cone metric spaces over the same Banach algebra A. Then a function $f : X \to X'$ is said to be continuous if for any $\{x_n\} \subset X$ converges to x implies $\{fx_n\} \subset X'$ converges to fx.*

Now we state the following properties of $F-$cone metric and $F-$cone metric space which were given by Fernandez et. al. (See [6]).

Remark 3.6. *In an $F-$cone metric space (X,F) over the Banach algebra A, $F(x,y,z) = \theta$ implies $x = y = z$ for $x,y,z \in X$ but the converse is not true.*

Lemma 3.7. *Let (X,F) be an $F-$cone metric space over the Banach algebra A. Then $F(x,x,y) \succ \theta$ whenever $x \neq y$.*

Proposition 3.8. *If (X,F) is an $F-$cone metric space over the Banach algebra A then $F(x,x,y) = F(y,y,x)$ for all $x,y \in X$.*

Definition 3.9. *Let (X,F) be an $F-$cone metric space over the Banach algebra A. Then for $x \in X$ and $c \gg \theta$, the $F-$balls with center x and radius $c \gg \theta$ are $B_F(x,c) = \{y \in X : F(x,x,y) \ll F(x,x,x)+c\}$.*

Definition 3.10. *Let (X,F) be an $F-$cone metric space over the Banach algebra A with coefficient $s \geq 1$. Let $B_F(x,c) = \{y \in X : F(x,x,y) \ll F(x,x,x)+c\}$ for all $x \in X$ and for all $c \gg \theta$. We put $B = \{B_F(x,c) : x \in X \text{ and } \theta \ll c\}$. Then B is a subbase for some topology τ on X.*

Theorem 3.11. *Let (X,F) be an $F-$cone metric space over the Banach algebra A and P be a solid cone of the Banach algebra A. Then (X,F) is a Hausdorff space with respect to the topology τ.*

4. Main Results

In this section we prove some common fixed point results for a pair of mappings in an $F-$cone metric space over the underlying ordered Banach algebra H.

Theorem 4.1. *Let (X,H,F) be a $\theta-$complete $F-$cone metric space and P be a solid cone of H. Let f be a continuous mapping of the $\theta-$complete $F-$cone metric space (X,H,F) into itself. Assume that there exists an element $k \in P$ with $\rho(k) < \frac{1}{s}$, s is the coefficient of X and a mapping $g : X \to X$ which commutes with f satisfying $g(X) \subset f(X)$. If f,g satisfies $F(gx,gx,gy) \preceq kF(fx,fx,fy)$ for all $x,y \in X$, then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Then there exists $x_1 \in X$ such that $f(x_1) = g(x_0)$ since $g(X) \subset f(X)$. Next we can find $x_2 \in X$ such that $f(x_2) = g(x_1)$. In a similar manner we can construct a sequence $\{x_n\}$ in X such that $f(x_n) = g(x_{n-1})$ for all $n \in \mathbb{N}$. Let

$$y_{n-1} = g(x_{n-1}) = f(x_n) \quad \forall n \geq 1 \tag{4.1}$$

Then,

$$\begin{aligned} F(y_n,y_n,y_{n+1}) &= F(gx_n,gx_n,gx_{n+1}) \\ &\preceq kF(fx_n,fx_n,fx_{n+1}) \\ &= kF(y_{n-1},y_{n-1},y_n) \ \forall n \in \mathbb{N} \end{aligned} \tag{4.2}$$

Now for $1 \leq n < m$ we get,

$$\begin{aligned} & F(y_n,y_n,y_m) \\ \preceq\ & s[2F(y_n,y_n,y_{n+1}) + F(y_m,y_m,y_{n+1}) \\ & -F(y_{n+1},y_{n+1},y_{n+1})] \\ \preceq\ & s[2F(y_n,y_n,y_{n+1}) + F(y_m,y_m,y_{n+1})] \\ \preceq\ & 2sF(y_n,y_n,y_{n+1}) + s^2[2F(y_{n+1},y_{n+1},y_{n+2}) + \\ & F(y_m,y_m,y_{n+2}) - F(y_{n+2},y_{n+2},y_{n+2})] \\ \preceq\ & 2sF(y_n,y_n,y_{n+1}) + 2s^2F(y_{n+1},y_{n+1},y_{n+2}) + \\ & s^2F(y_m,y_m,y_{n+2}) \\ & \dots \quad \dots \quad \dots \\ \preceq\ & 2[sF(y_n,y_n,y_{n+1}) + s^2F(y_{n+1},y_{n+1},y_{n+2}) + \dots + \\ & s^{m-n-1}F(y_{m-2},y_{m-2},y_{m-1})] + \\ & s^{m-n-1}F(y_{m-1},y_{m-1},y_m) \\ \preceq\ & 2[sF(y_n,y_n,y_{n+1}) + s^2F(y_{n+1},y_{n+1},y_{n+2}) + \dots + \\ & s^{m-n-1}F(y_{m-2},y_{m-2},y_{m-1}) + \\ & s^{m-n}F(y_{m-1},y_{m-1},y_m)] \quad [\because s \geq 1] \end{aligned}$$

Therefore for all $1 \leq n < m$ we get,

$$\begin{aligned} & F(y_n,y_n,y_m) \\ \preceq\ & 2[sF(y_n,y_n,y_{n+1}) + s^2F(y_{n+1},y_{n+1},y_{n+2}) + \dots + \\ & s^{m-n-1}F(y_{m-2},y_{m-2},y_{m-1}) + s^{m-n}F(y_{m-1},y_{m-1},y_m)] \end{aligned} \tag{4.3}$$

Now from (4.2) we get,

$$\begin{aligned} F(y_n,y_n,y_{n+1}) &\preceq kF(y_{n-1},y_{n-1},y_n) \\ &\preceq k^2F(y_{n-2},y_{n-2},y_{n-1}) \\ &\preceq k^3F(y_{n-3},y_{n-3},y_{n-2}) \\ &\dots \quad \dots \quad \dots \\ &\preceq k^nF(y_0,y_0,y_1) \end{aligned}$$

Therefore from (4.3) we get,

$$\begin{aligned} F(y_n,y_n,y_m) &\preceq 2[sk^n + s^2K^{n+1} + \dots + \\ & \quad s^{m-n}k^{m-1}]F(y_0,y_0,y_1) \\ &\preceq 2[(sk)^n + (sk)^{n+1} + \dots + \\ & \quad (sk)^{m-n}]F(y_0,y_0.y_1) \quad [\because s \geq 1] \\ &\preceq 2(sk)^n[e + sk + \dots + (sk)^{m-n-1}]F(y_0,y_0,y_1) \\ &\preceq 2(sk)^n(e-sk)^{-1}F(y_0,y_0,y_1) \end{aligned} \tag{4.4}$$

Since $\rho(k) < \frac{1}{s}$ it implies that $\rho(sk) < 1$, we have $||sk||^n \to 0$ as $n \to \infty$ by Remark 2.5.
Hence from (4.4) and by Lemma 2.9 it follows that $\{F(y_n,y_n,y_m)\}$ is a $c-$sequence. So $\{y_n\}$ is a $\theta-$Cauchy sequence in X. Since X is $\theta-$complete there exists some $z \in X$ such that $\{y_n\}$ converges to z. Now $y_n = gx_n = fx_{n+1}$ for all $n \geq 1$. Therefore we have $f(x_{n+1}) \to z$ and $g(x_n) \to z$ as $n \to \infty$.
Since f is continuous then clearly g is also continuous in X. Thus $g(f(x_{n+1})) \to gz$ and $f(g(x_n)) \to fz$ as $n \to \infty$. Since f and g commute so $fz = gz$ and so z is a coincidence point of f and g.

Now,

$$\begin{aligned} F(gz,gz,g^2z) &\preceq kF(fz,fz,fgz) \\ &= k(F(fz,fz,gfz)) \\ &= k(F(gz,gz,g^2z)) \end{aligned}$$

Since $\rho(k) < \frac{1}{s} < 1$ so it is possible only when $F(gz,gz,g^2z) = \theta$ which implies $g^2z = gz$. Now, $f(gz) = g(fz) = g(gz) = gz$ and therefore gz is a common fixed point of f and g.
If possible let u and v be two common fixed points of f and g, then $fu = gu = u$ and $fv = gv = v$. Thus $F(u,u,v) = F(gu,gu,gv) \preceq F(fu,fu,fv) = kF(u,u,v)$ and hence $F(u,u,v) = \theta$ implying that $u = v$. Then f and g have a unique common fixed point in X. □

Theorem 4.2. *Let (X,H,F) be a $\theta-$complete $F-$cone metric space and P be a solid cone of H. Let f be a continuous mapping of the $\theta-$complete $F-$cone metric space (X,H,F) into itself. Suppose that there exists an element $k \in P$ with $\rho(k) < \frac{1}{s+1}$, s is the coefficient of X and a mapping $g : X \to X$ which commutes with f satisfying $g(X) \subset f(X)$. If $F(gx,gx,gy) \preceq k[F(fx,fx,gx)+F(fy,fy,gy)]$ for all $x,y \in X$, then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Then there exists $x_1 \in X$ such that $f(x_1) = g(x_0)$ since $g(X) \subset f(X)$. Next we can find $x_2 \in X$ such that $f(x_2) = g(x_1)$. Proceeding in this way we can construct a sequence $\{x_n\}$ in X such that $f(x_n) = g(x_{n-1})$ for all $n \in \mathbb{N}$. Let

$$y_{n-1} = g(x_{n-1}) = f(x_n) \quad \forall n \geq 1 \tag{4.5}$$

Then,

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ = \; & F(gx_n,gx_n,gx_{n+1}) \\ \preceq \; & k[F(fx_n,fx_n,gx_n)+F(fx_{n+1},fx_{n+1},gx_{n+1})] \\ = \; & k[F(y_{n-1},y_{n-1},y_n)+F(y_n,y_n,y_{n+1})] \end{aligned} \tag{4.6}$$

which implies

$$\begin{aligned} F(y_n,y_n,y_{n+1}) &\preceq (e-k)^{-1}kF(y_{n-1},y_{n-1},y_n) \\ &= \beta F(y_{n-1},y_{n-1},y_n) \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\beta = (e-k)^{-1}k$. (4.7)

Now $\rho(\beta) = \rho((e-k)^{-1}k) \leq \rho(k)\rho((e-k)^{-1}) \leq \frac{\rho(k)}{1-\rho(k)} < \frac{1}{s}$. So by routine calculation we get $\{y_n\}$ is a $\theta-$Cauchy sequence in X. Since X is $\theta-$complete then there exists an element $z \in X$ such that $\{y_n\}$ converges to z. Now $y_n = gx_n = fx_{n+1}$ for all $n \geq 1$, thus we have $f(x_{n+1}) \to z$ and $g(x_n) \to z$ as $n \to \infty$.
Since f is continuous so $f(g(x_n)) \to fz$ as $n \to \infty$ that is $g(f(x_n)) \to fz$ as $n \to \infty$.

Now, $F(g(fx_n),g(fx_n),gz) \preceq k[F(ff(x_n),f(fx_n),g(fx_n)) + F(fz,fz,gz)]$. Letting $n \to \infty$ we get,

$$\begin{aligned} F(fz,fz,gz) &\preceq k[F(fz,fz,fz)+F(fz,fz,gz)] \\ &\preceq \beta F(fz,fz,fz) \\ &\preceq \beta F(fz,fz,gz) \end{aligned} \tag{4.8}$$

which in turn implies that $(e-\beta)F(fz,fz,gz) \preceq \theta$. Therefore we have $F(fz,fz,gz) = \theta$ which implies $fz = gz$.
Now, $F(gz,gz,g^2z) \preceq k[F(fz,fz,gz)+F(fgz,fgz,g^2z)]$ which in turn follows that

$$\begin{aligned} F(gz,gz,g^2z) &\preceq k[F(fz,fz,gz)+F(gfz,gfz,g^2z)] \\ &\preceq k[F(gz,gz,gz)+F(g^2z,g^2z,g^2z)] \\ &\preceq 2kF(gz,gz,g^2z) \end{aligned}$$

Hence $(e-2k)F(gz,gz,g^2z) \preceq \theta$. As $\rho(2k) < 1$ we have $F(gz,gz,g^2z) = \theta$ and consequently $g(gz) = gz$.

Now, $f(gz) = g(fz) = g(gz) = gz$ and therefore gz is a common fixed point of f and g in X. If possible let u and v be two common fixed points of f and g. Then $fu = gu = u$ and $fv = gv = v$.
Then,

$$\begin{aligned} F(u,u,v) &= F(gu,gu,gv) \\ &\preceq k[F(fu,fu,gu)+F(fv,fv,gv)] \\ &= k[F(u,u,u)+F(v,v,v)] \\ &\preceq 2kF(u,u,v) \end{aligned}$$

By similar argument as above we get $F(u,u,v) = \theta$ implying that $u = v$, which proves the uniqueness of the common fixed point of f and g. □

Theorem 4.3. *Let (X,H,F) be a $\theta-$complete $F-$cone metric space and P be a solid cone of H. Let f be a continuous mapping of the $\theta-$complete $F-$cone metric space (X,H,F) into itself. Suppose that there exists an element $k \in P$ with $\rho(k) < \frac{1}{2s(s+1)}$, s is the coefficient of X and a mapping $g : X \to X$ which commutes with f such that $g(X) \subset f(X)$. Let $F(gx,gx,gy) \preceq k[F(fx,fx,gy)+F(fy,fy,gx)]$ for all $x,y \in X$. Then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Then there exists $x_1 \in X$ such that $f(x_1) = g(x_0)$ since $g(X) \subset f(X)$. Next we can find $x_2 \in X$ such that $f(x_2) = g(x_1)$. Continuing in this way we can construct a sequence $\{x_n\}$ in X such that $f(x_n) = g(x_{n-1})$ for all $n \in \mathbb{N}$. Let

$$y_{n-1} = g(x_{n-1}) = f(x_n) \quad \forall n \geq 1 \tag{4.9}$$

Then,

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ = \; & F(gx_n,gx_n,gx_{n+1}) \\ \preceq \; & k[F(fx_n,fx_n,gx_{n+1})+F(fx_{n+1},fx_{n+1},gx_n)] \\ = \; & [F(y_{n-1},y_{n-1},y_{n+1})+F(y_n,y_n,y_n)] \\ \preceq \; & sk[2F(y_{n-1},y_{n-1},y_n)+F(y_{n+1},y_{n+1},y_n)-F(y_n,y_n,y_n)] \\ & +kF(y_n,y_n,y_n) \\ = \; & 2skF(y_{n-1},y_{n-1},y_n)+skF(y_n,y_n,y_{n+1})+kF(y_n,y_n,y_n) \\ \preceq \; & (2s+1)kF(y_{n-1},y_{n-1},y_n)+skF(y_n,y_n,y_{n+1}) \end{aligned}$$

Hence $F(y_n,y_n,y_{n+1}) \preceq (e-sk)^{-1}(2s+1)kF(y_{n-1},y_{n-1},y_n)$ for all $n \in \mathbb{N}$.

Therefore,

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ \preceq\ & \gamma F(y_{n-1},y_{n-1},y_n) \quad \forall n\in\mathbb{N}, \\ & \text{where } \ \gamma=(e-sk)^{-1}(2s+1)k. \end{aligned} \tag{4.10}$$

Now,

$$\begin{aligned} \rho(\gamma) &= \rho((e-sk)^{-1}(2s+1)k) \\ &\leq (2s+1)\rho((e-sk)^{-1})\rho(k) \\ &\leq (2s+1)\tfrac{\rho(k)}{1-\rho(sk)} \\ &= (2s+1)\tfrac{\rho(k)}{1-s\rho(k)} < \tfrac{1}{s} \end{aligned}$$

By routine calculation we see that $\{y_n\}$ is a $\theta-$Cauchy sequence in (X,H,F). Since X is $\theta-$complete then there exists some $z\in X$ such that $y_n\to z$ as $n\to\infty$. Now since $y_n=fx_{n+1}=gx_n\ \forall n\in\mathbb{N}$ then $f(x_{n+1})\to z$ and $g(x_n)\to z$ as $n\to\infty$. Since f is continuous so $f(g(x_n))\to fz$ as $n\to\infty$ that is $g(f(x_n))\to fz$ as $n\to\infty$.

Now, $F(g(fx_n),g(fx_n),gz)\preceq k[F(f(fx_n),f(fx_n),gz)+F(fz,fz,g(fx_n))]$. Taking $n\to\infty$ we get,

$$\begin{aligned} F(fz,fz,gz) &\preceq k[F(fz,fz,gz)+F(fz,fz,fz)] \\ &\preceq 2kF(fz,fz,gz) \end{aligned}$$

Thus $(e-2k)F(fz,fz,gz)\preceq\theta$ and so $F(fz,fz,gz)=\theta$. $[\because \rho(2k)=2\rho(k)<\frac{2}{2s(s+1)}=\frac{1}{s(s+1)}<1]$ Which shows that $fz=gz$.

Now,

$$\begin{aligned} F(gz,gz,g^2z) &\preceq k[F(fz,fz,g^2z)+F(fgz,fgz,gz)] \\ &= k[F(gz,gz,g^2z)+F(gfz,gfz,gz)] \\ &= k[F(gz,gz,g^2z)+F(g^2z,g^2z,gz)] \\ &= 2kF(gz,gz,g^2z) \end{aligned}$$

In a similar fashion we get $F(gz,gz,g^2z)=\theta$ implying that $g^2z=gz$. Now, $f(gz)=g(fz)=g(gz)=gz$. So gz is a common fixed point of f and g. in X. If possible let u and v be two common fixed points of f and g. Then $fu=gu=u$ and $fv=gv=v$.

Then,

$$\begin{aligned} F(u,u,v) &= F(gu,gu,gv) \\ &\preceq k[F(fu,fu,gv)+F(fv,fv,gu)] \\ &= k[F(u,u,v)+F(v,v,u)] \\ &= 2kF(u,u,v) \end{aligned}$$

By similar argument we get $F(u,u,v)=\theta$ showing that $u=v$. Thus f and g have a unique common fixed point in X. □

Theorem 4.4. *Let (X,H,F) be a $\theta-$complete $F-$cone metric space and P be a solid cone of H. Let f be a continuous mapping of the $\theta-$complete $F-$cone metric space (X,H,F) into itself. Assume that there exists $k_1,k_2,k_3,k_4,k_5\in P$ such that they commute with each other satisfying $s\rho(k_1)+\rho(k_2)+s(2s+1)\rho(k_3)+s\rho(k_4)+s\rho(k_5)<1$, s is the coefficient of X. If a mapping $g:X\to X$ commutes with f, such that $g(X)\subset f(X)$ and $F(gx,gx,gy)\preceq k_1F(fx,fx,gx)+k_2F(fy,fy,gy)+k_3F(fx,fx,gy)+k_4F(fy,fy,gx)+k_5F(fx,fx,fy)$ for all $x,y\in X$, then f and g have a unique common fixed point in X.*

Proof. Let $x_0\in X$. Then there exists $x_1\in X$ such that $f(x_1)=g(x_0)$ since $g(X)\subset f(X)$. Next we can find $x_2\in X$ such that $f(x_2)=g(x_1)$. Proceeding in this way we can construct a sequence $\{x_n\}$ in X such that $f(x_n)=g(x_{n-1})$ for all $n\in\mathbb{N}$. Let

$$y_{n-1}=g(x_{n-1})=f(x_n)\quad \forall n\geq 1 \tag{4.11}$$

Then,

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ =\ & F(gx_n,gx_n,gx_{n+1}) \\ \preceq\ & k_1F(fx_n,fx_n,gx_n)+k_2F(fx_{n+1},fx_{n+1},gx_{n+1}) \\ & +k_3F(fx_n,fx_n,gx_{n+1})+k_4F(fx_{n+1},fx_{n+1},gx_n) \\ & +k_5F(fx_n,fx_n,fx_{n+1}) \\ =\ & k_1F(y_{n-1},y_{n-1},y_n)+k_2F(y_n,y_n,y_{n+1})+ \\ & k_3F(y_{n-1},y_{n-1},y_{n+1})+k_4F(y_n,y_n,y_n)+ \\ & k_5F(y_{n-1},y_{n-1},y_n) \end{aligned}$$

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ \preceq\ & k_1F(y_{n-1},y_{n-1},y_n)+k_2F(y_n,y_n,y_{n+1}) \\ & sk_3[2F(y_{n-1},y_{n-1},y_n)+F(y_{n+1},y_{n+1},y_n)-F(y_n,y_n,y_n)] \\ & +k_4F(y_n,y_n,y_n)+k_5F(y_{n-1},y_{n-1},y_n) \\ \preceq\ & k_1F(y_{n-1},y_{n-1},y_n)+k_2F(y_n,y_n,y_{n+1}) \\ & 2sk_3F(y_{n-1},y_{n-1},y_n)+sk_3F(y_n,y_n,y_{n+1}) \\ & +k_4F(y_{n-1},y_{n-1},y_n)+k_5F(y_{n-1},y_{n-1},y_n) \end{aligned}$$

Thus $(e-k_2-sk_3)F(y_n,y_n,y_{n+1})\preceq(k_1+k_4+k_5+2sk_3)F(y_{n-1},y_{n-1},y_n)$, hence we have $F(y_n,y_n,y_{n+1})\preceq(e-k_2-sk_3)^{-1}(k_1+k_4+k_5+2sk_3)F(y_{n-1},y_{n-1},y_n)$.

Thus,

$$\begin{aligned} & F(y_n,y_n,y_{n+1}) \\ \preceq\ & \delta F(y_{n-1},y_{n-1},y_n) \quad \forall n\in\mathbb{N}, \text{ where} \\ & \delta=(e-k_2-sk_3)^{-1}(k_1+k_4+k_5+2sk_3) \end{aligned} \tag{4.12}$$

Now,

$$\begin{aligned} \rho(\delta) &= \rho((e-k_2-sk_3)^{-1}(k_1+k_4+k_5+2sk_3)) \\ &\leq \tfrac{1}{1-\rho(k_2+sk_3)}\rho((k_1+k_4+k_5+2sk_3)) \\ &\leq \tfrac{\rho(k_1)+2s\rho(k_3)+\rho(k_4)+\rho(k_5)}{1-\rho(k_2)+s\rho(k_3)} < \tfrac{1}{s} \end{aligned}$$

By similar argument as before we see that $\{y_n\}$ is a $\theta-$Cauchy sequence in X. Since X is $\theta-$complete there exists an element $z\in X$ such that $y_n\to z$ as $n\to\infty$.

Now since $y_n = fx_{n+1} = gx_n \; \forall n \in \mathbb{N}$ then $f(x_{n+1}) \to z$ and $g(x_n) \to z$ as $n \to \infty$. Since f is continuous so $f(g(x_n)) \to fz$ as $n \to \infty$ that is $g(f(x_n)) \to fz$ as $n \to \infty$.

Now,

$$\begin{aligned} & F(g(fx_n), g(fx_n), gz) \\ \preceq\; & k_1F(f(fx_n), f(fx_n), g(fx_n)) + k_2F(fz, fz, gz) + \\ & k_3F(f(fx_n), f(fx_n), gz) + k_4F(fz, fz, g(fx_n)) + \\ & k_5F(f(fx_n), f(fx_n), fz) \end{aligned}$$

Taking $n \to \infty$ we get,

$$\begin{aligned} & F(fz, fz, gz) \\ \preceq\; & k_1F(fz, fz, fz) + k_2F(fz, fz, gz) + \\ & k_3F(fz, fz, gz) + k_4F(fz, fz, fz) + \\ & k_5F(fz, fz, fz) \\ \preceq\; & (k_1 + k_2 + k_3 + k_4 + k_5)F(fz, fz, gz) \end{aligned}$$

Now,

$$\begin{aligned} & \rho(k_1 + k_2 + k_3 + k_4 + k_5) \\ \leq\; & \rho(k_1) + \rho(k_2) + \rho(k_3) + \rho(k_4) + \rho(k_5) \\ \leq\; & s\rho(k_1) + \rho(k_2) + s(2s+1)\rho(k_3) + \\ & s\rho(k_4) + s\rho(k_5) < 1 \quad [\because s \geq 1] \end{aligned}$$

Thus, $F(fz, fz, gz) = \theta \Rightarrow fz = gz$.
Also,

$$\begin{aligned} & F(gz, gz, g^2z) \\ \preceq\; & k_1F(fz, fz, gz) + k_2F(fgz, fgz, g^2z) + \\ & k_3F(fz, fz, g^2z) + k_4F(fgz, fgz, gz) + \\ & k_5F(fz, fz, fgz) \\ \preceq\; & k_1F(fz, fz, gz) + k_2F(gfz, gfz, g^2z) + \\ & k_3F(fz, fz, g^2z) + k_4F(gfz, gfz, gz) + \\ & k_5F(fz, fz, gfz) \\ =\; & k_1F(gz, gz, gz) + k_2F(g^2z, g^2z, g^2z) + \\ & k_3F(gz, gz, g^2z) + k_4F(g^2z, g^2z, gz) + \\ & k_5F(gz, gz, g^2z) \\ \preceq\; & (k_1 + k_2 + k_3 + k_4 + k_5)F(gz, gz, g^2z) \end{aligned}$$

Therefore, $F(gz, gz, g^2z) = \theta \Rightarrow g^2z = gz$. So we get, $f(gz) = g(fz) = g(gz) = gz$. Thus gz is a common fixed point of f and g in X. If possible let u and v be two common fixed points of f and g.

Then $fu = gu = u$ and $fv = gv = v$.

Then,

$$\begin{aligned} F(u, u, v) &= F(gu, gu, gv) \\ &\preceq k_1F(fu, fu, gu) + k_2F(fv, fv, gv) + \\ &\quad k_3F(fu, fu, gv) + k_4F(fv, fv, gu) + \\ &\quad k_5F(fu, fu, fv) \\ &= k_1F(u, u, u) + k_2F(v, v, v) + k_3F(u, u, v) + \\ &\quad k_4F(v, v, u) + k_5F(u, u, v) \\ &\preceq (k_1 + k_2 + k_3 + k_4 + k_5)F(u, u, v) \end{aligned}$$

Hence we get, $F(u, u, v) = \theta \Rightarrow u = v$ and therefore f and g have a unique common fixed point in X. □

Remark 4.5. *Theorem 4.1, Theorem 4.2 and Theorem 4.3 are special cases of Theorem 4.4.*

1. If we put $k_5 = k$ and $k_1 = k_2 = k_3 = k_4 = \theta$ in Theorem 4.4 we get the result corresponding to Theorem 4.1. In this case we get $s\rho(k) < 1$ that is $\rho(k) < \frac{1}{s}$.

2. If we put $k_1 = k_2 = k$ and $k_3 = k_4 = k_5 = \theta$ in Theorem 4.4 we get the result of Theorem 4.2. In this case we get $s\rho(k) + \rho(k) < 1$ that is $\rho(k) < \frac{1}{s+1}$.

3. If we put $k_3 = k_4 = k$ and $k_1 = k_2 = k_5 = \theta$ in Theorem 4.4 we get the result due to Theorem 4.3. In this case we get $s(2s+1)\rho(k) + s\rho(k) < 1$ that is $\rho(k) < \frac{1}{2s(s+1)}$.

Example 4.6. *Let $X = [0, \infty)$ and $H = \mathbb{R}$, where the normal cone P is given by $P = \{x \in H : x \geq 0\}$. Let $F : X \times X \times X \to H$ be defined by*

$$F(x,y,z) = \begin{cases} x+y+z, & \text{if } x \neq y \neq z \text{ or } x \neq y = z, \\ \frac{x+z}{2}, & \text{if } x = y \neq z, \\ \frac{x}{2}, & \text{if } x = y = z. \end{cases} \tag{4.13}$$

(F_1) If $x = y = z$ then clearly $F(x,x,x) = F(y,y,y) = F(z,z,z) = F(x,y,z)$. Now, $F(x,x,x) = F(y,y,y) = F(z,z,z) = F(x,y,z) \Rightarrow \frac{x}{2} = \frac{y}{2} = \frac{z}{2} \Rightarrow x = y = z$.

(F_2) Clearly $0 \leq F(x,x,x) \leq F(x,x,y) \leq F(x,y,z) \; \forall x,y,z \in X$ with
$x \neq y \neq z$.

(F_3) By routine verification it can be checked that $F(x,y,z) \leq 2(F(x,x,t) + F(y,y,t) + F(z,z,t) - F(t,t,t))$ for all $x,y,z,t \in X$.
Hence (X, F) is an $F-$cone metric space over H. Clearly it is neither N_p-cone nor N_b-cone metric on X.

The following is an example of $F-$cone metric space over a Banach algebra having a cone without being normal.

Example 4.7. *[6] Let $A = C^1_{\mathbb{R}}[0,1]$ and we define a norm on A by $||x|| = ||x||_\infty + ||x'||_\infty$ for $x \in A$. Define the multiplication operation on A by $(x.y)(t) = x(t)y(t)$ for all $t \in [0,1]$ and for all $x, y \in A$. Then the set $P = \{x \in A : x \geq 0\}$ is a cone in A without being normal. Also let us take $X = [0, \infty)$. Define a mapping $F : X \times X \times X \to A$ by $F(x,y,z)(t) = ((max\{x,z\})^2 + (max\{y,z\})^2, \alpha((max\{x,z\})^2 + (max\{y,z\})^2))e^t$ for all $x,y,z \in X$, where $\alpha > 0$ is a constant. Then (X, F) is an $F-$cone metric space over the Banach algebra A with the coefficient $s = 2$. But it is neither a N_p-cone metric space nor a N_b-cone metric space over Banach algebra.*

Example 4.8. *Let $X = [0, \infty)$ and $H = \mathbb{R}$, where the cone P is given by $P = \{x \in H : x \geq 0\}$. Let $F : X \times X \times X \to H$ be given by Example 4.6. Also let $g : X \to X$ be given by $g(x) = \frac{x}{20} \; \forall x \in X$ and $f : X \to X$ be given by $f(x) = \frac{x}{2} \; \forall x \in X$. Then $F(gx, gx, gy) \leq k_1F(fx, fx, gx) + k_2F(fy, fy, gy) + k_3F(fx, fx, gy) + k_4F(fy, fy, gx) + k_5F(fx, fx, fy) \; \forall x, y \in X$, where the constants are given by $k_1 = \frac{1}{10}$, $k_2 = \frac{17}{55}$, $k_3 = \frac{1}{100}$, $k_4 = \frac{1}{10}$ and $k_5 = \frac{1}{11}$. Therefore f and g satisfy all the conditions of Theorem 4.4. Here we see that (X, H, F) is a $\theta-$complete $F-$cone metric space and 0 is the unique common fixed point of f and g in X.*

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References

[1] Bakhtin, I.A., The contraction mapping principle in quasi-metric spaces, *Functional Analysis,* **30**, Ulyanovsk. Gos. Ped. Inst, Ulyanovsk (1989).

[2] Dey, D., Saha, M., Partial cone metric space and some fixed point theorems, *Turkic World Math. Soc. J. App. Eng. Math.*, **3**(1) (2013), 1-9.

[3] Du, W.S., A note on cone metric fixed point theory and its equivalence, *Nonlinear Anal.*, **72**(5) (2010), 2259-2261.

[4] Fernandez, J., Malviya, N., Shukla, S., Cone $b-$metric like spaces over Banach algebra and fixed point theorems with application, *Asian J. Math. Comput. Res.*, (accepted).

[5] Fernandez, J., Saxena, K., Malviya, N., Fixed points of expansive maps in partial cone metric spaces, *Gazi Univ. J. Sci.*,**27**(4) (2014), 1085-1091.

[6] Fernandez, J., Malviya, N., Radenović, S., Saxena, K., $F-$cone metric spaces over Banach algebra, *Fixed Point Theory Appl. 2017*, DOI 10. 1186/s13663-017-0600-5 (2017).

[7] Fernandez, J., Saxena, K., Malviya, N., On cone b_2-metric spaces over Banach algebra, *Sao Paulo J. Math. Sci.*, (accepted).

[8] Fernandez, J., Modi, G., Malviya, N., Some fixed point theorems for contractive maps in $N-$cone metric spaces, *Math. Sci.* , **9** (2015), doi:10.1007/s40096-015-0145-x, 33-38.

[9] Fernandez, J., Malviya , N., Fisher, B., The asymptotically regularity and sequences in partial cone $b-$metric spaces with application, *Filomat*, **30**(10) (2016), doi:10.2298/FIL1610749F, 2749-2760.

[10] Fernandez, J., Saelee, S., Saxena, K., Malviya, N., Kumam, P., The $A-$cone metric space over Banach algebra with application of generalized Lipschitz and expansive maps in fixed point theory and integral equations, *Cogent Math.*, **4**, 1282690 (2017).

[11] Fernandez, J., Saxena, K., Modi, G., The $N-$cone metric space over Banach algebra and some fixed point theorems, (communicated).

[12] Fernandez, J., Saxena, K., Malviya, N., The N_b-cone metric space over Banach algebra, (communicated).

[13] Fernandez, J., Saxena, K., Malviya, N., The N_p-cone metric space over Banach algebra and some fixed point theorems, (communicated).

[14] Huang, H., Radenović, S., Common fixed point theorems of generalized Lipschitz mappings in cone metric spaces over Banach algebras, *Appl. Math. Inf. Sci.*, **9**(6) (2015), 2983-2990.

[15] Huang, H., Radenović, S., Common fixed point theorems of generalized Lipschitz mappings in cone $b-$metric spaces over Banach algebras and applications, *J. Nonlinear Sci. Appl.*, **8** (2015), 787-799.

[16] Huang, L.G., Zhang, X., Cone metric spaces and fixed point theorem for contractive mappings, *J. Math. Anal. Appl.*, **332**(2) (2007), 1468-1476.

[17] Kadelburg, Z., Radenović, S., Rakočević, V., A note on the equivalence of some metric and cone metric fixed point results, *Appl. Math. Lett.*, **24** (2011), 370-374.

[18] Kadelburg, Z., Radenović, S., A note on various types of cones and fixed point results in cone metric spaces, *Asian J. Math. Appl. 2013*, Article ID ama0104 (2013).

[19] Laha, A.K., Saha, M., Fixed point on $\alpha - \psi$ multivalued contractive mappings in cone metric space, *Acta Et Commentationes Universitatis Tartuensis de Mathematica,* **20**(1) (2016).

[20] Liu, H., Xu, S., Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings,*Fixed Point Theory Appl. 2013*, **320** (2013).

[21] Malviya, N., Fisher, B., $N-$cone metric space and fixed points of asymptotically regular maps, *Filomat J. Math.*, (preprint).

[22] Matthews, S.G., Partial metric topology, *Ann. N.Y. Acad. Sci.*, **728**(1) (1994), 183-197.

[23] Radenović, S., Rhoades, B.E., Fixed point theorem for two non-self mappings in cone metric spaces, *Comput. Math. Appl.*, **57** (2009), 1701-1707.

[24] Rudin, W., Functional Analysis, *2nd Edn. McGraw-Hill, NewYork* (1991).

[25] Xu, S., Radenović, S., Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl. 2014*, **102** (2014).

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