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Existence and uniqueness of solutions for nonlinear fractional integrodifferential equations with non-local boundary conditions

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Abstract

In this paper we study on existence of solutions for a nonlinear fractional integrodifferential equations with nonlocal boundary conditions by using Krasnaoselskii's fixed point theorem and Schaefer's fixed point theorem and also we obtain uniqueness of solutions for the same problem by using Banach contraction principle. Example is provided for illustrating our main results.

Keywords

Fractional differential equations, Nonlocal boundary conditions, Caputo fractional derivative, Banach contraction principle, Krasnaoselskii's fixed point theorem, Schaefer's fixed point theorem.

AMS Subject Classification

34A08, 26A33, 34B10, 47H10.

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1. Introduction

Let $(\mathbb{R}^n, \|\cdot\|)$ be a Banach space. Consider the continuous function $\mathscr{F} : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$. Denote $C([0,T],\mathbb{R}^n)$ be the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}^n$ with the norm $\|z\| = max\{|z(t)| : t\in[0,T]\}$,

In this paper, we study the existence and uniqueness of solutions for the following class of nonlinear fractional integrodifferential equation of the form

$$ {}^cD^{\gamma}_{0+}z(t) = \mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t)) \quad (1.1)$$

subject to the two point boundary conditions,

$$Pz(0)+Qz(T) = R \quad (1.2)$$

where $0<\gamma<1$, P and Q are real constants and also $P+Q\neq 0$. ${}^cD^{\gamma}_{0+}$ is the Caputo fractional derivative of order γ.for $\rho,\chi : [0,T]\times[0,T]\to[0,+\infty)$,where μ,ν are linear integral operators such as,

$$(\mu z)(t) = \int_0^t \rho(t,s)z(s)ds,\ (\nu z)(t) = \int_0^t \chi(t,s)z(s)ds$$

In past years most of the papers have been carried out involving both Riemann-Liouville and Caputo (1967) fractional derivatives which has been appreciable progress in ordinary and partial differential equations. In 2010, an absorbing frame of reference to the subject, merge all declared notions of fractional derivatives and integrals, was introduced in Agrawal (2010) [3] and later studied in Bourdin et al.(2014), Klimek and Lupa (2013), Odzijewicz et al.(2012, 2013). In general, both differential and integral equations are combined to get integro-differential equations. The recent results of fractional boundary value problems with integro-differential equations can be found in (see [12], [13]). Nonlocal conditions come up when values of the function on the boundary is connected to values inside the domain. The differential equation with nonlocal conditions has been fundamentally examined by Byszewski [6]. Moreover, nonlocal BVPs for fractional differential equations have gained significant observation (see [5], [30]) which consists of two ,three and multi point and nonlocal boundary value problems as special cases. see ([4], [10], [13], [17]).

2. Preliminaries

In this section, let us recall some basic definitions and preliminaries facts that will be used in the remainder of this paper.

Definition 2.1. *If $w \in C([a,b])$ and $\gamma > 0$, then the Riemann-Liouville fractional integral is defined by*

$$I^{\gamma}_{a^+} w(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{w(s)}{(t-s)^{(1-\gamma)}} ds$$

where $\Gamma(\cdot)$ is the Gamma function defined for any complex number τ as

$$\Gamma(\tau) = \int_0^{\infty} t^{(\tau-1)} e^{(-t)} dt$$

Definition 2.2. *The Caputo fractional derivative is defined for a continuous function $w : (a,b) \to R$ is defined by*

$${}^cD^{\gamma}_{a^+} w(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{w^{(n)}(s)}{(t-s)^{(\gamma-n+1)}} ds$$

where $n = [\gamma]+1$, (the notation $[\gamma]$ denotes the integer part of the real number γ)

Lemma 2.3. *[15] Let $\gamma > 0, w(t) \in C(0,1) \cap L(0,1)$, then the homogenous fractional differential equation*

$${}^cD^{\gamma}_{0+} w(t) = 0,$$

has a solution

$$w(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in R, i = 0,1,\ldots,n-1$, and $n = [\gamma]+1$

Lemma 2.4. *[15] Let $\gamma > 0$,then*

$$I^{\gamma}_{0+} {}^cD^{\gamma}_{0+} w(t) = w(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0,1,\ldots n-1$, and $n = [\gamma]+1$.

Lemma 2.5. *[15] Let $i, j \geq 0$, $k \in L_1[0,T]$, Then*

$$I^i_{0+} I^j_{0+} k(t) = I^{i+j}_{0+} k(t) = I^j_{0+} I^i_{0+} k(t)$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $k \in C[0,T]$, then (5) is true for all $t \in [0,T]$.

Lemma 2.6. *[15] If $j > 0$, $k \in C[0,T]$, then ${}^cD^{\gamma}_{0+} I^{\gamma}_{0+} k(t) = k(t)$ for all $t \in [0,T]$.*

Lemma 2.7. *Let $0 < \gamma \leq 1$ and $k, w \in C([0,T], \mathbb{R}^n)$. Then the unique solution of the boundary value problem for fractional differential equation*

$${}^cD^{\gamma}_{0+} z(t) = y(t), \qquad t \in [0,T] \tag{2.1}$$

$$Pz(0) + Qz(T) = R \tag{2.2}$$

is given by

$$z(t) = \int_0^T G(t,s) y(s) ds + J, \tag{2.3}$$

where,

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} - \frac{1}{\Gamma(\beta)}(P+Q)^{-1}Q \\ \qquad\qquad \times (T-s)^{\beta-1} & 0 \leq s \leq t, \\ -\frac{1}{\Gamma(\beta)}(P+Q)^{-1}Q(T-s)^{\beta-1} & t \leq s \leq T \end{cases}$$

$J = (P+Q)^{-1}R$.

Proof. Assume that z is a solution of the boundary value problem (2.1), (2.2) then using Lemma 2.4, we have

$$z(t) = I^{\gamma}_{0+} y(t) - c_0, \qquad c_0 \in \mathbb{R}^n \tag{2.4}$$

$$z(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) ds - c_0$$

using boundary conditions and we obtain

$$z(0) = -c_0 \text{ and } z(T) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\alpha-1} y(s) ds - c_0$$

substitute these values into (2.2), we get

$$c_0 = (P+Q)^{-1} Q \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} y(s) ds - (P+Q)^{-1} R$$

substitute c_0 in (2.4), we get

$$z(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) ds - (P+Q)^{-1} Q \times \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} y(s) ds + (P+Q)^{-1} R$$

$$z(t) = I^{\gamma}_{0+} y(t) - (P+Q)^{-1} Q I^{\gamma}_{0+} y(T) + (P+Q)^{-1} R$$

which can be written as (2.3). Lemma is proved. □

Lemma 2.8. *Suppose that $\mathscr{F} \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ then the function $z(t)$ is solution of fractional boundary value problem (1.1), (1.2) if and only if $z(t)$ is solution of the fractional integral equation.*

$$z(t) = \int_0^T G(t,s) \mathscr{F}(s, z(s), (\mu z)(x), (\nu z)(x)) ds + (P+Q)^{-1} R. \tag{2.5}$$

Proof. Let $z(t)$ be a solution of the boundary value problem $(1.1),(1.2)$ then by same method as used in Lemma 2.7, we can prove that it is a solution of the fractional integral equation (2.5).

Conversely, let $z(t)$ satisfy (2.5) and denote the right hand side of equation (2.5) by $m(t)$. Then by Lemma 2.5 and 2.6, we obtain

$$\begin{aligned} m(t) =& \int_0^T G(t,s)\mathscr{F}(s,z(s),(\mu z)(x),(\nu z)(x))ds \\ &+ (P+Q)^{-1}R \\ =& I_{0+}^{\gamma}\mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t)) + (P+Q)^{-1}R \end{aligned}$$

this implies that

$$\begin{aligned} {}^cD_{0+}^{\gamma}m(t) =& {}^cD_{0+}^{\gamma}I_{0+}^{\gamma}\mathscr{F}(s,z(t),(\mu z)(t),(\nu z)(t)) \\ &+ {}^cD_{0+}^{\gamma}(P+Q)^{-1}R \\ =& \mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t)) \end{aligned}$$

Hence, $z(t)$ is a solution of fractional differential equation (1.1). Also, it is satisfy the condition (1.2). □

Remark 2.9. *Under natural conditions on $w(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $w(t)$ as $\gamma \to n$.*

Remark 2.10. *[15]The Caputo derivative of order $\gamma > 0$ with $n-1 < \gamma < n$ of the power function $w(t) = t^{\delta}$ for $\delta \geq 0$ satisfies*

$$D^{\gamma}t^{\delta} = \begin{cases} \frac{\Gamma(\delta+1)}{\Gamma(\delta-\gamma+1)}t^{(\delta-\gamma)} & if(\delta > n-1) \\ 0 & if(\delta \leq n-1) \end{cases}$$

3. Main Results

In this section, we deal with the existence and uniqueness of solution for the system $(1.1)-(1.2)$ using fixed point techniques.

Now, we list the following hypotheses for our comfort:

(H_1) There exists positive functions $V_{\mathscr{F}_1}(t)$,$V_{\mathscr{F}_2}(t)$,$V_{\mathscr{F}_{3(t)}}$ such that

$$\begin{aligned} &\|\mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t)) - \mathscr{F}(t,y(t),(\mu y)(t),(\nu y)(t))\| \\ &\leq V_{\mathscr{F}_1}(t)\|z-y\| + V_{\mathscr{F}_2}(t)\|(\mu z)-(\mu y)\| \\ &+ V_{\mathscr{F}_3}(t)\|(\nu z)-(\nu y)\| \end{aligned}$$

for each $t \in [0,T]$ and all $z,y \in \mathbb{R}^n$

(H_2) Further,

$$\rho_0 = \sup_{t\in[0,T]}\left|\int_0^t \rho(t,s)ds\right|, \; \chi_0 = \sup_{t\in[0,T]}\left|\int_0^t \chi(t,s)ds\right|$$

$$I_{V_{\mathscr{F}}}^{\gamma} = \sup_{t\in[0,T]}\left\{\left|I^{\gamma}V_{\mathscr{F}_1}(t)\right|, \left|I^{\gamma}V_{\mathscr{F}_2}(t)\right|, \left|I^{\gamma}V_{\mathscr{F}_3}(t)\right|\right\}$$

(H_3) The function $\mathscr{F} : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(H_4) There exists a constant $\sigma > 0$ such that

$$\|\mathscr{F}(t,z,(\mu z),(\nu z)\| \leq \sigma(t)$$

for each $t \in [0,T]$, $\sigma \in L^1([0,T],\mathbb{R}^+)$ and all $z \in \mathbb{R}^n$. Then the boundary value problem (1.1),(1.2) has at least one solution on $[0,T]$.

Our first result is based on Banach fixed point theorem.

Definition 3.1. *[11] (Banach Fixed Point Theorem) If X is a nonempty closed subset of a Banach space $C([0,T],\mathbb{R}^n)$ and $\Theta : C([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n)$ is a contraction mapping, then Θ has a fixed point.*

Theorem 3.2. *If*

$$\zeta = \left(1+\left\|(P+Q)^{-1}Q\right\|\right)\left\{[1+\rho_0+\chi_0]I_{V_{\mathscr{F}}}^{\gamma}\right\} < 1 \tag{3.1}$$

then the boundary value problem $(1.1),(1.2)$ has unique solution on $[0,T]$.

Proof. Modify the problem $(1.1),(1.2)$ into a fixed point problem. Consider the operator

$$\Theta : C([0,T],\mathbb{R}^n) \longrightarrow C([0,T],\mathbb{R}^n)$$

defined by

$$\begin{aligned} \Theta(z)(t) =& \int_0^T G(t,s)\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds \\ &+ (P+Q)^{-1}R \end{aligned} \tag{3.2}$$

Clearly, the fixed point of the operator Θ are solution of the problem $(1.1),(1.2)$. We shall use the Banach contraction principle to prove that Θ defined by (3.2) has a fixed point. We shall prove that Θ is a contraction.

Let $z,y \in C([0,T],\mathbb{R}^n)$. Then, for each $t \in [0,T]$ we have $\|\Theta(z)(t)-\Theta(y)(t)\|$

$$\begin{aligned} \leq& \int_0^T \|G(t,s)\| \Big\| \mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s)) \\ &- \mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s)) \Big\| ds \end{aligned}$$

Substitute $G(t,s)$ and Separate the integral, We get

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Bigg\| \mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s)) - \mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s))\Bigg\| ds + \frac{1}{\Gamma(\gamma)}\Bigg\|(P+Q)^{-1}$$

$$\times Q\Bigg\|\int_0^T (T-s)^{\gamma-1}\Bigg\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s)) - \mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s))\Bigg\| ds$$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Bigg[V_{\mathscr{F}_1}(s)\|z-y\| + V_{\mathscr{F}_2}(s) \times \|(\mu z)-(\mu y)\| + V_{\mathscr{F}_3}(s)\|(\nu z)-(\nu y)\|\Bigg] ds + \frac{1}{\Gamma(\gamma)}$$

$$\times \left\|(P+Q)^{-1}Q\right\| \int_0^T (T-s)^{\gamma-1}\Bigg[V_{\mathscr{F}_1}(s)\|z-y\| + V_{\mathscr{F}_2}(s)\|(\mu z)-(\mu y)\| + V_{\mathscr{F}_3}(s)\|(\nu z)-(\nu y)\|\Bigg] ds$$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Bigg[V_{\mathscr{F}_1}(s)+\rho_0 V_{\mathscr{F}_2}(s)+\chi_0 V_{\mathscr{F}_3}(s)\Bigg] \|z-y\| ds + \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\int_0^T (T-s)^{\gamma-1}$$

$$\times \Bigg[V_{\mathscr{F}_1}(s)+\rho_0 V_{\mathscr{F}_2}(s)+\chi_0 V_{\mathscr{F}_3}(s)\Bigg]\|z-y\| ds$$

$$\leq \left[I^\gamma V_{\mathscr{F}_1}(t)+\rho_0 I^\gamma V_{\mathscr{F}_2}(t)+\chi_0 I^\gamma V_{\mathscr{F}_3}(t)\right]\|z-y\| + \left\|(P+Q)^{-1}Q\right\|\Bigg[I^\gamma V_{\mathscr{F}_1}(T)+\rho_0 I^\gamma V_{\mathscr{F}_2}(T) + \chi_0 I^\gamma V_{\mathscr{F}_3}(T)\Bigg]\|z-y\|$$

$$\leq [1+\rho_0+\chi_0]\, I^\gamma_{V_{\mathscr{F}}}\|z-y\| + \left\|(P+Q)^{-1}Q\right\| \times [1+\rho_0+\chi_0]\, I^\gamma_{V_{\mathscr{F}}}\|z-y\| \qquad t\in[0,T]$$

$$\leq \left(1+\left\|(P+Q)^{-1}Q\right\|\right)\left\{[1+\rho_0+\chi_0] I^\gamma_{V_{\mathscr{F}}}\right\}\|z-y\|$$

Thus

$$\|\Theta(z)(t)-\Theta(y)(t)\| \leq \zeta\|z-y\|.$$

Accordingly by (3.2) Θ is a contraction. As a outcome of Banach fixed point theorem, we conclude that Θ has a fixed point which is a solution of the problem (1.1), (1.2).
The theorem is proved. □

Our second result is based on Krasnoselskii's fixed point theorem

Definition 3.3. *[12](Krasnoselskii's Fixed Point Theorem) Let X be a bounded closed convex subset of a Banach space $C([0,T],\mathbb{R}^n)$ and let M,N be operators such that*

(i) $Mz+Ny\in X$ whenever $z,y\in X$,

(ii) M is compact and continuous,

(iii) N is a contraction mapping.

Then there exists $u\in X$ suchthat $u=Mu+Nu$

Theorem 3.4. *Assume that $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is jointly continuous and satisfies (H1) and (H2). If*

$$\varsigma_g = \left\|(P+Q)^{-1}Q\right\|\left\{[1+\rho_0+\chi_0]I^\gamma_{V_f}\right\} < 1$$

then the fractional integrodifferential equation (1.1) has atleast one solution.

Proof. We shall use Krasnoselskii's fixed point theorem to prove that Θ defined by (3.2) has a fixed point. The proof will be given in various steps.
Consider $B_\kappa = \{z\in C([0,T],\mathbb{R}^n) : \|z\|\leq\kappa\}$.

$$\kappa \geq \frac{\|\sigma\|_{L^1}}{\Gamma(\gamma+1)}T^\gamma\left[1+\left\|(P+Q)^{-1}Q\right\|\right] + \left\|(P+Q)^{-1}R\right\|$$

from equation (3.2),We define the operator M and N as follows:

$$(Mz)(t) = \tfrac{1}{\Gamma(\gamma)}\textstyle\int_0^t (t-s)^{\gamma-1}\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds$$

$$(Nz)(t) = -\tfrac{1}{\Gamma(\gamma)}(A+B)^{-1}B\textstyle\int_0^T (T-s)^{\gamma-1} \times\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds + (P+Q)^{-1}R$$

For $z,y\in B_\kappa$,
Step 1 : $Mz+Ny\in B_\kappa$ whenever $z,y\in B_\kappa$
$\|(Mz)(t)+(Ny)(t)\|$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\| ds + \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\int_0^T (T-s)^{\gamma-1} \times \|\mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s))\| ds + \left\|(P+Q)^{-1}R\right\|$$

$$\leq \frac{1}{\Gamma(\gamma)}\|\sigma\|_{L^1}\int_0^t (t-s)^{\gamma-1}ds + \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\| \times \|\sigma\|_{L^1}\int_0^T (T-s)^{\gamma-1}ds + \left\|(P+Q)^{-1}R\right\|$$

$$\leq \frac{\|\sigma\|_L{}^1}{\Gamma(\gamma+1)}[T^\gamma + \left\|(P+Q)^{-1}Q\right\|T^\gamma] + \left\|(P+Q)^{-1}R\right\|$$

$$\leq \frac{\|\sigma\|_{L^1}}{\Gamma(\gamma+1)}T^\gamma\left[1+\left\|(P+Q)^{-1}Q\right\|\right] + \left\|(P+Q)^{-1}R\right\|$$

$$\leq \kappa$$

Therefore, $Mz+Ny\in B_\kappa$.
Step 2.M is compact and continuous.
Let $t_1,t_2\in[0,T]$, $t_1<t_2$ and B_κ be a bounded set of $C([0,T],\mathbb{R}^n)$,

$z \in B_\kappa$, Then
$\|(Mz)(t_2)-(Mz)(t_1)\|$

$$\leq \frac{1}{\Gamma(\gamma)}\left[\int_0^{t_2}(t_2-s)^{\gamma-1}\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|ds + \int_0^{t_1}(t_1-s)^{\gamma-1}\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|ds\right]$$
$$\leq \frac{1}{\Gamma(\gamma)}\left[\int_0^{t_1}\left((t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\right)| \times \|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|ds + \int_{t_1}^{t_2}(t_2-s)^{\gamma-1} \|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|ds\right]$$
$$\leq \frac{\|\sigma\|_{L^1}}{\Gamma(\gamma)}\left[\int_0^{t_1}\left\|(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\right\|ds + \int_{t_1}^{t_2}\left\|(t_2-s)^{\gamma-1}\right\|ds\right]$$
$$\leq \frac{\|\sigma\|_{L^1}}{\Gamma(\gamma+1)}|t_2^\gamma - t_1^\gamma|$$

As $t_1 \to t_2$, the right hand side of above inequality tends to zero. So M is relatively compact on B_κ. Accordingly by Arzela-Ascoli theorem, M is compact on B_κ.

Step 3. N is a contraction mapping
$\|(Nz)(t)-(Ny)(t)\|$

$$\leq \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\left[\int_0^T(T-s)^{\gamma-1} \times \|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|ds - \int_0^T(T-s)^{\gamma-1} \times \|\mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s))\|ds\right]$$
$$\leq \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\int_0^T(T-s)^{\gamma-1} \times \left[\left\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s)) - \mathscr{F}(s,y(s),(\mu y)(s),(\nu y)(s))\right\|\right]ds$$
$$\leq \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\int_0^T(T-s)^{\gamma-1}\left[V_{\mathscr{F}_1}(s)\|z-y\| + V_{\mathscr{F}_2}(s)\|(\mu z)-(\mu y)\| + V_{\mathscr{F}_3}(s)\|(\nu z)-(\nu y)\|\right]ds$$
$$\leq \frac{1}{\Gamma(\gamma)}\left\|(P+Q)^{-1}Q\right\|\int_0^T(T-s)^{\gamma-1} \times \left[V_{\mathscr{F}_1}(s)+\rho_0 V_{\mathscr{F}_2}(s)+\chi_0 V_{\mathscr{F}_3}(s)\right]\|z-y\|ds$$
$$\leq \left\|(P+Q)^{-1}Q\right\|\left[I^\gamma V_{\mathscr{F}_1}(T)+\rho_0 I^\gamma V_{\mathscr{F}_2}(T) + \chi_0 I^\gamma V_{\mathscr{F}_3}(T)\right]\|z-y\|$$
$$\leq \left\|(P+Q)^{-1}Q\right\|\left\{[1+\rho_0+\chi_0]I^\gamma_{V_{\mathscr{F}}}\right\}\|z-y\|$$

$\|(Nz)(t)-(Ny)(t)\| \leq \varsigma_g\|z-y\|$
Thus, all the assumption of this theorem are satisfied. As a result of Krasnoselskii's fixed point theorem , we have that the boundary value problem $(1.1),(1.2)$ has atleast one solution on $[0,T]$. This Completes the proof. □

Our third result is based on Schaefer's fixed point theorem.

Definition 3.5. *[23](Schaefer's Fixed Point Theorem) Let $C([0,T],\mathbb{R}^n)$ be a Banach space and $\Theta: C([0,T],\mathbb{R}^n)\to C([0,T],\mathbb{R}^n)$ completely continuous operator. If the set*

$$X(\Theta)=\{x\in C([0,T],\mathbb{R}^n): x=\beta\Theta x \text{ for some } \beta\in[0,T]\}$$

Theorem 3.6. *Assume that (H2) and (H3) are satisfied. Then the boundary value problem (1.1)-(1.2) has atleast one solution on $[0,T]$.*

Proof. Schaefer's fixed point theorem is used to prove that Θ defined by (3.2) has a fixed point.The proof will be given in various steps.

Step 1. Operator Θ is continuous.
Let $\{z_n\}$ be a sequence such that $z_n\to z$ in $C([0,T],\mathbb{R}^n)$.Then for each $t\in[0,T]$
$\|\Theta(z_n)(t)-\Theta(z)(t)\|$

$$\leq \int_0^T\|G(t,s)\|\left\|\mathscr{F}(s,z_n(s),(\mu z_n)(s),(\nu z_n)(s)) - \mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\right\|ds$$
$$\leq \frac{1}{\Gamma(\gamma)}\left[\int_0^t(t-s)^{\gamma-1}\left\|\mathscr{F}(s,z_n(s),(\mu z_n)(s),(\nu z_n)(s)) - \mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\right\|ds + \left\|(P+Q)^{-1}R\right\| \times \int_0^T(T-s)^{\gamma-1}\left\|\mathscr{F}(s,z_n(s),(\mu z_n)(s),(\nu z_n)(s)) - \mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\right\|ds\right]$$
$$\leq \frac{1}{\Gamma(\gamma+1)}\left[T^\gamma(1+\left\|(P+Q)^{-1}Q\right\|)\right] \times \left\|\mathscr{F}(s,z_n(s),(\mu z_n)(s),(\nu z_n)(s))\right.$$

$$-\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\Big\|$$

Since $\mathscr{F}$ is continuous function, we have
$\|\Theta(z_n)(t)-\Theta(z)(t)\|$

$$\leq \frac{1}{\Gamma(\gamma+1)}\left[T^{\gamma}(1+\|(P+Q)^{-1}Q\|)\right] \times\Big\|\mathscr{F}(s,z_n(s),(\mu z_n)(s),(\nu z_n)(s)) -\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\Big\|\to 0$$

as $n\to\infty$
Step 2.Θ maps bounded sets into bounded sets in $C([0,T],\mathbb{R}^n)$. Indeed it is enough to show that for any $\eta>0$, there exists a positive constant q such that for each,

$$z\in B_\eta=\left\{z\in C([0,T],\mathbb{R}^n):\|z\|\leq\eta\right\}$$

we have $\|\Theta(z)\|\leq q$ and $(H3)$ we have for $t\in[0,T]$

$$\|\Theta(z)(t)\|\leq\int_0^T\|G(t,s)\|\,\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|\,ds+\|(P+Q)^{-1}R\|$$

Hence

$$\|\Theta(z)(t)\|\leq\frac{\|\sigma\|_{L^1}T^\gamma}{\Gamma(\gamma+1)}\left[1+\|(P+Q)^{-1}Q\|\right]+\|(P+Q)^{-1}R\|$$

Thus

$$\|\Theta(z)(t)\|\leq\frac{\|\sigma\|_{L^1}T^\gamma}{\Gamma(\gamma+1)}\left[1+\|(P+Q)^{-1}Q\|\right]+\|(P+Q)^{-1}R\|=q$$

Step 3. Θ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R}^n)$.
Let $t_1,t_2\in[0,T]$, $t_1<t_2$, B_η be a bounded set of $C([0,T],\mathbb{R}^n)$ as in step 2, and let $z\in B_\eta$. Then
$\|\Theta(z)(t_2)-\Theta(z)(t_1)\|$

$$\begin{aligned}&\leq\int_0^{t_2}\|G(t_2,s)\|\,\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|\,ds-\int_0^{t_1}\|G(t_1,s)\|\,\|\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))\|\,ds\\&\leq\frac{1}{\Gamma(\gamma)}\Bigg[\int_0^{t_1}\left[(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\right]\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds+\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\\&\quad\times\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds\Bigg]\\&\leq\frac{\|\sigma\|_{L^1}}{\Gamma(\gamma+1)}\left[t_2^\gamma-t_1^\gamma+2(t_2-t_1)^\gamma\right]\end{aligned}$$

As $t_1\to t_2$, the right hand side of the above inequality tends to Zero. As consequence of step 1 to 3 together with the Arzela-Ascoli Theorem, we can conclude that the operator $\Theta:C([0,T],\mathbb{R}^n)\to C([0,T],\mathbb{R}^n)$ is completly continuous.
Step 4. A priori bounds.
Now it remains to show that that the set

$$\Lambda=\{z\in C([0,T],\mathbb{R}^n):z=\omega\Theta(z),\quad\text{for some}\quad 0<\omega<1\}$$

is bounded.
Let $z\in\Lambda$ then $z=\omega(\Theta z)$ for some $0<\omega<1$. Thus for each $t\in[0,T]$ we have

$$z(t)=\omega\left[\int_0^T G(t,s)\mathscr{F}(s,z(s),(\mu z)(s),(\nu z)(s))ds+(P+Q)^{-1}R\right]$$

This implies by $(H3)$ and step 2 that for each $t\in[0,T]$ we have

$$|\Theta(z)(t)|\leq\frac{\|\sigma\|_{L^1}T^\gamma}{\Gamma(\gamma+1)}\left[1+\|(P+Q)^{-1}Q\|\right]+\|(P+Q)^{-1}R\|$$

Thus for every $t\in[0,T]$, we have

$$\|z\|\leq\frac{\|\sigma\|_{L^1}T^\gamma}{\Gamma(\gamma+1)}\left[1+\|(P+Q)^{-1}Q\|\right]+\|(P+Q)^{-1}R\|=I$$

This shows that the set Λ is bounded. As a outcome of Schaefer's fixed point theorem, we conclude that Θ has a fixed point which is a solution of the problem $(1.1)-(1.2)$. □

4. Example

In this section we give the example of our main results. We examine the following nonlinear fractional integrodifferential equations with two point boundary condition

$$^cD^{\frac{1}{2}}z(t)=\frac{e^{-t}|z(t)|}{(4+e^t)(1+|z(t)|)}+\frac{1}{9}\int_0^t\frac{1}{(t+1)^3}sin\frac{\sqrt{s}}{t}ds+\frac{1}{7}\int_0^t\frac{1}{(t+2)^2}cos\frac{\sqrt{s}}{t}ds,\qquad t\in[0,1]\tag{4.1}$$

$$x(0)+0.25x(1)=7\tag{4.2}$$

Problem $(4.1)-(4.2)$ is of the form $(1.1)-(1.2)$ with $\gamma=\frac{1}{2}$,

$$\mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t))=\frac{e^{-t}|z(t)|}{(4+e^t)(1+|z(t)|)}+\frac{1}{9}(\mu z)(t)+\frac{1}{7}(\nu z)(t)$$

where $(\mu z)(t)=\int_0^t\frac{1}{(t+1)^3}sin\frac{\sqrt{s}}{t}ds$ and $(\nu z)(t)=\int_0^t\frac{1}{(t+2)^2}cos\frac{\sqrt{s}}{t}ds$ and also $t\in[0,1]$ we have

$$\begin{aligned}&\|\mathscr{F}(t,z(t),(\mu z)(t),(\nu z)(t))-\mathscr{F}(t,y(t),(\mu y(t),(\nu y)(t))\|\\&\leq\frac{e^{-t}}{4+e^t}\|z(t)-y(t)\|+\frac{1}{9}\|(\mu z)(t)-(\mu y)(t)\|\\&\quad+\frac{1}{7}\|(\nu z)(t)-(\nu y)(t)\|\\&\leq\frac{1}{5}\|z-y\|+\frac{1}{9}\|(\mu z)-(\mu y)\|+\frac{1}{7}\|(\nu z)-(\nu y)\|\end{aligned}$$

from (16), $P=1$, $Q=0.25$, $R=7$, $T=1$.
Thus assumptions $(H1)-(H2)$ holds with,
$1+\left\|(P+Q)^{-1}Q\right\|=1.2$
$\rho_0=0.125$, $\chi_0=0.111$ and $1+\rho_0+\chi_0=1.24$
$I^{\gamma}_{V_{\mathscr{F}}}=0.226$
We get $\varsigma=0.336<1$
utilize theorem (3.1) we get (4.1) has a unique solution.

5. Conclusion

On the whole we have investigated the existence and uniqueness of solution to the Caputo type fractional differential equation with two point boundary conditions. The first sufficient condition proves the existence and uniqueness of the solution of (1.1) is derived by utilizing Banach fixed point theorem. The second sufficient condition gives the existence of solution of (1.1) is obtained via Krasnoselskii's fixed point theorem and the third sufficient condition is obtained by Schaefer's fixed point theorem. At last, example is provided to illustrate the applications of the abstract results.

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