Malaya Journal of Matematik, Vol. 7, No. 4, 771-774, 2019

https://doi.org/10.26637/MJM0704/0023

On various weakly prime fuzzy ideals of semirings

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Abstract

The notions of the weakly prime fuzzy ideals and three weakly fuzzy $m_0(m_1,m_2)$-systems of semirings are introduced. Six different prime fuzzy ideals are characterized by fuzzy points, their respective (weakly)fuzzy m-systems, respective m-systems and respective prime ideals in semirings. To validate our results, the examples are provided.

Keywords

Weakly $0-(1-2)$-prime fuzzy ideals, fuzzy m-system.

AMS Subject Classification

16Y60, 03E72, 16D25.

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Article History: Received **12** August **2019**; Accepted **18** November **2019**

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Contents

1. Introduction

P.Dheena and G.Mohanraj generalized prime fuzzy ideals [2, 3] of semirings and near-subtraction semigroup. G.Mohanraj et al. discussed prime fuzzy ideals of ordered semigroup and hemirings[6, 8].

2. Preliminaries

For basic ideas of semirings,ideals and fuzzy ideals of semirings can be found in [12].

Lemma 2.1. *[13] If P is an ideal of R, P is a 2-prime ideal $(1-(0))$-prime ideal of R if and only if P^c is an m_2-system $(m_1-system, m_0-system)$ of R.*

Theorem 2.2. *[2] Let f be a fuzzy set of R.Then f is a fuzzy ideal(fuzzy k-ideal) of R if and only if f_t is a ideal(k-ideal) of R for $t\in(0,1]$ whenever non-empty.*

Lemma 2.3. *[11] For any $a\in R,(a_t)$ is the fuzzy ideal generated by a fuzzy point a_t, then*

(1)$(a_t)=t\chi_{(a)}$
(2)$(a_t)_r=t\chi_{(a)_r}$
(3)$(a_t)_l=t\chi_{(a)_l}$
where $(a)_r$ and $(a)_l$ are the right ideal,left ideal generated by a.

Theorem 2.4. *[10] The following statements are equivalent for fuzzy ideals f.*
(1) f is a weakly prime fuzzy ideal.
(2) $a_t\cdot 1\cdot b_t\subseteq f$ implies $a_t\in f$ or $b_t\in f$ for $t\in[0,1]$ and $a,b\in R$.
(3) For any $a_t,b_t\in 1$, $(a_t)\cdot(b_t)\subseteq f$ implies $a_t\in f$ or $b_t\in f$.
(4) For the right ideals A and B in R,$t\chi_A\cdot t\chi_B\subseteq f$ implies $t\chi_A\subseteq f$ or $t\chi_B\subseteq f$.
(5) For the left ideals A and B in R,$t\chi_A\cdot t\chi_B\subseteq f$ implies $t\chi_A\subseteq f$ or $t\chi_B\subseteq f$.
(6) For the right ideals A and left ideals L in R, $t\chi_A\cdot t\chi_L\subseteq f$ implies $t\chi_A\subseteq f$ or $t\chi_L\subseteq f$.

Lemma 2.5. *[10] For any fuzzy sets A and B of R and $t\in[0,1]$, $t\chi_A\cdot t\chi_B=t\chi_{AB}$*

Lemma 2.6. *[2] If f is any fuzzy ideal of R, then $\overline{f}$ is a fuzzy k-ideal of R.*

3. Various 0-prime fuzzy ideals

Form here onwards R denotes semirings.

Definition 3.1. *The fuzzy set f is called weakly fuzzy m_0-system if there is $x\in R$ such that $f(axb)>t$ when $f(a)>t$ and $f(b)>t$.*

Theorem 3.2. *The following statements are equivalent for a fuzzy ideals f.*
(1) f is a weakly 0-prime fuzzy ideal.
(2) For $a,b \in R$ and for $t \in [0,1]$ $a_t \cdot 1 \cdot b_t \subseteq f$ implies $a_t \in f$ or $b_t \in f$, for $t \in [0,1]$.
(3) $1-f$ is weakly fuzzy m_0-system.
(4) $(1-f)^t$ is m_0-system, for $t \in [0,1]$ for $t \in [0,1]$.
(5) f_t^c is a m_0-system, for all $t \in [0,1]$ for $t \in [0,1]$.
(6) f_t is 0-prime ideal of R, for $t \in [0,1]$.

Proof. (1)⇒(2): By Theorem 2.4,(2) holds.
(2)⇒(3): Suppose $1-f(a) > t$ and $1-f(b) > t$. Then $f(a) < 1-t$ and $f(a) < 1-t$. Thus $a_{1-t}\overline{\in}f$ and $b_{1-t}\overline{\in}f$. By(2) $a_{1-t} \cdot 1 \cdot b_{1-t} \nsubseteq f$. Then, there is $y \in R$ such that $f(ayb) < 1-t$. Thus, $1-f(ayb) > t$. Therefore, $1-f$ is weakly fuzzy m_0-system.
(3)⇒(4): Now, $x,y \in (1-f)^t$ implies $1-f(x) > t$ and $1-f(y) > t$. By(3), $(1-f)(xry) > t$ for some $r \in R$, implies $xry \in (1-f)^t$. Thus, $(1-f)^t$ is m_0-system.
(4)⇒(5): Now,

$$\begin{aligned}(1-f)^t &= \{x | 1-f(x) > t\}\\ &= \{x | f(x) < 1-t\}\\ &= f_{1-t}^c.\end{aligned}$$

Thus, by (4) f_t^c is m_0-system
(5)⇒(6): By Lemma 2.1, (6) holds.
(6)⇒(1) : If there are ideals A and B such that $t\chi_A \cdot t\chi_B \subseteq f$, then by Lemma 2.5, $A \cdot B \subseteq f_t$. Thus $A \subseteq f_t$ or $B \subseteq f_t$ imply $t\chi_A \subseteq f$ or $t\chi_B \subseteq f$. Thus, f is a weakly 0-prime fuzzy ideal. □

Theorem 3.3. *The fuzzy ideal f is 0-prime fuzzy ideal if and only if $t\chi_A \cdot s\chi_B \subseteq f$ implies $t\chi_A \subseteq f$ or $s\chi_B \subseteq f$ for the ideals A and B in R and for $t,s \in [0,1]$.*

Proof. Suppose that f is 0-prime fuzzy ideal. For any ideals A and B in R, by Theorem 2.2, $t\chi_A$, $s\chi_B$ are fuzzy ideals. Then, $t\chi_A \cdot s\chi_B \subseteq f$ implies $t\chi_A \subseteq f$ or $s\chi_B \subseteq f$.

Conversely, if there are fuzzy ideals g and h such that $g \cdot h \subseteq f$ with $h \nsubseteq f$, then there exists $y \in R$ such that $s = h(y) > f(y)$. For any $x_t \in g$, $t\chi_{(x)} \cdot s\chi_{(y)} \subseteq g \cdot h \subseteq f$. Thus $x_t \in f$ for all $x_t \in g$. Thus $g \subseteq f$. Therefore, f is 0-prime fuzzy ideal. □

Remark 3.4. *(i) Every 0-prime fuzzy ideal is weakly 0-prime fuzzy ideal.*
(ii) But, weakly 0-prime fuzzy ideal need not be 0-prime fuzzy ideal by the Example as in [10].

4. Various 1-prime fuzzy ideals

Definition 4.1. *The fuzzy ideal f of R is called a weakly 1-prime fuzzy ideal of R if $t\chi_A \cdot t\chi_B \subseteq f$ implies $t\chi_A \subseteq f$ or $t\chi_B \subseteq f$ for any k- ideals A and for any ideals B in R and $t \in [0,1]$.*

Definition 4.2. *The fuzzy set f is said to be weakly fuzzy m_1-system if for any $t \in [0,1)$ and $a,b \in R$, $f(a) > t, f(b) > t$ implies that there exists $a_1 \in \overline{(a)}$ and $b_1 \in (b)$ such that $f(a_1b_1) > t$.*

Theorem 4.3. *The following statements are equivalent for a fuzzy ideals f.*
(1) f is a weakly 1-prime fuzzy ideal.
(2) $\overline{(a_t)} \cdot (b_t) \subseteq f$ implies $a_t \in f$ or $b_t \in f$, for $t \in [0,1]$
(3) $1-f$ is weakly fuzzy m_1-system of R.
(4) $(1-f)^t$ is a m_1-system, for $t \in [0,1]$.
(5) f_t^c is a m_1-system, for $t \in [0,1]$.
(6) f_t is 1-prime ideal, for $t \in [0,1]$.

Proof. (1)⇒(2): By Lemma 2.3 and 2.6, (2) holds.
(2)⇒(3): If $1-f(a) > t$ and $1-f(b) > t$, then $a_{1-t}\overline{\in}f$ and $b_{1-t}\overline{\in}f$. By (2), $\overline{(a_{1-t})} \cdot (b_{1-t}) \nsubseteq f$.Thus, there is $a_1 \in \overline{(a)}$, $b_1 \in (b)$ such that $1-f(a_1b_1) > t$. Hence, $1-f$ is weakly fuzzy m_1-system.
(3)⇒(4): Now, $a,b \in (1-f)^t$ implies $a_1b_1 \in (1-f)^t$ for $a_1 \in \overline{(a)}$, $b_1 \in (b)$. Thus, $(1-f)^t$ is m_1-system.
(4)⇒(5): Now, $f_t^c = (1-f)^{1-t}$ implies that (5) holds.
(5)⇒(6): The result follows from by Lemma 2.1.
(6)⇒(1): If there exist k-ideal A and ideal B such that $t\chi_A \cdot t\chi_B \subseteq f$, then by Lemma 2.5, $A \cdot B \subseteq f_t$. Thus by (6) $A \subseteq f_t$ or $B \subseteq f_t$ imply $t\chi_A \subseteq f$ or $t\chi_B \subseteq f$. Therefore, f is a weakly 1-prime fuzzy ideal. □

Theorem 4.4. *The following statements are equivalent for a fuzzy ideals f.*
(1) f is a 1-prime fuzzy ideal.
(2) $t\chi_A \cdot s\chi_B \subseteq f$ implies $t\chi_A \subseteq f$ or $s\chi_B \subseteq f$ for any k-ideal A for any ideal B in R and for $t,s \in [0,1]$.
(3) $\overline{(a_t)} \cdot (b_s) \subseteq f$ implies $a_t \in f$ or $b_s \in f$, for $t,s \in [0,1]$.
(4) $1-f$ is a fuzzy m_1-system of R.

Proof. (1)⇒(2): For any k-ideal A and for any ideal B in R, such that $t\chi_A \cdot s\chi_B \subseteq f$, by Lemma 2.6 $t\chi_A \subseteq f$ or $s\chi_B \subseteq f$.
(2)⇒(3): Now, $\overline{(a_t)} \cdot (b_s) \subseteq f$, then by Lemma 2.3& 2.6 and Theorem 2.2, $t\chi_{\overline{(a)}} \cdot s\chi_{(b)} \subseteq f$. Thus, $a_t \in f$ or $b_s \in f$.
(3)⇒(4): Suppose $1-f(a) > t$ and $1-f(b) > s$. Then $a_{1-t}\overline{\in}f$ and $b_{1-s}\overline{\in}f$ imply $(1-t)\chi_{\overline{(a)}} \cdot (1-s)\chi_{(b)} \nsubseteq f$. Thus there exists $a_1 \in \overline{(a)}$, $b_1 \in (b)$ with $a_1b_1 \in \overline{(a)} \cdot (b)$ such that $f(a_1b_1) < (1-t) \wedge (1-s)$. Therefore $1-f(a_1b_1) > t \vee s$. Hence, $1-f$ is a fuzzy m_1-system.
(4)⇒(1):If there are k-ideal A and an ideal B such that $t\chi_A \cdot s\chi_B \subseteq f$ with $t\chi_A \nsubseteq f$ and $s\chi_A \nsubseteq f$,then $a_t\overline{\in}f$ and $b_s\overline{\in}f$ for $a \in A$ and $b \in B$.Then $(1-f)(a) > 1-t$ and $(1-f)(b) > 1-s$ implies $f(a_1b_1) < t \wedge s$ for $a_1 \in \overline{(a)}$ and $b_1 \in (b)$ which contradicts $t\chi_A \cdot s\chi_B \subseteq f$. Thus, f is 1-prime fuzzy ideal. □

Theorem 4.5. *[12] The fuzzy ideal f of R is 1-prime fuzzy ideal if and only if*
(i) $|\mathrm{Im} f| = \{1,t\}$
(ii) f_1 is 1-prime ideal.

Remark 4.6. *(i) Every* 1*-prime fuzzy ideal is weakly* 1*-prime fuzzy ideal by Theorem 4.3 and 4.5.*
(ii) But, weakly 1*-prime fuzzy ideal fails to be a* 1*-prime fuzzy ideal by the Example 4.7.*
(iii) Every weakly 0*-prime fuzzy ideal is a weakly* 1*-prime fuzzy ideal.*
(iv) But, weakly 1*-prime fuzzy ideal fails to be a weakly* 0*-prime fuzzy ideal by the Example 4.7.*
(v) Every 0*-prime fuzzy ideal is* 1*-prime fuzzy ideal.*
(vi) But, 1*-prime fuzzy ideal fails to be* 0*-prime fuzzy ideal by the Example 4.7.*

Example 4.7. *Consider the semiring* $B(7,4) = \{0,1,2,3,4,5,6\}$ *as in [1].*

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	4
2	2	3	4	5	6	4	5
3	3	4	5	6	4	5	6
4	4	5	6	4	5	6	4
5	5	6	4	5	6	4	5
6	6	4	5	6	4	5	6

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	5	4	6
3	0	3	6	6	6	6	6
4	0	4	5	6	4	5	6
5	0	5	4	6	5	4	6
6	0	6	6	6	6	6	6

Now,we defined

$$f(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.8 & \text{if } x \in \{3,6\} \\ 0.5 & \text{if } x \in \{4,5\} \\ 0.3 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 0.3 & \text{if } x \in \{1,2\} \\ 1 & \text{otherwise} \end{cases}$$

Clearly, f *is weakly* 1*-prime fuzzy ideal. By Theorem 4.5* f *is not a* 1*-prime fuzzy ideal. Now ,*$\{0,3,4,5,6\}$ *is* 1*-prime ideal but not* 0*-prime ideal. Then by Theorem 3.2,* f *is not weakly* 0*-prime fuzzy ideal.*

By Theorem 4.5, g *is* 1*-prime fuzzy ideal but not a* 0*-prime fuzzy ideal.*

5. Various 2-prime fuzzy ideals

Definition 5.1. *The fuzzy ideal* f *of* R *is called a weakly* 2*-prime fuzzy ideal of* R *if* $t\chi_A \cdot t\chi_B \subseteq f$ *implies* $t\chi_A \subseteq f$ *or* $t\chi_B \subseteq f$ *for the k- ideals* A *and* B *in* R *and* $t \in [0,1]$.

Definition 5.2. *The fuzzy set* f *is said to be weakly fuzzy* m_2*-system if for any* $t \in [0,1)$ *and* $a, b \in R$, $f(a) > t$, $f(b) > t$ *implies that there exists* $a_2 \in \overline{(a)}$ *and* $b_2 \in \overline{(b)}$ *such that* $f(a_2 b_2) > t$.

Theorem 5.3. *The following statements are equivalent for a fuzzy ideals* f.
(1) f *is a weakly* 2*-prime fuzzy ideal.*
(2) $\overline{(a_t)} \cdot \overline{(b_t)} \subseteq f$ *implies* $a_t \in f$ *or* $b_t \in f$ *for* $t \in [0,1]$
(3) $1 - f$ *is a weakly fuzzy* m_2*-system of* R.
(4) $(1-f)^t$ *is a* m_2*-system, for* $t \in [0,1]$.
(5) f_t^c *is a* m_2*-system, for* $t \in [0,1]$.
(6) f_t *is a* 2*-prime ideal, for* $t \in [0,1]$.

Proof. (1)⇒(2): By Lemma 2.3 and 2.6, (2) holds.
(2)⇒(3): Now, $1 - f(a) > t$ and $1 - f(b) > t$ implies $a_{1-t} \overline{\in} f$ and $b_{1-t} \overline{\in} f$. By (2) $\overline{(a_{1-t})} \cdot \overline{(b_{1-t})} \nsubseteq f$. Thus, $a_1 \in \overline{(a)}$, $b_1 \in \overline{(b)}$ such that $f(a_1 b_1) < 1 - t$. Therefore, $1 - f(a_1 b_1) > t$. Hence, $1 - f$ is weakly fuzzy m_2-system.
(3)⇒(4):Now, $a, b \in (1-f)^t$ implies $a_1 b_1 \in (1-f)^t$ for $a_1 \in \overline{(a)}$, $b_1 \in \overline{(b)}$. Thus, $(1-f)^t$ is m_2-system.
(4)⇒(5): Now, $f_t^c = (1-f)^{1-t}$ implies that (5) holds.
(5)⇒(6): By Lemma 2.1, (6) holds.
(6)⇒(1): If there are k-ideals A and B such that $t\chi_A \cdot t\chi_B \subseteq f$, then by Lemma 2.5, $A \cdot B \subseteq f_t$. Thus by (6) $A \subseteq f_t$ or $B \subseteq f_t$ imply $t\chi_A \subseteq f$ or $t\chi_B \subseteq f$. Thus, f is a weakly 2-prime fuzzy ideal. □

Theorem 5.4. *The following statements are equivalent for a fuzzy ideals* f.
(1) f *is a* 2*-prime fuzzy ideal.*
(2) $t\chi_A \cdot s\chi_B \subseteq f$ *implies* $t\chi_A \subseteq f$ *or* $s\chi_B \subseteq f$ *for the* k*-ideals* A *and* B *in* R *and for* $t, s \in [0,1]$.
(3) $\overline{(a_t)} \cdot \overline{(b_s)} \subseteq f$ *implies* $a_t \in f$ *or* $b_s \in f$, *for* $t, s \in [0,1]$.
(4) $1 - f$ *is a fuzzy* m_2*-system of* R.

Proof. (1)⇒(2): For the k-ideals A and B in R, such that $t\chi_A \cdot s\chi_B \subseteq f$, then by Lemma 2.6, $t\chi_A \subseteq f$ or $s\chi_B \subseteq f$.
(2)⇒(3): Now, $\overline{(a_t)} \cdot \overline{(b_s)} \subseteq f$, then by Lemma 2.3 and 2.6, $t\chi_{\overline{(a)}} \cdot s\chi_{\overline{(b)}} \subseteq f$. Thus, $a_t \in f$ or $b_s \in f$.
(3)⇒(4): Now, $1 - f(a) > t$ and $1 - f(b) > s$ imply $a_{1-t} \overline{\in} f$ and $b_{1-s} \overline{\in} f$. By (3),$(1-t)\chi_{\overline{(a)}} \cdot (1-s)\chi_{\overline{(b)}} \nsubseteq f$. Then, there exists $a_2 \in \overline{(a)}$, $b_2 \in \overline{(b)}$ with $a_2 b_2 \in \overline{(a)} \cdot \overline{(b)}$ such that $f(a_2 b_2) < (1-t) \wedge (1-s)$. Thus, $1 - f(a_2 b_2) > t \vee s$. Therefore, $1 - f$ is a fuzzy m_2-system.
(4)⇒(1): If there exist fuzzy k-ideals g and h of R such that $g \cdot h \subseteq f$ with $g \nsubseteq f$ and $h \nsubseteq f$,then there exists $a, b \in R$ such that $s = g(a) > f(a)$ and $t = h(b) > f(b)$.Thus $a_s \overline{\in} f$ and $b_t \overline{\in} f$.Then $\overline{(a_s)} \cdot \overline{(b_t)} \nsubseteq f$ which contradicts $g \cdot h \subseteq f$.Thus f is 2-prime fuzzy ideal. □

Remark 5.5. *(i) Every* 2*-prime fuzzy ideal is weakly* 2*-prime fuzzy ideal by Theorem 5.3.*
(ii) But, weakly 2*-prime fuzzy ideal fails to be a* 2*-prime fuzzy ideal by the Example 5.6.*
(iii) Every weakly 0*-prime fuzzy ideal is a weakly* 2*-prime fuzzy ideal.*
(iv) But, weakly 2*-prime fuzzy ideal fails to be a weakly* 0*-prime fuzzy ideal by the Example 5.6.*

Example 5.6. *Consider the semiring $B(7,4) = \{0,1,2,3,4,5,6\}$ as in Example 4.7 .Now,we defined*

$$f(x) = \begin{cases} 1 & if\ x=0 \\ 0.8 & if\ x=4 \\ 0.4 & otherwise \end{cases}$$

Clearly, f is weakly 2-prime fuzzy ideal but not a 2-prime fuzzy ideal and weakly 0-prime fuzzy ideal.

References

[1] Alarcon F.E and Anderson D.D, Commutative semiring and their lattices of ideals, *Houston Journal of Mathematics*, 20(4)(1994), 571–590.

[2] Dheena.P and Mohanraj.G, (λ,μ)-fuzzy ideals in Semirings, *Advances in Fuzzy Mathematics*, 6(2)(2011), 183–192.

[3] Dheena.P, and Mohanraj.G, Fuzzy weakly prime ideals of near-subtraction semigroups, *Annals of Fuzzy Mathematics and Informatics*, 4(2)(2012), 235–242.

[4] Dheena.P and Mohanraj.G, On $(\lambda,\mu)-$fuzzy prime ideals of semirings, *The Journal of Fuzzy Mathematics*, 20(3)(2012), 889–898.

[5] Dheena.P and Coumaressane.S, Fuzzy 2-$(0-or1)$-prime ideals in semirings, *Bulletin Korean Mathematical Society*, 43(3)(2006), 559–573.

[6] Mohanraj.G, Krishnasamy.D and Hema.R, On fuzzy *m*-systems and *n*-systems of ordered semigroups, *Annals of Fuzzy Mathematics and Informatics*, 7(1)(2014), 173–179.

[7] Mohanraj.G, Hema.R and Prabu.E, On various weak fuzzy prime ideals of ordered semigroup, Proceedings of the International Conference on Mathematical Sciences published by Elsevier, (2014), 475–479.

[8] Mohanraj.G and Prabu.E, Weakly fuzzy prime ideal of hemiring, Proceedings of the International Conference on Mathematical Sciences published by Elsevier,(2014), 480–483.

[9] Mohanraj.G and Prabu.E ,Generalized fuzzy right *h*-ideals of hemirings redefined by fuzzy sums and products,Bulletin of the International Mathematical Virtual Institute, 7(3)(2017), 527–538.

[10] Mohanraj.G and Azhaguvelavan.S , On weakly prime (λ,μ)-fuzzy ideals of semirings, *Mathematical Sciences International Research Journal(IMRF)*, 7(2018), 138–144.

[11] Mohanraj.G and Azhaguvelavan.S, Weakly prime fuzzy ideals of hemirings, Submitted.

[12] Mohanraj.G and Azhaguvelavan, Various generalized prime fuzzy ideals of semirings, *AIP Conference Proceedings*, (2019), 020020-1-020020-8.

[13] Nandakumar.P, $1-(2)$-prime ideals in semirings, *Kyungpook Mathematical Journal*, 50(2010), 117–122.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666

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