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Toeplitz properties of ω-order preserving partial contraction mapping

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Abstract. In this paper, spectral mapping theorem for the point spectrum on infinitesimal generator of a C_0-semigroup was further investigated. Toeplitz properties of semigroup considering ω-order preserving partial contraction mapping ($\omega - OCP_n$) as a semigroup of linear operator was established to obtained new results. We also consider $A \in \omega - OCP_n$ which is the infinitesimal generator of a C_0-semigroup using the Spectral Mapping Theorem (SMT) to obtain the relationships between the spectrum of A and the spectrum of each of the operators $\{T(t),\ t \geq 0\}$.

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1. Introduction and Background

The emphasis of spectral theory in functional analysis is important because it studies the structure of a linear operator on the basis of its spectral properties such as the location of the spectrum, the behaviour of the resolvent and the asymptotics of its eigenvalues. It is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. Suppose X is Banach space, $X_n \subseteq X$ is a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup and its also Toeplitz matrix. This paper consist of results of Toeplitz ω-preserving partial contraction mapping generating a spectral mapping theorem. Balakrishnan [1], established fractional powers of closed operators. Banach [2], introduced the concept of Banach spaces. Bojanczyk *et al.* [3], obtained some results on stability of the Bareiss and related Toeplitz factorization algorithms. Böttcher and Grudsky [4], deduced some results on Teoplitz matrices, asymptotics linear algebra and functional analysis. Engel and Nagel

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[5], obtained one-parameter semigroup for linear evolution equations. Greiner *et al.* [6], showed some results on the spectral bond generator of semigroup of positive operators. Hasegawa [7], introduced some results on the convergence of resolvents of operators. Neerven [8], established the asymptotic behavior of semigroup of linear operator. Pazy [9], presented semigroup of linear operators and applications to partial differential equations. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established its properties, also in [11], Rauf *et al.* deduced some results of stability and spectra properties on semigroup of linear operator. Slemrod [12], explained asymptotic behavior of C_0-semigroup as determined by the spectrum of the generator. Vrabie [13], proved some results of C_0-semigroup and its applications. Yosida [14], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1. *(C_0-Semigroup) [13] A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.*

Definition 2.2. *(ω-OCP_n) [10] A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.*

Definition 2.3. *(Resolvent Set) [5] We define the resolvent set of A denoted by $\rho(A)$ set of all $\lambda \in \mathbb{C}$ such that λI - A is one-to-one with range equal to X*

Definition 2.4. *(Spectrum) [5] The spectrum of A denoted by $\sigma(A)$ is defined as the complement of the resolvent set.*

Definition 2.5. *(Toeplitz matrix) [4] Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant for any $n \times n$ and for any $m \times n$ matrices.*

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have

for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 4
Let X be the Banach space of Continuous function on [0,1] which are equal to zero at $x = 1$ with the supremum norm. Define

$$(T(t)f)(x) = \begin{cases} f(x+t) & if \;\; x+t \leq 1 \\ 0 & if \;\; x+t > 1 \end{cases}$$

$T(t)$ is obviously a C_0-Semigroup of Contractions on X. Its infinitesimal generator $A \in \omega$-OCP_n is given by

$$D(A) = \{f : f \in C'([0,1]) \cap X_1 f' \in X\}$$

and

$$Af = f' \quad for \quad f \in D(A).$$

one checks easily that for every $\lambda \in \mathbb{C}$ and $g \in X$ the equation $\lambda f - f' = g$ has a unique solution $f \in X$ given by

$$f(t) = \int_t^1 e^{\lambda(t-s)} g(s) ds.$$

Therefore $\sigma(A) = \phi$. on the other hand, since for every $t \geq 0$, $T(t)$ is a bounded linear operator, $\sigma(T(t)) \neq \phi$ for all $t \geq 0$ and the relation $\sigma(T(t)) = exp\{t\sigma(A)\}$ does not hold for any $t \geq 0$.

Theorem 2.6. *(Hille-Yoshida) [11] A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if*

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3. Main Results

This section presents results of spectral mapping theorem for point spectrum generated by Toeplitz ω-OCP_n:

Theorem 3.1. *Let $T(t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then we have the spectral inclusion relation*

$$\sigma(T(t)) > \exp(t\sigma(A)), \quad \forall t \geq 0.$$

Proof. Firstly we need to show that A is a Toeplitz matrix. Assume b is a trigonometric polynomial of the form

$$\varphi(t) = \sum_{j=-r}^{r} b_j t^j$$

$t \in T(t)$, and let X and Y be infinite matrices of all entries of which are sero outside the upper left $r \times r$ block, that is

$$P_r X P_r = X, \quad P_r Y P_r = Y.$$

without loss of generality assume that $r \geq 1$. Put

$$A_n = T_n(a) + P_n X P_n + \omega_n Y \omega_n,$$

where $A \in \omega - OCP_n$. Obviously, A_n is a band matrix with at most $2r + 1$ non-zero diagonals.So, let

$$M = \max(\|T(a) + X\|, \|T(\hat{a} + Y\|), \quad M_0 = \|T(a)\|.$$

Since

$$\|T(a)\| = \|T(a)\| = \|a\|_\infty,$$

then we have

$$\|T(a) + X\| \geq \|T(a)\| = \|T(a)\|,$$

$$\|T(\tilde{a}) + Y\| \geq \|T(\tilde{a})\| = \|T(\tilde{a})\| = \|T(a)\|$$

we always have $M \geq M_0$, so that $\|A_n\| \to M$ as $n \to \infty$.

It is easy to see that

$$\int_0^t T(s)_x ds \in D(A)$$

or all $t \geq 0$. In fact, a direct application of the definition of the generator shows that

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x, \quad \forall x \in X. \tag{3.1}$$

$$A\left(\int_0^t T(s)xds\right) = \int_0^t T(s)Axdx. \tag{3.2}$$

By applying (3.1) and (3.2) to the semigroup $T(t) - \lambda := \{e^{-\lambda t}T(t)\}_{t\geq 0}$ generated by $A - \lambda$, for all $\lambda \in \mathbb{C}$ and $t \geq 0$ we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)xds = (e^{\lambda t} - T(t))x, \quad \forall x \in X$$

and $A \in \omega - OCP_n$, so that

$$\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)xds = (e^{\lambda t} - T(t))x \quad \forall x \in D(A) \tag{3.3}$$

and $A \in \omega - OCP_n$.

Suppose $e^{\lambda t} \in \varphi(T(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote the inverse of $e^{\lambda t} - T(t)$ by $K_{\lambda t}$. Since $K_{\lambda t}$ commutes with $T(t)$ and hence also with A, then we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)K_{\lambda,tx}ds = x, \quad \forall x \in X$$

and $A \in \omega - OCP_n$, so that

$$\int_0^t e^{\lambda(t-s)}T(s)K_{\lambda,t}(\lambda - A)xds = x, \quad \forall x \in D(A)$$

and $A \in \omega - OCP_n$.

This shows that the bounded operator B_λ defined by

$$B_\lambda x := \int_0^t e^{\lambda(t-s)}T(s)K_{\lambda,t}xds$$

is a two-sided inverse of $\lambda - A$. It follows that $\lambda \in \varphi(A)$ is in the spectral inclusion relation which achieved the proof.

■

Theorem 3.2. *Assume $T(t)_{t\geq 0}$ is a semigroup of linear operator on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then*

$$\sigma_p(T(t)) \setminus \{0\} \exp(t\sigma_p(A)), \quad \forall t \geq 0.$$

Proof. Suppose $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is an eigenvector corresponding to λ, the identity (3.3) shows that $T(t)x = e^{\lambda t}x$, that is $e^{\lambda t}$ is an eigenvalue of $T(t)$ with eigenvector x. This proves the inclusion $\supset$.

The inclusion $\subset$ is proved as follows. The case $t = 0$ being trivial, we fix $t > 0$. If $\lambda \in \sigma_p(T(t)) \setminus \{0\}$, then $\lambda = e^{\mu t}$ for some $\mu \in \mathbb{C}$. If x is an eigenvector, then

$$T(t)_x = e^{\mu t}x$$

implies that the map

$$s \mapsto e^{-\mu s}T(s)x$$

is a periodic with period t.

Since this map is not identically zero, the uniqueness theorem for the Fourier transform implies that at least one of its Fourier coefficients is non-zero. Thus, there exists an integer $k \in \mathbb{Z}$ such that

$$x_k := \frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}(e^{\mu s}T(s)x)ds \neq 0.$$

we shall show that $\mu_k := \mu + 2\pi ik/t$ is an eigenvalue of A with eigenvector x_k.

By the t-periodicity of $s \mapsto e^{-\mu s}T(s)x$, for all $Rev > \omega_0(T(t))$, we have

$$\begin{aligned} R(v, A)x &= \int_0^\infty e^{-vs}T(s)xds \\ &= \sum_{n=0}^{\infty}\int_{nt}^{(n+1)t} e^{-vs}T(s)xds \\ &= \sum_{n=0}^{\infty}\int_0^t e^{-vs}T(s)(e^{-vnt}T(nt)x)ds \qquad (3.4) \\ &= \sum_{n=0}^{\infty} e^{(\mu-v)nt}\int_0^t e^{-vs}T(s)xds \\ &= \frac{1}{1-e^{(\mu-v)t}}\int_0^t e^{-vs}T(s)xds. \end{aligned}$$

Since the integral on the right hand side is an entire function, this shows that the map $v \mapsto R(v, A)x$ admits a holomorphic continuation to $\mathbb{C} \setminus \{\mu + 2\pi in/t : n \in \mathbb{Z}\}$. Denoting this extension by $F_x(\cdot)$, (3.4) and the definition of x_k we have

$$\lim_{v \to \mu k} (v - \mu k) F_x(v) = x_k.$$

Also, by (3.4) and the t-periodicity of $s \mapsto e^{-\mu s} T(s)x$,

$$\begin{aligned}\lim_{v \to \mu k} (\mu - A)((v - \mu k) F_x(v)) &= \lim_{v \to \mu k} (\mu - A)((v - \mu k) F_x(v)) \\ &= \lim_{v \to \mu k} \frac{v - \mu k}{1 - e^{(\mu - v)t}} \left((I - e^{-vt} T(t)) + (\mu k - v) \int_0^t e^{-vs} T(s) x ds \right) \\ &= \frac{1}{t}(0 + 0) = 0.\end{aligned}$$

From the closedness of A , it follows that $x_k \in D(A)$, $A \in \omega - OCP_n$ and $(\mu k - A)_{xk} = 0$.

The spectral mapping theorem also holds for the residual spectrum. This follows from a duality argument for which we need the following definitions.

Since $T(t)$ is a C_0-semigroup on X, then we define

$$X^{\odot} := \{x^* \in X^* : \lim_{t \to 0} \|T^*(t)_x - x^*\| = 0\},$$

where $T^*(t) := (T(T))^*$ is the adjoint operator. It is easy to see that $X^{\odot}$ is a closed $T^*(t)$ - invariant subspace of X^*, and the restriction $T^{\odot}$ of T^* to $X^{\odot}$ is a C_0-semigroup on $X^{\odot}$. We denote its generator by $A^{\odot} \in \omega - OCP_n$.

We claim that $\sigma_p(A^*) = \sigma_p(A^{\odot})$, where $A^* \in \omega - OCP_n$ is the adjoint of the generator A of $T(t)$, and $\sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t))$, $t \geq 0$ and $A \in \omega - OCP_n$.

We start with the first of these assertion. For all $x^* \in D(A^*)$, $x \in X$ and $A \in \omega - OCP_n$, we have

$$\left.\begin{aligned}\langle T^*(t)x^* - x^*, x \rangle &= \langle x^*, T(t)x - x \rangle \\ &= \langle A^* x^*, \textstyle\int_0^t T(t) x dt \rangle \\ &= \textstyle\int_0^t \langle A^* x^*, T(t)x \rangle ds \\ &= \textstyle\int_0^t \langle T^*(t) A^* x^*, x \rangle dt.\end{aligned}\right\} \tag{3.5}$$

Therefore,

$$|\langle T^*(t)x^* - x^*, x \rangle| \leq t \|x\| \|A^* x^*\| \sup_{0 \leq s \leq t} \|T(s)\|.$$

By taking the supremum over all $x \in X$ of norm ≤ 1, it follows that

$$\lim_{t \to 0} \|T^*(t)x^* - x^*\| = 0,$$

that is $x^* \in X^{\odot}$. This proves that $D(A^*) \subset X^{\odot}$.

Now assume that $A^* x^* = \lambda x^*$ for some $x^* \in D(A^*)$ and $A^* \in \omega - OCP_n$. Then $x^* \in X^{\odot}$ and (3.5) shows that

$$\left\langle \frac{1}{t}(T^{\odot}(t)x^* - x^*) - \lambda x^*, x \right\rangle = \frac{\lambda}{t} \int \langle T^{\odot}(t)x^* - x^*, x \rangle dt$$

and therefore,

$$\left\| \frac{1}{t}(T^{\odot}(t)x^* - x^*) - \lambda x^* \right\| \leq |\lambda| \|x^*\| \sup_{0 \leq s \leq t} \|T(s)x - x\|.$$

Letting $t \to 0$, this shows that $x^* \in D(A^{\odot})$ and

$$A^{\odot} x^* = \lambda x^*,$$

so $\lambda \in \sigma_p(A^{\odot})$.

Conversely, if $\lambda \in \sigma_p(A^{\odot})$ and $A^{\odot}x^{\odot} = \lambda x^{\odot}$, for some $x^{\odot} \in D(A^{\odot})$ and $A^{\odot} \in \omega - OCP_n$, then for all $x \in D(A)$ we have

$$\begin{aligned}\langle x^{\odot}, Ax\rangle &= \lim_{t\to 0}\frac{1}{t}\langle x^{\odot}, T(t)x - x\rangle\\ &= \lim_{t\to 0}\frac{1}{t}\langle T^{\odot}(t)x^{\odot} - x^{\odot}, x\rangle = \langle A^{\odot}x^{\odot}, x\rangle\\ &= \lambda\langle x^{\odot}, x\rangle.\end{aligned}$$

This shows that $x^{\odot} \in D(A^*)$ and $A^*x^{\odot} = \lambda x^{\odot}$, so that $\lambda \in \sigma_p(A^*)$.

Next we prove that

$$\sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t)) \quad for\ all\ t \geq 0.$$

Clearly, we have the inclusion

$$\sigma_p(T^{\odot}(t)) \subset \sigma_p(T^*(t)).$$

Since $T^{\odot}(t)$ is a restriction of $T^*(t)$.

Conversely, if $T^*(t) = \lambda x^*$ for some non-zero $x^* \in X^*$, then for all $\mu \in \varphi(A^*) = \varphi(A)$ we have $R(\mu, A^*)x^* \in D(A^*) \subset X^{\odot}$ and

$$T^{\odot}(t)R(\mu, A^*)x^* = R(\mu, A^*)T^*(t)x^* = \lambda R(\mu, A^*)x^*.$$

Hence, $R(\mu, A^*)x^*$ is an eigenvector of $T^{\odot}(t)$ with eigenvector λ. Hence, the proof is complete.

■

Theorem 3.3. *Assume $T(t)$ is a C_0-semigroup on a Banach space X, with generator $A \in \omega - OCP_n$ which is Toeplitz. Then,*

$$\left.\begin{array}{l}(i)\sigma_r(T(t)) \setminus \{0\} = \exp(t\sigma_r(A)).\\ (ii)Suppose \quad A \in \omega - OCP_n \quad is\ a\ closed\ linear\ operator\ on \quad X, and \sigma(A) = \sigma_r(A) \cup \sigma_a(A).\end{array}\right\} \quad (3.6)$$

Proof. By (3.6) above, we have

$$\sigma_r(T(t)) = \sigma_p(T^*(t)) = \sigma_p(T^{\odot}(t))$$

and

$$\sigma_r(A) = \sigma_p(A^*) = \sigma_p(A^{\odot}).$$

It now follows from Theorem 3.2 applied to the C_0-semigroup $T^{\odot}(t)$, which proves (i).

To proof (ii), assume that $\lambda \in \sigma(A) \setminus \sigma_r(A)$. Then $\lambda - A$ has dense range. If $\lambda - A$ is not injective, then $\lambda \in \sigma_p(A) \subset \sigma_a(A)$ and we are done. Suppose therefore that $\lambda - A$ is injective.

Assume for the moment that there exists a constant $c > 0$ such that

$$\|(\lambda - A)x\| \geq c\|x\| \quad \forall x \in D(A) \quad and \quad A \in \omega - OCP_n.$$

Then the range of $\lambda - A$ is closed. Indeed, if $y_n \to y$ with

$$y_n = (\lambda - A)x_n$$

then

$$\|x_n - x_m\| \leq c^{-1}\|(\lambda - A)(x_n - x_m)\| = \|y_n - y_m\|$$

so the sequence (x_n) is Cauchy, with limit x. The closedness of A implies that $x \in D(A)$, $A \in \omega - OCP_n$ and $(\lambda - A)x = y$, proving that y belongs to the range of $\lambda - A$. Thus, the range of $\lambda - A$ is closed. Since it is also dense, it foloows that it is of X. Since $\lambda - A$ is injective, the inverse $R_\lambda := (\lambda - A)^{-1}$ is well-defined as a closed linear operator on X whose domain is all of X. Hence R_λ is bounded by closed graph theorem. Thus, $\lambda - A$ is invertible, a contraction. It follows that a constant $c > 0$ as above does not exists. But then there is a sequence x_n of norm one vector, $x_n \in D(A)$ for all n, such that

$$\lim_{n\to\infty} (\lambda - A)x_n = 0,$$

which proves that $\lambda \in \sigma_a(A)$.

■

Theorem 3.4. *Let $A \in \omega - OCP_n$ Toeplitz be a closed linear operator on a Banach space X. Then the topological boundary $Q_\sigma(A)$ of the spectrum $\sigma(A)$ is contained in the approximate point spectrum $\sigma_a(A)$.*

Proof. Let $\lambda \in Q_\sigma(A)$ be fixed and let $(\lambda_n) \subset \varphi(A)$ be a sequence such that $\lambda_n \to \lambda$. It follows from uniform boundedness theorem and suppose $A \in \omega - OCP_n$ is a closed linear operator on X. Then for all $\lambda \in \varphi(A)$, we have

$$\|R(\lambda, A)\| \geq \frac{1}{dist(\lambda, \sigma(A))}.$$

Then there exists an $x \in X$ such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)x\| \to \infty.$$

Assume

$$x_n := \|R(\lambda_n, A)x\|^{-1} R(\lambda_n, A)x.$$

Then

$$\|x_n\| = 1$$

and

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = \lim_{n\to\infty} \|R(\lambda_n, A)x\|^{-1}. \cdot \|(\lambda_n - \lambda)R(\lambda_n, A)x - x\| = 0.$$

Hence, the proof is complete. ■

4. Conclusion

In this paper, it has been established that Toeplitz ω-order preserving partial contraction mapping (ω-OCP_n) generates results on spectral mapping theorem for point spectrum.

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