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Vertex semi-middle graph of a graph

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Abstract
In this communication, the vertex semi-middle graph of a graph $M_v(G)$ is introduced. We obtain a characterization of graphs whose $M_v(G)$ is planar, outerplanar and minimally non-outerplanar. Further, we obtain $M_v(G)$ is Eulerian, crossing number one and crossing number two.

Keywords
Crossing number, Middle graph, Planar, Semientire graph.

AMS Subject Classification
05C10, 05C45, 05C75.

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Contents

1. Introduction

By graph, we mean a finite, undirected graph without loops or multiple edges and planar. We refer the terminology of [1]. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G)UE(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it. This concept was introduced in [3] and was studied by Kulli and Patil [4, 5]. The *edgedegree* [6] of an edge $e=\{u,v\}$ is $d(u)+d(v)$. Degree of a region is the number of vertices lies on a region. Let v_1,v_2,v_3 be the pendant vertices of $K_{1,3}$. The graph $K_{1,3}(P_n)$ is obtained from $K_{1,3}$ by attaching one time to any one pendant vertex of $K_{1,3}$ as shown in Fig.1.
In the paper [7], defined the concept of vertex semientire block graph. We motivated this concept to define the vertex semi-middle graph of a graph. Let $G(V,E)$ be a planar graph with R regions. The vertex semi-middle graph of a graph G, denoted by $M_v(G)$ is a graph whose vertex set is $V(G)UE(G)UR(G)$ and two vertices of $M_v(G)$ are adjacent if and only if they corresponds to two adjacent edges of G or one corresponds to a vertex and other to an edge incident with it or one corresponds to a vertex other to a region in which vertex lies on the region.

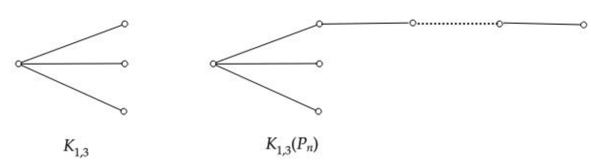

Fig. 1.

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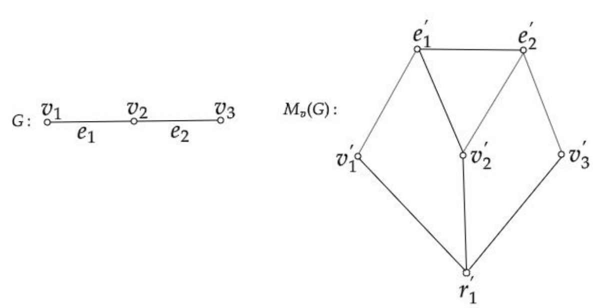

Fig. 2.

2. Preliminaries.

The following results will be useful in our results.

Theorem 2.1. *[1] A finite graph G is Eulerian if and only if all its vertex degree are even.*

Theorem 2.2. *[3] For any (p,q) graph G, middle graph of a graph $M(G)$ has $(p+q)$ vertices and $q+\sum_{i=1}^{q}\frac{1}{2}\{d(e_i)\}$ edges. Where $d(e_i)$ is the edgedegree of a edge e_i.*

Theorem 2.3. *[1] A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

3. Vertex semi-middle graph of a graph

We begin with some observations.

Observation 3.1. Every pendant vertex of G is also a pendant vertex of $M(G)$.

Observation 3.2. Let $e_i \in E(G)$ with edgedegree n then in $M_v(G)$, $deg(e_i^{'}) = n$.

Theorem 3.1. *For any graph G, $M_v(G)$ is always non-separable.*

Proof. We establish the following cases.
Case 1. Consider G be any tree. Let $v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}$ be the vertices of $M_v(G)$ corresponds to the vertices $v_1, v_2, v_3 .... v_n$ of G and $e_1^{'}, e_2^{'}, e_3^{'} .... e_{n-1}^{'}$ be the vertices of $M_v(G)$ corresponds to the edges $e_1, e_2, e_3 .... e_{n-1}$ of G. By the Observation 3.1, $M(G)$ contains the pendant vertices. Further, in $M_v(G)$ region vertex $r_1^{'}$ adjacent to the vertices $v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}$ without cut vertex. Clearly $M_v(G)$ is non-separable.
Case 2. Consider G be any cycle. Let $v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}$ be the vertices of $M_v(G)$ corresponds to the vertices $v_1, v_2, v_3 .... v_n$ of G and $e_1^{'}, e_2^{'}, e_3^{'} .... e_n^{'}$ be the vertices of $M_v(G)$ corresponds to the edges $e_1, e_2, e_3 .... e_n$ of G. In $M_v(G)$ region vertices $r_1^{'}, r_2^{'}$ adjacent to the vertices $v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}$ without cut vertex. Clearly $M_v(G)$ is non-separable. □

Proposition 3.1. Let $v_i \in V[G]$ and $\deg(v_i) = n$ then in $M_v(G)$, $\deg(v_i^{'}) = n + r_v$. where r_v is the number of regions in which vertex v lies.

Theorem 3.2. *For any graph G, p vertices, q edges and l regions then $M_v(G)$ has $(p+q+r)$ vertices and $q+\sum_{i=1}^{q}\frac{1}{2}\{d(e_i)\}$ $+\sum_{j=1}^{r} d(r_j)$ edges. Where $d(e_i)$ is the edgedegree of a edge e_i and $d(r_j)$ is the degree of a region r_j.*

Proof. By the definition of $M_v(G)$, the $V[M_v(G)]$=$V(G) U E(G) U R(G)$. Hence $V[M_v(G)]$ = $(p+q+r)$.
Further, by Theorem 2.2, $E[M(G)]$ = $q+\sum_{i=1}^{q}\frac{1}{2}\{d(e_i)\}$. The degree of a region is the sum of the number of vertices lies on the each region in G which is $\sum d(r_j)$. The number of edges in $M_v(G)$ is equal to the sum of edges in $M(G)$ and $\sum d(r_j)$. Hence $E[M_v(G)] = q+\sum_{i=1}^{q}\frac{1}{2}\{d(e_i)\}+\sum_{j=1}^{r} d(r_j)$. □

Theorem 3.3. *For any graph G, $M_v(G)$ is planar if and only if G is a path.*

Proof. Consider $M_v(G)$ is planar. We have the following cases.
Case 1. Suppose G is star $K_{1,3}$, $G = K_{1,3} : v_1, v_2, v_3, v_4$ and $deg(v_1) = 3$. Further $V[M_v(G)]$=$\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, e_1^{'}, e_2^{'}, e_3^{'}, r_1^{'}\}$. By the definition of middle graph $M(K_{1,3})$ is planar. Further in $M_v(G)$, the region vertex $r_1^{'}$ is adjacent to the vertices $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}$ of $M(G)$. $M_v(K_{1,3})$ is homeomorphic to K_5, by Theorem 2.3 which is non-planar, a contradiction.
Case 2. Consider G is a cycle, $G = C_n : v_1, v_2, v_3 .... v_n, n > 2$. Further, $V[M_v(G)] = \{v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'} .... e_n^{'}, r_1^{'}, r_2^{'}\}$. By definition of middle graph, $M(C_n)$ is planar. Further in $M_v(G)$, region vertices $r_1^{'}, r_2^{'}$ adjacent to the vertices $v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}$. Clearly, $M_v(G)$ is a non-planar. Which is a contradiction.
Conversely, suppose G is a path, $G = P_n : v_1, v_2, v_3 .... v_n, n > 1$. Further, $V[M_v(G)] = \{v_1^{'}, v_2^{'}, v_3^{'} .... v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'} .... e_{n-1}^{'}, r_1^{'}\}$. By the definition of middle graph, $M(G)$ is planar. For the $M_v(G)$ of a path P_n, $\{v_1^{'}e_1^{'}r_1^{'}v_2^{'}, v_2^{'}e_2^{'}r_1^{'}v_3^{'}, v_3^{'}e_3^{'}r_1^{'}v_4^{'} .... v_n^{'}e_n^{'}r_1^{'}v_{n+1}^{'}\} \in V[M_v(G)]$, in which each set $\{v_n^{'}e_n^{'}r_1^{'}v_{n+1}^{'}\}$ forms a planar graph. Hence $M_v(G)$ is planar. □

Proposition 3.2. The $M_v(G)$ of a G is 1-minimally non-outerplanar if and only if $G = P_3$.

Proposition 3.3. The $M_v(G)$ of a G is 2-minimally non-outerplanar if and only if $G = P_4$.

Theorem 3.4. *For any graph G, $M_v(G)$ is outerplanar if and only if G is P_2.*

Proof. Consider $G = P_2$, then $M_v(G) = C_4$. Since C_4 is outerplanar, hence $M_v(G)$ is outerplanar.
Conversely, suppose $M_v(G)$ is outerplanar and G is connected. We now prove that $G = P_2$. On the contrary, assume $G = P_3$. Then G has two edges e_1 and e_2. By Proposition 3.2 $M_v(G)$ =1-minimally non-outerplanar and hence $M_v(G)$ is not outerplanar, a contradiction.

□

Theorem 3.5. *The $M_v(G)$ of a conneccted graph G is k-minimally non-outerplanar $k \geq 1$ if and only if G is P_{k+2}.*

Proof. Suppose G is P_{k+2}, $k \geq 1$ to establish the result, we apply mathematical induction on k. Consider $k = 1$ then by Proposition 3.2, is 1-minimally non-outerplanar.
Consider the result is valid for $k = m$, therefore if G is P_{m+2} then $M_v(G)$ is m-minimally non-outerplanar.
Suppose $k = m+1$ then G is P_{m+3}. We now prove that $M_v(G)$ is $(m+1)$ minimally non-outerplanar.
Let $G = P_{m+3}$, and v_1 be an end vertex of G. Let $G_1 = G - v_1 = P_{m+2}$. By inductive hypothesis, $M_v(G_1)$ is m-minimally non-outerplanar.
Let $e_i = (v_i, v_j)$ be an endedge and r_i be the region of G_1. Then e_i is an endedge incident with the cutvertex v_i. The

vertices $e_i^{'}$, $r_i^{'}$ and $v_j^{'}$ in $M_v(G_1)$ are on the boundary of the exterior region on some cycle C. Now join the vertex v_1 to the vertex v_j of G_1 such that the resulting graph is G.
Let $e_j = (v_j, v_1)$ be an endedge and r_i be the region of G. The formation of $M_v(G)$ is an extension of $M_v(G_1)$ with additional vertices e_j and v_1 such that $e_j^{'}$ adjacent with $e_i^{'}$, $v_j^{'}$ and $v_1^{'}$. Similarly $r_i^{'}$ is adjacent with $v_i^{'}$, $v_j^{'}$ and $v_1^{'}$. Clearly $v_j^{'}$ is an inner vertex of $M_v(G)$, but it is not an inner vertex of $M_v(G_1)$. Thus $M_v(G)$ is $(m+1)$ - minimally non-outerplanar.
Conversely, assume $M_v(G)$ is k-minimally non - outerplanar, then by Theorem 3.3, $M_v(G)$ is planar. Thus G is a path.
Suppose G is a path. We obtain the following cases.
Case 1. Suppose $G = P_{k+1}$, $k \geq 1$. In particular if $k = 1$ then $G = P_2$ by the Theorem 3.4, $M_v(P_2)$ is outerplanar, a contradiction.
Case 2. Suppose $G = P_{k+3}$, in particular, if $k = 1$ then $G = P_4$ by the Proposition 3.3, $M_v(P_4)$ is 2-minimally non-outerplanar, a contradiction. Hence G is P_{k+2}. □

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Theorem 3.6. *For any graph G, $M_v(G)$ has crossing number one if and only if G is C_3 or G is $K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, where $n_1, n_2, n_3 \geq 0$.*

Proof. Suppose that $M_v(G)$ has crossing number one. Now, we deal with the subsequent cases.
Case 1. Suppose $G = C_4$: v_1, v_2, v_3, v_4. Further, $V[M_v(G)]$ = $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, r_1^{'}, r_2^{'}\}$. By the definition of middle graph, $M(G)$ is planar. Further in $M_v(G)$, $r_1^{'}$, $r_2^{'}$ are adjacent to $v_1^{'}$, $v_2^{'}$, $v_3^{'}$, $v_4^{'}$ and gives crossing number two, a contradiction.
Case 2. Suppose $G = K_{1,4}$: v_1, v_2, v_3, v_4, v_5 and $deg(v_i) = 4$. Further, $V[M_v(G)]$ =$\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, r_1^{'}\}$. By the definition of middle graph, $Cr[M(K_{1,4})] = 1$. Further in $M_v(G)$, $r_1^{'}$ adjacent to the $v_1^{'}$, $v_2^{'}$, $v_3^{'}$, $v_4^{'}$, $v_5^{'}$ of $M(G)$. Which gives a crossing number three, a contradiction.
Conversely, suppose $G = K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$: $v_1, v_2, v_3, v_4, v_{p_{n_1}}, v_{p_{n_2}}, v_{p_{n_3}}$ for $n_1, n_2, n_3 \geq 0$. Further, $V[M_v(G)]$=$\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v^{'}_{p_{n_1}}, v^{'}_{p_{n_2}}, v^{'}_{p_{n_3}}, e_1^{'}, e_2^{'}, e_3^{'}, e^{'}_{v_{n_1}}, e^{'}_{v_{n_2}}, e^{'}_{v_{n_3}}, r_1^{'}\}$. By the definition of middle graph, $M(G)$ is planar, without loss of generality we consider the inner vertices in $M(G)$ are $e_1^{'}$, $v_2^{'}$. In $M_v(G)$, the edges between $v_2^{'}$ and $r_1^{'}$ is crossing over the edges already drawn in $M(G)$. Hence $M_v(G)$ has crossing number one. □

Theorem 3.7. *For any graph G, $M_V(G)$ has crossing number two if and only if G is C_4 or $B_{2,2}$ or subdivision of any edge in $B_{2,2}$ or $C_3(P_{n_1})$, where $n_1 \geq 0$.*

Proof. Suppose that $M_v(G)$ has crossing number two. We now establish the subsequent cases.
Case 1. Suppose G=C_5: v_1, v_2, v_3, v_4, v_5. Further, $V[M_v(G)]$= $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, e_5^{'}, r_1^{'}, r_2^{'}\}$. By the definition of middle graph, $M(G)$ is planar. Further in $M_v(G)$, $r_1^{'}$, $r_2^{'}$ are adjacent to $v_1^{'}$, $v_2^{'}$, $v_3^{'}$, $v_4^{'}$, $v_5^{'}$ and gives crossing number four, a contradiction.
Case 2. Suppose that G be a $B_{2,3}$ or subdivision of any edge in $B_{2,3}$. Let $r_1^{'}$ be the region vertex of $M_v(G)$ corresponds to the region r_1 of G. By the definition of middle graph, $M(G)$ gives a crossing number one. Further in $M_v(G)$, $r_1^{'}$ is adjacent to the vertices of $M(G)$. Which gives crossing number more than two, a contradiction.
Case 3. Suppose $G = C_3(P_{n_1}, P_{n_2}) : v_1, v_2, v_3, v_{p_{n_1}}, v_{p_{n_2}}$ for $n_1, n_2 \geq 0$. Further, $V[M_v(G)] = \{v_1^{'}, v_2^{'}, v_3^{'}, v^{'}_{p_{n_1}}, v^{'}_{p_{n_2}}, e_1^{'}, e_2^{'}, e_3^{'}, e^{'}_{v_{n_1}}, e^{'}_{v_{n_2}}, r_1^{'}, r_2^{'}\}$. By the definition of middle graph, $M(G)$ is planar, without loss of generality we consider the inner vertices in $M(G)$ are $e_2^{'}$, $v_3^{'}$, $e_4^{'}$, $v_5^{'}$. In $M_v(G)$, the edges between $v_3^{'}$ and $r_1^{'}$, $v_3^{'}$ and $r_2^{'}$, $v_5^{'}$ and $r_2^{'}$ are crossing over the edges already drawn in $M(G)$. Hence, $M_v(G)$ has crossing number three, a contradiction.
Conversely, suppose $G = C_3(P_{n_1}) : v_1, v_2, v_3, v_{p_{n_1}}$ for $n_1 \geq 0$. Further, $V[M_v(G)] = \{v_1^{'}, v_2^{'}, v_3^{'}, v^{'}_{p_{n_1}}, e_1^{'}, e_2^{'}, e_3^{'}, e^{'}_{v_{n_1}}, r_1^{'}, r_2^{'}\}$. By the definition of middle graph, $M(G)$ is planar, without loss of generality we consider the inner vertices in $M(G)$ are $e_3^{'}, v_4^{'}$. In $M_v(G)$, the edges between $v_4^{'}$ and $r_2^{'}$ is crossing over the edges already drawn in $M(G)$. Also, the edges between $v_2^{'}$ and $r_2^{'}$ crossing over the edge between $v_3^{'}$ and $r_1^{'}$. Hence, $M_v(G)$ has crossing number two. □

Theorem 3.8. *For any graph G, $M_v(G)$ is Eulerian if and only if the following conditions holds.*
i) Edge degree of the edge is even.
ii) Degree of the region is even.
iii) The degree of the vertex v is even and it lies on even number of regions.
iv) The degree of the vertex v is odd and it lies on odd number of regions.

Proof. Suppose G is Eulerian. We have the following cases.
Case 1. Consider the edge with edge degree odd, by Observation 3.2, the degree of the corresponding vertex in $M_v(G)$ becomes odd. By the Theorem 2.1, $M_v(G)$ is non-eulerian, a contradiction.
Case 2. Suppose the degree of the region is odd, in G region r_1 contains odd number of vertices. By the definition, the degree of the corresponding vertex in $M_v(G)$ becomes odd. By Theorem 2.1, $M_v(G)$ is non-eulerian, a contradiction.
Case 3. Consider the vertex lie on odd regions with even degree. By Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes odd. By the Theorem 2.1, $M_v(G)$ is non-eulerian, a contradiction.
Case 4. Consider the vertex lies on even regions with odd degree. By Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes odd. By the Theorem 2.1, the $M_v(G)$ is non-eulerian, a contradiction.
Conversely, suppose above conditions holds.
Case 1. Consider the edge with even degree. By Observation

3.2, the degree of the corresponding vertex in $M_v(G)$ becomes even.

Case 2. Suppose the degree of the region is even. In G region r_1 contains even number of vertices. By definition, the degree of the corresponding vertex in $M_v(G)$ becomes even.

Case 3. Consider the vertex lie on even regions with even degree. By Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes even.

Case 4. Consider the vertex lies on odd regions with odd degree. By the Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes even.

From all the above cases, degree of every verex in $M_v(G)$ is even. Hence by Theorem 2.1, $M_v(G)$ is eulerian.

□

4. Conclusions

In this paper, we discuss the concept of vertex semi-middle graph of a graph. Further, we discuss the planarity, Eulerian, crossing number one and two of $M_v(G)$.

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