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Analysis of an $M/G/1$ retrial queue with second optional service and customer feedback, under Bernoulli vacation schedule

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Abstract

A single server retrial queueing system with second optional service under Bernoulli vacation schedule is investigated. The customer is permitted to balk if his service is not immediate upon arrival and allowed to join the orbit for repeating his service. Instead, if the server is free the customer's service is started immediately. Every customer is provided with a first phase of essential service followed by a second phase of optional service. After a service completion if the system is found to be empty then the server begins a vacation period. On the other hand if the system is not empty, the server chooses to either continue serving the customer with probability $(1-a)$ or goes on vacation with probability $a(0 \leq a \leq 1)$. After a service completion, a customer opts to either exit the system or chooses to join the orbit for repeating service. The joint generating functions of orbit size and server status are derived using supplementary variable technique. Some important performance measures have been derived and the effect of various parameters on the system performance has been analysed numerically. Stochastic decomposition law has been established in the absence of balking.

Keywords

Retrial queue, Balking, Second optional service, Bernoulli vacation, Feedback.

AMS Subject Classification

60K25, 90B22, 68M20.

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Contents

1. Introduction

Retrial queueing system is characterised by the phenomenon that an arriving customer who finds the server busy upon arrival may join the virtual group of blocked customers, called orbit and retry for service after a random amount of time. Study of retrial queues has gained more importance due to the potential real time applications in telephone services, computer and communication networks. In the so called classical retrial policy the interval between successive repeated customers are exponentially distributed with rate $n\theta$ when the number of customers in the retrial group i.e., orbit size is n, studied by Falin [5]. However, there is a second kind of discipline, in which intervals separating successive repeated attempts are independent of the orbit size. This policy is known as the constant retrial policy. The latter discipline was introduced by Fayolle [7], who investigated an $M/M/1$ retrial queue in which repeated customers form a queue and only the customers at the head of the orbit can request a service after an exponentially distributed retrial time. Farahmand [6] calls this discipline a retrial queue with FCFS orbit. Artalejo and Gomez-Corral [1] introduced the linear retrial policy by incorporating both the possibilities assuming that time intervals between successive repeated attempts are exponentially distributed random variables with parameter $\theta_n = \alpha[1-\delta_{n,0}]+n\theta$, when the orbit size is n. Yang and Templeton [19] and Artalejo [2] have done an extensive survey on retrial queueing systems. Gomez- Corral [10] has discussed a single server retrial queueing system with general retrial times.

Queueing systems with server vacations have been widely studied in the past. Miller [14] was the first to study an $M/G/1$ queueing system where the server is unavailable during some random length of time (termed as vacation). A vacation may refer to a maintenance activity, additional tasks etc. Survey on vacation queues can be found in Doshi [4] and Takagi [17]. Different vacation policies like the single vacation, multiple vacation, limited number of vacations, Bernoulli vacation and K optional vacations have been widely discussed by researchers. Keilson and Servi [11] introduced Bernoulli vacation schedule where, after a service completion if the system is empty the server begins a vacation. On the other hand, if the system is found to have waiting customers, the server opts to begin a vacation with probability $'a'$ or decides to provide service with probability $1-a$. At the end of a vacation period, service begins if there is a customer in the queue else, the server waits for an arrival.

C.M. Krishna and Y.H. Lee [13] have studied a two phase service queueing system. Some real time applications of two phase queues are found in distributed system control such as load balancing, routing, scheduling in a real-time environment and reconfiguration require two phase execution at a central server. That is, jobs come into the server which then probes the distributed system for status information. This is the first phase. The second phase consists of the server performing individual service on each job, example deciding which processor to allocate that job to, or how to reconfigure the system in reaction to the incoming jobs. Bharat Doshi [3] has analyzed a two phase queueing system with general service times. Krishna Kumar et. al. [12] introduced an $M/G/1$ retrial queueing system with two phase service and preemptive resume. Gautam Choudhury et. al. [9] have analysed an $M/G/1$ unreliable server queue with two phases of service and Bernoulli vacation schedule under randomised vacation policy.

Feedback is a phenomenon where each customer after service either immediately returns to the orbit for another service with probability f or leaves the system forever with probability $1-f$, where $0 \leq f < 1$. Feedback in the retrial queueing systems may occur in many practical situations, for example, telecommunication systems where messages turned out as errors at the destination are sent again for transmission. Feedback was introduced by Takacs [18]. A two phase queueing system with Bernoulli feedback has been studied by Gautam Choudhury and Madhuchanda Paul [9].

In this paper, we have studied the M/G/1 retrial queueing system with two phases of service of which the second phase is optional. Here the customer is permitted to balk if his service is not immediate upon arrival. The server goes for vacation under the Bernoulli vacation schedule. A typical application of our system can be found in Cloud computing. Cloud computing is the delivery of computing resources over the Internet. Cloud services allow individuals and businesses to use software and hardware that are managed by service providers from remote locations. To get the service of cloud the user has to be in queue until he is served. Each arriving user requests the Cloud Service Provider for access, if the server is available, the arriving user will start receiving service. Otherwise the user waits in virtual queues and continuously tries for access while one of them would be successful at the instant when the server is free. In the queueing context, the cloud users and service provider correspond to the customers and server respectively. The service provider may undertake some additional or maintenance activities which can be considered as a server vacation. When the service provider is busy or engaged, the user may be virtually waiting to try repeatedly and gain access of service or may quit which correspond to the orbit and balking behavior of a customer. A customer who has completed service may return back to the system for repeated service which corresponds to the feedback of a customer.

The rest of the paper is organised as follows. In section 2, we describe the mathematical model of the system under study and the stability condition. Section 3 deals with the steady state distribution of the queueing model and the probability generating functions of the system size and orbit size. In section 4 we have derived some performance measures of the system in the steady state. Section 5 deals with the stochastic decomposition of the model and finally section 6 exhibits the effect of various parameters on the system performance measures by means of graphs.

2. Model description and stability condition

We consider a single server retrial queueing system with two phases of service: A first phase of essential service (FES) provided for all customers and a second phase of optional service (SOS) provided only for the customers who opt for it. Customers arrive to the system according to a Poisson process with rate λ. When the primary arrival finds the server free, it's service starts immediately. On the other hand, if the server is busy or on vacation the primary arrival either balks the system with probability b or joins the pool of waiting customers called orbit with probability $1-b$. The customers in the orbit, independent of each other, continually make repeated attempts to receive service. The retrial time of the customer in the orbit is assumed to be generally distributed with distribution function $R(x)$ and Laplace Stieltjes transform (LST) $R^*(\theta)$. The conditional completion rate of retrial time is $\theta(x)dx = \dfrac{dR(x)}{1-R(x)}$.

A single server provides both the phases of service. The FES is compulsory for all the customers. On completing

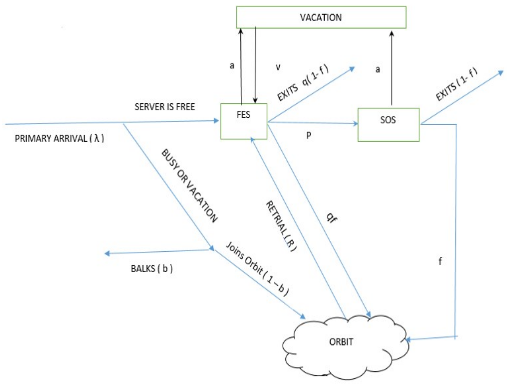

Figure 1. Two phase queue with balking, vacation and feedback

an FES, the customer chooses the SOS with probability p. On the other hand he may choose to either leave the system with probability $q(1-f)$ or join the orbit for repeated service as feedback customer with probability qf. A customer who opted for SOS may choose to leave the system with probability $1-f$ or may decide to join the orbit with probability f, upon completion of SOS. The service times are assumed to be generally distributed with distribution function $S_i(x)$ and LST $S_i^*(\theta)$, conditional completion rates $\mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)}, i = 1,2$. The server is assumed to go on a single Bernoulli vacation as in Keilson and Servi [11] discussed in Section 1. The vacation time V is assumed to be generally distributed with distribution function $V(x)$ and LST $V^*(\theta)$. The conditional completion rate of the vacation time is $\nu(x)dx = \frac{dV(x)}{1-V(x)}$. On return from vacation, the server starts providing service for a customer, if any or awaits a new arrival.

The state of the system at time t can be defined by the Markov process $\{C(t), X(t), \xi(t); t \geq 0\}$ where

$$C(t) = \begin{cases} 0, \text{ if server is free} \\ 1, \text{ if server is busy with FES} \\ 2, \text{ if server is busy with SOS} \\ 3, \text{ if server is on vacation,} \end{cases}$$

$X(t)$ corresponds to the number of customers in the orbit at time t and $\xi(t)$ represents the elapsed retrial time if $C(t) = 0$ and $X(t) > 0$, the elapsed service time in FES, SOS if $C(t) = 1,2$ and $X(t) \geq 0$ respectively and elapsed vacation time if $C(t) = 3$ and $X(t) \geq 0$.

Ergodicity condition:

We obtain the necessary and sufficient condition for the system to be stable. In the following theorem, we establish the ergodicity condition of the embedded Markov chain at departure or vacation completion epochs. Let $\{t_n; n \in N\}$ be the sequence of epochs at which either a service completion occurs or a vacation period ends. The sequence of random vectors $X_n = (C(t_n+), X(t_n+))$ forms a Markov chain, which is the embedded Markov chain for our queueing system. Its state space is $S = \{0,1,2,3\} \times \{0,1,2,3\cdots\} - (0,0)$.

Theorem 2.1. *Let X_n be the orbit length at the time of either n^{th} customer's departure or vacation completion, $n \geq 1$. Then $\{X_n; n \geq 1\}$ is ergodic if and only if $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f < R^*(\lambda)$.*

Proof. It is not difficult to see that $\{X_n; n \geq 1\}$ is an irreducible and aperiodic Markov chain. To prove ergodicity, we shall use Foster's Criterion: An irreducible and aperiodic Markov chain is ergodic if there exists a non- negative function $f(j), j \in N, \varepsilon > 0$ such that the mean drift, $\Psi_j = E[f(X_{n+1}) - f(X_n)/X_n = j]$ is finite for all $j \in N$, except perhaps a finite number j. In our case, we consider the function $f(j) = j$. Then we have,

$$\Psi_j = \begin{cases} \lambda(1-b)[E(S_1) + pE(S_2) + E(V)] + f, \text{for, } j = 0 \\ \lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f - R^*(\lambda), \\ \text{for } j = 1,2,3,\dots \end{cases}$$

Clearly, the inequality $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f < R^*(\lambda)$ is a sufficient condition for ergodicity. The same inequality is also necessary for ergodicity. As noted in Sennott et. al. [15], we can guarantee non-ergodicity of the Markov chain $\{X_n; n \geq 1\}$, if it satisfies Kaplan's condition namely $\Psi_j < \infty$ for all $j \geq 0$ and there exists $j_0 \in N$ such that $\Psi_j \geq 0$ for $j \geq j_0$. Notice that, in our case, Kaplan's condition is satisfied because $r_{ij} = 0$ for $j < i-1$ and $i > 0$, where $R = (r_{ij})$ is the one step transition matrix of $\{X_n; n \geq 1\}$.
Then, $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f \geq R^*(\lambda)$ implies the non-ergodicity of the Markov chain. □

Remark 2.2. *Since the arrival stream is a Poisson process, it can be shown from Burke's theorem that the steady state probabilities of $\{C(t), X(t); t \geq 0\}$ exist and is positive if and only if $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f < R^*(\lambda)$. From the mean drift $\Psi_j = \lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f, j \geq 1$, we have the reasonable conclusion that the term $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f$ has four components: new arrivals during the server providing the first phase of service $\lambda(1-b)E(S_1)$, new arrivals during the server providing the second phase of service $\lambda(1-b)pE(S_2)$, the arrivals while the server is on vacation $\lambda(1-b)aE(V)$ and finally the feedback customers f. Further, $R^*(\lambda)$ is the expected number of orbiting customers who enter service successfully, given that the previous service time leaves j customers in the orbit. For stability, we require that new customers arrive during service and vacation time more slowly than orbiting customers seeking service, at the commencement of service. That is, $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f < R^*(\lambda)$.*

3. Steady state distribution

For the Markov process $\{X(t); t \geq 0\}$, we define the probability $P_0(t) = P\{C(t) = 0, X(t) = 0\}$ and the probability densities,
$P_n(x,t) = P\{C(t) = 0, X(t) = n, x \leq \xi(t) < x+dx\}, \quad t \geq 0, \quad x \geq 0, \quad n \geq 1,$
$Q_{i,n}(x,t) = P\{C(t) = i, X(t) = n, x \leq \xi(t) < x+dx\}, \quad t \geq 0, \quad x \geq 0, \quad n \geq 0, \quad i = 1,2,$
$V(x,t) = P\{C(t) = 3, X(t) = n, x(t) \leq \xi(t) < x+dx\}, \quad t \geq 0, \quad x \geq 0, \quad n \geq 0.$
We assume that the stability condition, $\lambda(1-b)[E(S_1) + pE(S_2) + aE(V)] + f < R^*(\theta)$ is satisfied, so that the limiting probability $P_0 = lim_{t\to\infty} P_0(t)$ and the limiting probability densities given by,
$P_n(x) = lim_{t\to\infty} P_n(x,t), \quad x \geq 0, \quad n \geq 1,$
$Q_{i,n}(x) = lim_{t\to\infty} Q_{i,n}(x,t), \quad x \geq 0, \quad i = 1,2, \quad n \geq 0,$
$V_n(x) = lim_{t\to\infty} V_n(x,t), \quad x \geq 0, \quad n \geq 0$ exist.
Using the method of supplementary variable, we obtain the system of equations that govern the dynamics of the system behaviour as:

$$\lambda P_0 = \int_o^\infty V_0(x)\nu(x)dx, \tag{3.1}$$

$$\frac{d}{dx}P_n(x) + [\lambda + \theta(x)]P_n(x) = 0, \qquad n \geq 1, \tag{3.2}$$

$$\frac{d}{dx}Q_{i,0}(x) + [\lambda(1-b) + \mu_i(x)]Q_{i,0} = 0, \qquad i = 1,2, \tag{3.3}$$

$$\frac{d}{dx}Q_{i,n}(x) + [\lambda(1-b) + \mu_i(x)]Q_{i,n} = \lambda(1-b)Q_{i,n-1}(x), \qquad n \geq 1, i = 1,2, \tag{3.4}$$

$$\frac{d}{dx}V_0(x) + [\lambda(1-b) + \nu(x)]V_0(x) = 0, \tag{3.5}$$

$$\frac{d}{dx}V_n(x) + [\lambda(1-b) + \nu(x)]V_n(x) = \lambda(1-b)V_{n-1}(x), n \geq 1. \tag{3.6}$$

The steady state boundary conditions are,

$$\begin{aligned} P_n(0) = \int_o^\infty V_n(x)\nu(x)dx + (1-a)(1-f)\left\{q\int_o^\infty Q_{1,n}(x)\mu_1(x)dx + \int_o^\infty Q_{2,n}(x)\mu_2(x)dx\right\} \\ + (1-a)f\left\{q\int_o^\infty Q_{1,n-1}(x)\mu_1(x)dx + \int_o^\infty Q_{2,n-1}(x)\mu_2(x)dx\right\}, n \geq 1, \end{aligned} \tag{3.7}$$

$$Q_{1,0}(0) = \int_o^\infty P_1(x)\theta(x)dx + \lambda P_0, \tag{3.8}$$

$$Q_{1,n}(0) = \int_o^\infty P_{n+1}(x)\theta(x)dx + \lambda\int_o^\infty P_n(x)dx, n \geq 1, \tag{3.9}$$

$$Q_{2,n}(0) = p\int_o^\infty Q_{1,n}(x)\mu_1(x)dx, n \geq 1, \tag{3.10}$$

$$V_0(0) = q(1-f)\int_o^{\infty} Q_{1,0}(x)\mu_1(x)dx + (1-f)\int_o^{\infty} Q_{2,0}(x)\mu_2(x)dx, \tag{3.11}$$

$$V_n(0) = aq(1-f)\int_o^{\infty} Q_{1,n}(x)\mu_1(x)dx + a(1-f)\int_o^{\infty} Q_{2,n}(x)\mu_2(x)dx$$

$$+aqf\int_o^{\infty} Q_{1,n-1}(x)\mu_1(x)dx + af\int_o^{\infty} Q_{2,n-1}(x)\mu_2(x)dx, n \geq 1. \tag{3.12}$$

The normalization condition is

$$P_0 + \sum_{n=1}^{\infty}\int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty}\sum_{i=1}^{2}\int_0^{\infty} Q_{i,n}(x))dx + \sum_{n=0}^{\infty}\int_0^{\infty} V_n(x)dx = 1. \tag{3.13}$$

We define the Probability generating functions as,

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n; \qquad Q_i(x,z) = \sum_{n=0}^{\infty} Q_{i,n}(x)z^n, i = 1,2; \qquad \text{and} \qquad V(x,z) = \sum_{n=0}^{\infty} V_n(x)z^n.$$

Theorem 3.1. *If $\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]+f < R^*(\theta)$, then the steady state distributions of $\{X(t);t \geq 0\}$ are derived as,*

$$P(x,z) = \frac{z\lambda P_0\left\{V^*(\lambda(1-b))\{1-S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))](1-f+fz)[aV^*(\lambda(1-b)(1-z))+1-a]\} + (1-a)[1-V^*(\lambda(1-b)(1-z))]\right\} \times e^{-\lambda x}[1-R(x)]}{V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))] \times [aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}}, \tag{3.14}$$

$$Q_1(x,z) = \frac{\lambda P_0\left\{V^*(\lambda(1-b))(1-z)R^*(\lambda)+[z+(1-z)R^*(\lambda)](1-a)[1-V^*(\lambda(1-b)(1-z))]\right\} \times e^{-\lambda(1-b)(1-z)x}[1-S_1(x)]}{V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))] \times [aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}}, \tag{3.15}$$

$$Q_2(x,z) = \frac{p\lambda P_0\left\{V^*(\lambda(1-b))(1-z)R^*(\lambda)+[z+(1-z)R^*(\lambda)](1-a)[1-V^*(\lambda(1-b)(1-z))]\right\} \times S_1^*(\lambda(1-b)(1-z))e^{-\lambda(1-b)(1-z)x}[1-S_2(x)]}{V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))] \times [aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}}, \tag{3.16}$$

$$V(x,z) = \frac{\lambda P_0\left\{S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))](1-f+fz)\left\{a[1-z]V^*(\lambda(1-b))R^*(\lambda) + (1-a)[z+(1-z)R^*(\lambda)]\right\} - (1-a)z\right\} \times e^{-\lambda(1-b)(1-z)x}[1-V(x)]}{V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))] \times [aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}}. \tag{3.17}$$

Proof. Multiplying equations (3.2) - (3.6) by z^n and summing over all n, we get

$$\frac{\partial P(x,z)}{\partial x}+[\lambda+\theta(x)]P(x,z)=0, \quad x>0, \tag{3.18}$$

$$\frac{\partial Q_i(x,z)}{\partial x}+[\lambda(1-b)(1-z)+\mu_i(x)]Q_i(x,z)=0, \quad for \quad i=1,2, \quad x>0, \tag{3.19}$$

$$\frac{\partial V(x,z)}{\partial x}+[\lambda(1-b)(1-z)+\nu(x)]V(x,z)=0, \quad x>0, \tag{3.20}$$

Multiplying equations (3.7) - (3.12) by z^n and summing over all values of n, we obtain

$$P(0,z)=\int_o^\infty V(x,z)\nu(x)dx+(1-a)(1-f+fz)\left\{q\int_o^\infty Q_1(x,z)\mu_1(x)dx+\int_o^\infty Q_2(x,z)\mu_2(x)dx\right\} \\ -\lambda P_0-(1-a)V_0(0), \tag{3.21}$$

$$Q_1(0,z)=\frac{1}{z}\int_0^\infty P(x,z)\theta(x)dx+\lambda\int_0^\infty P(x,z)dx+\lambda P_0, \tag{3.22}$$

$$Q_2(0,z)=p\int_0^\infty Q_1(x,z)\mu_1(x)dx, \tag{3.23}$$

$$V(0,z)=a(1-f+fz)\left\{q\int_o^\infty Q_1(x,z)\mu_1(x)dx+\int_o^\infty Q_2(x,z)\mu_2(x)dx\right\}+[1-a]V_0(0). \tag{3.24}$$

Solving the partial differential equations (3.18) - (3.20) we get

$$P(x,z)=P(0,z)e^{-\lambda x}[1-R(x)], \tag{3.25}$$

$$Q_i(x,z)=Q_i(0,z)e^{-\lambda(1-b)(1-z)x}[1-S_i(x)], for \quad i=1,2, \tag{3.26}$$

$$V(x,z)=V(0,z)e^{-\lambda(1-b)(1-z)x}[1-V(x)]. \tag{3.27}$$

Using (3.26) in (3.23) we get

$$Q_2(0,z)=pQ_1(0,z)S_1^*(\lambda(1-b)(1-z)). \tag{3.28}$$

Substituting (3.25) in (3.22) we get

$$Q_1(0,z)=\frac{1}{z}\{\lambda P_0 z+[z+(1-z)R^*(\lambda)]P(0,z)\}. \tag{3.29}$$

Using (3.28) and (3.29) in (3.24) we get

$$V(0,z)=\frac{a[1-f+fz]}{z}\Big\{[\lambda P_0 z+P(0,z)[z+(1-z)R^*(\lambda)]][q \\ +pS_2^*(\lambda(1-b)(1-z))]S_1^*(\lambda(1-b)(1-z))\Big\}+(1-a)V_0(0). \tag{3.30}$$

Solving equation (3.1) we get

$$V_0(0) = \frac{\lambda P_0}{V^*(\lambda(1-b))}. \tag{3.31}$$

Combining (3.26) - (3.29) and using (3.31) in (3.21), after some algebraic manipulations, we get

$$P(0,z) = \frac{\begin{array}{r}\lambda P_0 z\Big\{V^*(\lambda(1-b))\Big\{1-S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]\\ \times(1-f+fz)[aV^*(\lambda(1-b)(1-z))+1-a]\Big\}\\ +(1-a)[1-V^*(\lambda(1-b)(1-z))]\Big\}\end{array}}{\begin{array}{r}V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}. \tag{3.32}$$

Using (3.32) in (3.28) - (3.30) and simplifying we get the required results (3.14) - (3.17). □

For the limiting probability generating functions $P(x,z)$, $Q_i(x,z), i=1,2$ and $V(x,z)$, we define the partial probability generating functions as, $P(z)=\int_0^\infty P(x,z)dx$; $Q_i(z)=\int_0^\infty Q_i(x,z)dx, i=1,2$ and $V(z)=\int_0^\infty V(x,z)dx$. Here, $P(z)$ is the probability generating function of the orbit size when the server is idle, $Q_i(z)$ is the probability generating function of the orbit size when the server is busy with phase i service, $i=1,2$ and $V(z)$ is the probability generating function when the server is on vacation. Define the probability generating function of the number of customers in the system as, $K(z)=P_0+P(z)+zQ_1(z)+zQ_2(z)+V(z)$ and the probability generating function of the number of customers in the orbit is defined as, $H(z)=P_0+P(z)+Q_1(z)+Q_2(z)+V(z)$, where P_0 is the probability that the server is idle in the system. The following theorem gives the main results of our model under consideration.

Theorem 3.2. *If $\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]+f<R^*(\lambda)$, then the partial generating functions are given as,*

$$P(z) = \frac{\begin{array}{r}zP_0\Big\{V^*(\lambda(1-b))\Big\{1-S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]\\ \times[1-f+fz][aV^*(\lambda(1-b)(1-z))+1-a]\Big\}\\ +[1-a][1-V^*(\lambda(1-b)(1-z))]\Big\}\times[1-R^*(\lambda)]\end{array}}{\begin{array}{r}V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]\\ \times[aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.33}$$

$$Q_1(z) = \frac{\begin{array}{r}P_0\Big\{V^*(\lambda(1-b))[1-z]R^*(\lambda)+[z+(1-z)R^*(\lambda)](1-a)\\ \times[1-V^*(\lambda(1-b)(1-z))]\Big\}e^{-\lambda(1-b)(1-z)x}[1-S_1(x)]\end{array}}{\begin{array}{r}(1-b)(1-z)V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.34}$$

$$Q_2(z) = \frac{\begin{array}{r}pP_0\Big\{V^*(\lambda(1-b))[1-z]R^*(\lambda)+[z+(1-z)R^*(\lambda)](1-a)\\ \times[1-V^*(\lambda(1-b)(1-z))]\Big\}S_1^*(\lambda(1-b)(1-z))e^{-\lambda(1-b)(1-z)x}[1-S_2(x)]\end{array}}{\begin{array}{r}(1-b)(1-z)V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.35}$$

$$V(z)=\frac{\begin{array}{c}P_0\Big\{S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))](1-f+fz)\\ \Big\{a[1-z]V^*(\lambda(1-b))R^*(\lambda)+(1-a)[z+(1-z)R^*(\lambda)]\Big\}-(1-a)z\Big\}\\ \times e^{-\lambda(1-b)(1-z)x}[1-V(x)]\end{array}}{\begin{array}{c}(1-b)(1-z)V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.36}$$

and the generating functions of the number of customers in the system and in the orbit are given as,

$$K(z)=\frac{\begin{array}{c}P_0\Bigg\{[q+pS_2^*(\lambda(1-b)(1-z))]S_1^*(\lambda(1-b)(1-z))\Big\{[1-V^*(\lambda(1-b)(1-z))]\\ \times\big\{(1-a)[z+(1-z)R^*(\lambda)](1-f)+aV^*(\lambda(1-b))R^*(\lambda)[1-f+fz]\big\}\\ +V^*(\lambda(1-b))R^*(\lambda)\{(1-b)[aV^*(\lambda(1-b)(1-z))+1-a][1-f+fz]-z\}\Big\}\\ +z[bV^*(\lambda(1-b))R^*(\lambda)-b(1-a)(1-V^*(\lambda(1-b)(1-z)))(1-R^*(\lambda))]\Bigg\}\end{array}}{\begin{array}{c}(1-b)V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.37}$$

$$H(z)=\frac{\begin{array}{c}P_0\Big\{V^*(\lambda(1-b))R^*(\lambda)[1-(1-b)z]+(1-a)[1-V^*(\lambda(1-b)(1-z))]\\ \times[(1-b)[1-R^*(\lambda)]z+R^*(\lambda)]+V^*(\lambda(1-b))S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]\\ [1-f+fz]R^*(\lambda)-b[aV^*(\lambda(1-b)(1-z))+1-a]V^*(\lambda(1-b))S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]\\ \times(1-f+fz)R^*(\lambda)-V^*(\lambda(1-b))S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))]R^*(\lambda)-[z+(1-z)R^*(\lambda)]\\ \times S_1^*(\lambda(1-b)(1-z))[q+pS_2^*(\lambda(1-b)(1-z))](1-a)(1-V^*(\lambda(1-b)(1-z)))f\Big\}\end{array}}{\begin{array}{c}(1-b)V^*(\lambda(1-b))\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-b)(1-z))\\ \times[q+pS_2^*(\lambda(1-b)(1-z))][aV^*(\lambda(1-b)(1-z))+1-a](1-f+fz)-z\}\end{array}}, \tag{3.38}$$

where,

$$P_0=\frac{V^*(\lambda(1-b))\{R^*(\lambda)-\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]-f\}}{V^*(\lambda(1-b))R^*(\lambda)[1+\lambda b(E(S_1)+pE(S_2)+aE(V))-f]+\lambda(1-a)E(V)[1-b(1-R^*(\lambda))-f]}. \tag{3.39}$$

Proof. Integrating equations (3.14) - (3.17) with respect to x from 0 to ∞, we get the results as in (3.33) - (3.36). Using equations (3.33) - (3.36) after considerable algebraic manipulations we get the probability generating function of the number of customers in the system $K(z)$ and that in the orbit $H(z)$ as in equations (3.37) and (3.38) respectively. Finally, the unknown P_0 is determined using the normalising condition, $P_0+P(1)+Q_1(1)+Q_2(1)+V(1)=1$. By setting $z=1$ in $K(z)$ or in $H(z)$ and applying L'Hospital's rule we get P_0 as in equation (3.39). □

4. Performance measures

In this section, we have derived some performance measures of the system in steady state. Let U be the steady state probability that the server is busy serving a customer i.e., server utilisation, I be the steady state probability that the server is idle during the retrial time or on vacation, L_s be the mean number of customers in the system and L_q be the mean number of customers in the orbit.

$U=Q_1(1)+Q_2(1)$,

$$=\frac{\lambda P_0\{[V^*(\lambda(1-b))R^*(\lambda)+\lambda(1-a)(1-b)E(V)][E(S_1)+pE(S_2)]\}}{V^*(\lambda(1-b))\{R^*(\lambda)-\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]-f\}}, \tag{4.1}$$

$$I=P_0+P(1)+V(1)$$

$$=\frac{P_0\{V^*(\lambda)R^*(\lambda)[1-f-\lambda(E(S_1)+pE(S_2))]+\lambda(1-a)(1-f)E(V)-\lambda 2(1-a)[E(S_1)+pE(S_2)]E(V)\}}{V^*(\lambda)\{R^*(\lambda)-\lambda[E(S_1)+pE(S_2)+aE(V)]-f\}}. \tag{4.2}$$

The average number of customers in the system under steady state condition is derived as,

$$
\begin{aligned}
L_s &= K'(1) \\
&= \frac{C_1}{2\Big\{V^*(\lambda(1-b))R^*(\lambda)[1-f+\lambda b[E(S_1)+pE(S_2)+aE(V)]]+\lambda(1-a)E(V)[1-f-b(1-R^*(\lambda))]\Big\}} \\
&\quad - \frac{D_1}{2\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]+f-R^*(\lambda)},
\end{aligned} \tag{4.3}
$$

where

$$
\begin{aligned}
C_1 = \Big\{&V^*(\lambda(1-b))R^*(\lambda)\{\lambda^2(1-b)b[E(S_1^2)+pE(S_2^2)+aE(V^2)+2pE(S_1)E(S_2)+2apE(S_2)E(V) \\
&+2aE(S_1)E(V)]+2\lambda\{[1-(1-b)f][E(S_1)+pE(S_2)]+abfE(V)\}\}+\lambda^2(1-b)(1-a) \\
&(\times)\{(1-f)[E(V^2)+2E(V)[E(S_1)+pE(S_2)]]-b(1-R^*(\lambda))E(V^2)\} \\
&+2\lambda(1-a)(1-R^*(\lambda))(1-f-b)E(V)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
D_1 = \{&\lambda^2(1-b)^2[E(S_1^2)+pE(S_2^2)+aE(V^2)+2pE(S_1)E(S_2)+2apE(S_2)E(V)+2aE(S_1)E(V)] \\
&+2\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)][1-R^*+f]+2[1-R^*(\lambda)f]\}.
\end{aligned}
$$

The average number of customers in the orbit under steady state condition is given by,

$$
\begin{aligned}
L_q &= H'(1) \\
&= \frac{C_2}{2\Big\{V^*(\lambda(1-b))R^*(\lambda)[1+\lambda b[E(S_1)+pE(S_2)+aE(V)]-f]+\lambda(1-a)[R^*(\lambda)+(1-b)(1-R^*(\lambda))+f]E(V)\Big\}} \\
&\quad - \frac{D_2}{2\Big\{\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)]+f-R^*(\lambda)\Big\}},
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
C_2 = &V^*(\lambda(1-b))R^*(\lambda)[\lambda^2(1-b)b[E(S_1^2)+pE(S_2^2)+aE(V^2)+2pE(S_1)E(S_2) \\
&+2apE(S_2)E(V)+2aE(S_1)E(V)]-2\lambda f(1-b)[E(S_1)+E(S_2)]+2\lambda abfE(V)] \\
&+\lambda^2(1-a)(1-b)\{[(1-b)(1-R*(\lambda))+R*(\lambda)-f]E(V^2)-2f[E(S_1)+E(S_2)]E(V)\} \\
&+2\lambda(1-a)(1-R(\lambda))[1-b-f]E(V),
\end{aligned}
$$

$$
\begin{aligned}
D_2 = \{&\lambda^2(1-b)^2[E(S_1^2)+pE(S_2^2)+aE(V^2)+2pE(S_1)E(S_2)+2apE(S_2)E(V)+2aE(S_1)E(V)] \\
&+2\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)][1+f-R^*(\lambda)]+2(1-R^*(\lambda))f\}.
\end{aligned}
$$

The probability of orbit being empty, P_{EO} is defined by, $P_{EO}=P_0+Q_{1,0}+Q_{2,0}+V_0$, where $Q_{i,0}$ for $i=1,2$ are the probabilities that the orbit is empty when the server is busy with FES and SOS respectively, V_0 is the probability that the orbit is empty when the server is on vacation and P_0 is the probability of an empty system. The above quantities are derived to be,

$$
Q_{1,0} = \frac{P_0[1-S_1^*(\lambda(1-b))]}{(1-b)(1-f)V^*(\lambda(1-b))S_1^*(\lambda(1-b))[q+pS_2^*(\lambda(1-b))]}, \tag{4.5}
$$

$$
Q_{2,0} = \frac{pP_0[1-S_2^*(\lambda(1-b))]}{(1-b)(1-f)V^*(\lambda(1-b))[q+pS_2^*(\lambda(1-b))]}, \tag{4.6}
$$

$$
V_0 = \frac{P_0[1-V^*(\lambda(1-b))]}{(1-b)V^*(\lambda(1-b))}, \tag{4.7}
$$

$$
P_{EO} = \frac{P_0\{1-[b(1-f)V^*(\lambda(1-b))+f]S_1^*(\lambda(1-b))[q+pS_2^*(\lambda(1-b))]\}}{(1-b)(1-f)V^*(\lambda(1-b))[q+pS_2^*(\lambda(1-b))]S_1^*(\lambda(1-b))}. \tag{4.8}
$$

5. Stochastic decomposition

Stochastic decomposition has been widely observed in $M/G/1$ type queues with generalized vacations(see Doshi [4], Takagi [17], Furhmann and Cooper [8]). The number of customers in the system at a random epoch is distributed as the sum of two independent random variables, one of which corresponds to the number of customers in the ordinary queueing system without vacations. The other random variable is usually interpreted as the number of customers in the system given that the server is on vacation. Stochastic decomposition has been observed to hold for some $M/G/1$ retrial queueing models (Artalejo [1]; Krishna Kumar et. al. [12]).

In our model, stochastic decomposition becomes inapplicable due to balking. But, in the absence of balking i.e., $b=0$, we have established the stochastic decomposition in an elegant way. Shanthi Kumar [16] has remarked that all cases of balking and reneging cannot be accommodated in the stochastic decomposition of $M/G/1$ queues. Our retrial queue with second optional service and Bernoulli vacation schedule can be viewed as an $M/G/1$ queue with generalised vacations in which the vacation begins at the end of each service period. Let $\Pi(z)$ be the probability generating function of the number of customers in the $M/G/1$ queueing system with second optional service and feedback in steady state at a random point in time, $\chi(z)$ be the probability generating function of the number of customers in the generalised vacation system at a random point in time given that the server is on vacation or idle, and K(z) be the probability generating function of the random variable being decomposed. Then the stochastic decomposition law can be Mathematically stated as,

$$K(z)=\Pi(z)\times\chi(z). \tag{5.1}$$

where

$$\Pi(z)=\frac{[1-z]S_1^*(\lambda(1-z))[q+pS_2^*(\lambda(1-z))][1-f-\lambda(E(S_1)+pE(S_2))]}{\{S_1^*(\lambda(1-z))[q+pS_2^*(\lambda(1-z))](1-f+fz)-z\}} \tag{5.2}$$

and

$$\chi(z)=\frac{P_0+P(z)+V(z)}{P_0+P(1)+V(1)}$$
$$=\frac{P_0(1-f)\{S_1^*(\lambda(1-z))[q+pS_2^*(\lambda(1-z))][1-f+fz]-z\}\{[1-z]V^*(\lambda)R^*(\lambda)+[z+(1-z)R^*(\lambda)](1-a)(1-V^*(\lambda(1-z)))\}}{V^*(\lambda)\{[z+(1-z)R^*(\lambda)]S_1^*(\lambda(1-z))[q+pS_2^*(\lambda(1-z))]\times(1-f+fz)[aV^*(\lambda(1-z))+1-a]-z\}\{1-f-\lambda(E(S_1)+pE(S_2))\}}. \tag{5.3}$$

Hence, it is seen that the stationary system size distribution of the $M/G/1$ queue with two phases of service and feedback under Bernoulli vacation is the convolution of the PGFs of two independent random variables: one of which is the stationary system size distribution of the $M/G/1$ queueing system with two phases of service and feedback and the other is the number of customers in the system during Bernoulli vacation schedule.

6. Numerical illustrations

In this section, we present the numerical analysis of the qualitative behaviour of the performance measures of the queueing system, by means of graphs. We study the effect of the system parameters arrival rate λ, retrial rate θ and the feedback probability f on the following performance measures:

- the probability P_0 that the system is empty
- the average number of customers L_s in the system
- server utilisation U

In Figures $(2)-(10)$, the service times of FES and SOS, the retrial from the orbit and vacation time are assumed

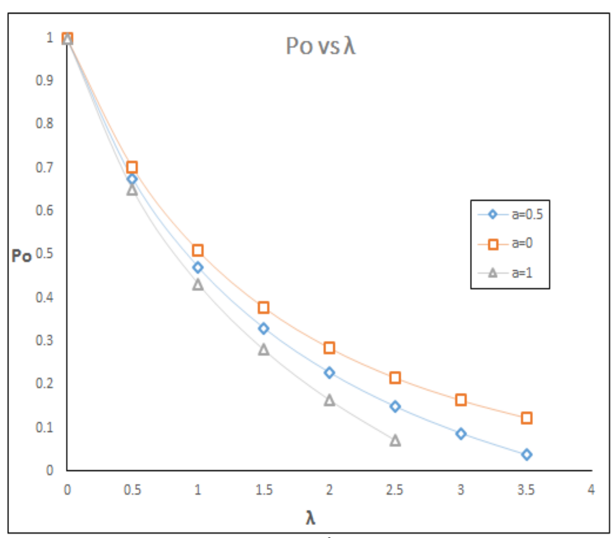

Figure 2. P_o versus λ for $a=0,0.5,1$

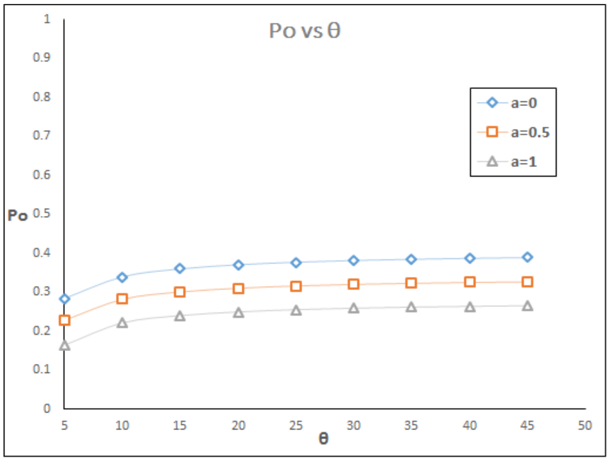

Figure 3. P_0 versus θ for $a = 0, 0.5, 1$

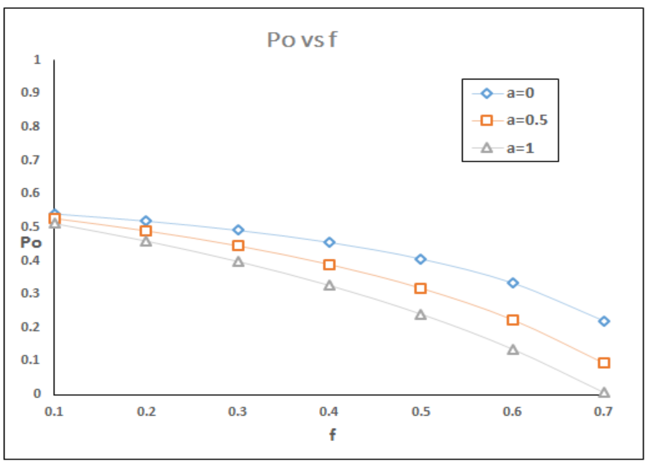

Figure 4. P_0 versus f for $a = 0, 0.5, 1$

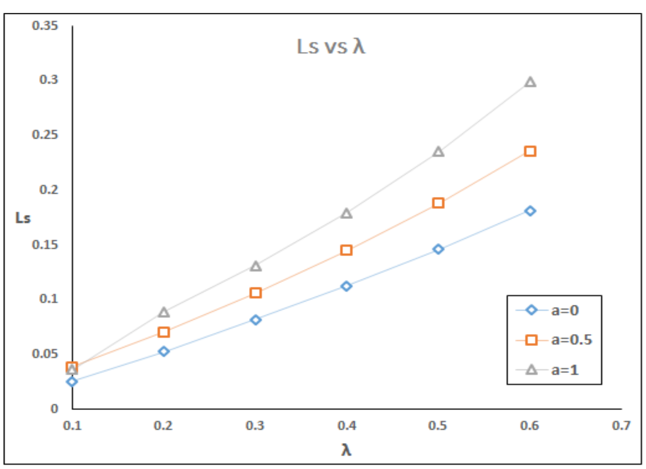

Figure 5. L_s versus λ for $a = 0, 0.5, 1$

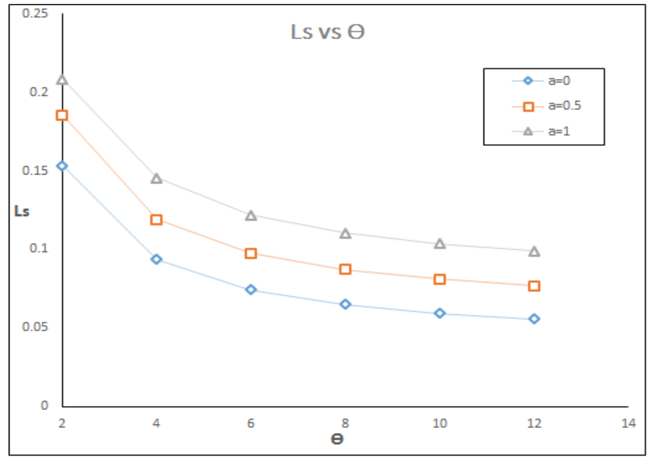

Figure 6. L_s versus θ for $a = 0, 0.5, 1$

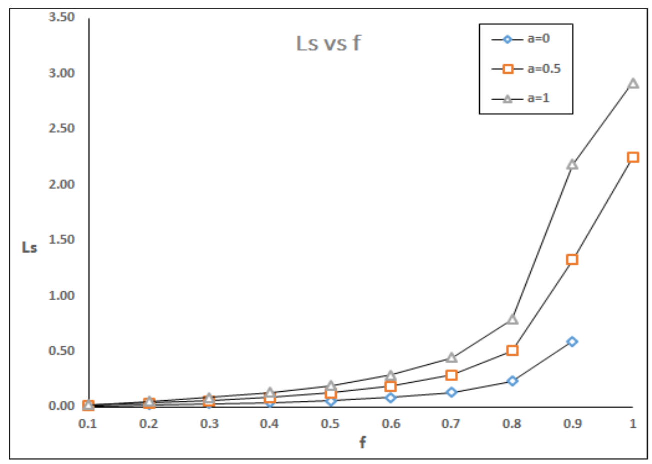

Figure 7. L_s versus f for $a = 0, 0.5, 1$

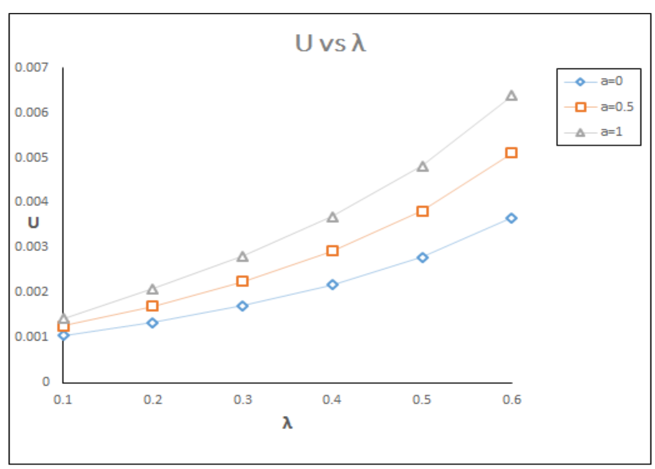

Figure 8. U versus λ for $a = 0, 0.5, 1$

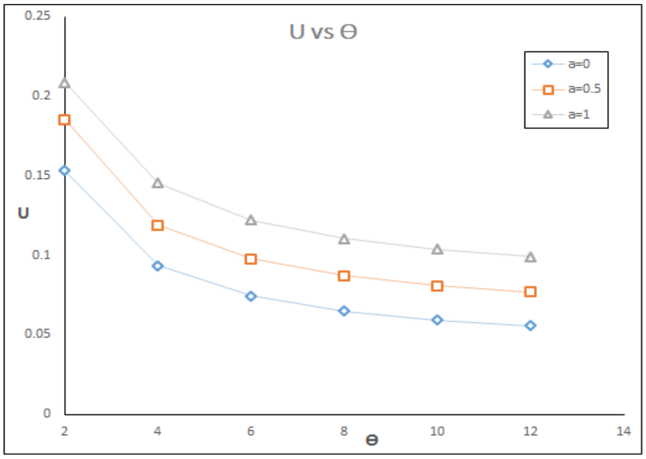

Figure 9. U versus θ for $a = 0, 0.5, 1$

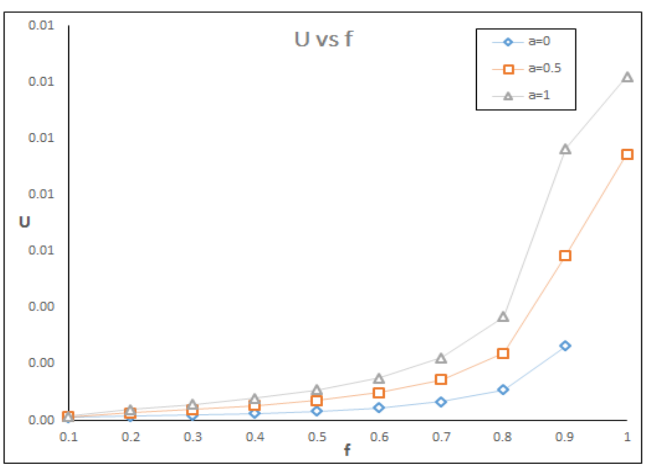

Figure 10. U versus f for $a = 0, 0.5, 1$

to follow exponential distribution with probability density functions $S_1(x) = \mu_1 e^{-\mu_1 x}$, $S_2(x) = \mu_2 e^{-\mu_2 x}$, $R(x) = \theta e^{-\theta x}$, $V(x) = \nu_1 e^{-\nu_1 x}$ respectively. Further, the parameters λ, μ_1, μ_2, ν, a, p, b and f are chosen satisfying the stability condition $\lambda(1-b)[E(S_1)+pE(S_2)+aE(V)] < R^*(\lambda)$.

In figures (2) - (4), the trend of the probability P_0 is plotted against λ, θ and f for the values of b=0.8, ν =2, λ=1, θ = 5, f=0.25, p=0.6. Figure (2) shows that P_0 decreases for increasing values of λ. Figure (3) shows that P_0 increases steadily for increasing values of the retrial rate θ. Figure (4) shows that P_0 decreases as expected for increasing values of the feedback probability f.

Figures (5) - (7) display the effect of the parameters on the system size L_s for the values of b=0.6, ν =3, θ = 5, λ=0.1, f=0.25, p=0.8. Figure (5), shows that L_s increases rapidly for increasing values of the arrival rate λ. Figure (6), displays the fact L_s decreases for increasing value of the retrial rate θ. Figure (7) shows that L_s increases steadily for increasing values of the feedback probability f.

Figures (8) - (10) exhibit the effect of the parameters on the server Utilisation for the values of b=0.6, ν =3, θ = 5, λ=0.1, f=0.25, p=0.8. Figure (8), shows that U increases for increasing values of the arrival rate λ. Figure (9), depicts the fact that U decreases steadily for increasing values of the retrial rate θ. Figure (10) displays the fact that U increases steadily for increasing values of the feedback probability f.

7. Conclusion

A single server retrial queue with second optional service, balking, Bernoulli vacation and feedback has been studied using supplementary variable technique. The joint generating functions of orbit size and server status are derived. Some system performance measures and orbit characteristics are also computed. Stochastic decomposition law has been established in the absence of balking. Numerical study of the parameters on the performance measures have been illustrated.

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