Cutting plane method in fuzzy environment

K.K. Mishra[1]*

Abstract
Linear programming is one of the most widely used decision making tools for solving real world problems. Real world situations are characterized imprecision rather than exactness. In this paper classical cutting plan method is extended to solve fuzzy number linear programming problem to find integer solution. The trapezoidal fuzzy numbers are defuzzificated by using linear ranking method proposed by Maleki [22]. This method is easy to apply. This method is explained with suitable numerical examples.

Keywords
Fuzzy linear programming problem, Trapezoidal fuzzy number, Ranking method.

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[1]*Department of Mathematics, Gandhi Institute for Technological Advancement, Bhubaneswar- 752054, Odisha, India.*
***Corresponding author**: [1] kkm.math1973@gmail.com*
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Contents

1. Introduction

In many practical problems the decision variables make sense only if they have integer values. It is often necessary to assign men, machines, and vehicles to activities in integer quantities. A linear programming problem in which some or all of the decision variables are restricted to assume non-negative integer values is commonly referred to as integer linear programming problem.

Integer programming problem is of particular importance in business and industry where, quite often, the fractional solutions are unrealistic because the units are not divisible. Hence, a new procedure has been developed in this direction for the case of LPP subjected to the additional restriction that the decision variables must have integer values. A systematic procedure for solving pure integer programming problem was first developed by R.E. Gomory in 1958. Later on, he extended the procedure to solve mixed integer programming problem named as cutting plane algorithm. In this method,

we first solve the IPP as ordinary LPP by ignoring the integer restriction and then introducing additional constraints one after the other to cut certain part of the solution space until an integer valued solution is obtained.

This paper is organized as follows. In section 2, some fundamental concepts on fuzzy number and the ranking functions are given. In section 3, Formulation of fuzzy number integer programming problem. In section 4, the procedure of the proposed fuzzy number Gomory's method is discussed. In section 5, two numerical examples are given and in section 6, the paper is concluded.

2. Fundamental of fuzzy set theory

The term fuzzy was proposed by Zadeh [20] in 1962. In [21], he published the paper entitled fuzzy sets in the year 1965. The fuzzy set theory is developed to improve the over-simplified model, thereby developing a more robust and flexible model in order to solve real-world complex systems involving human aspects. Furthermore, it helps the decision maker not only to consider the existing alternatives under given constraints (optimize a given system), but also to develop new alternatives (design a system). The fuzzy set theory has been applied in many fields, such as operations research, management science, control theory, artificial intelligence/expert system, human behavior, etc. In this study we will concentrate on fuzzy number mathematical programming problems. To do so, we will first introduce the required knowledge of the fuzzy set theory, fuzzy arithmetic and linear ranking function

in this chapter. For more details on this theory, we suggest the reader to refer [1–21].

Definition 2.1. *Let X is a collections of objects denoted generically by X, then a fuzzy set in X is a set of ordered pairs $\underline{A} = \{x, \mu_{\underline{A}}(x) | x \in X, \mu_{\underline{A}}(x) \in [0,1]\}$, where $\underline{A}$ is called the membership function.*

Definition 2.2. *The support of a fuzzy set $\underline{A}$ is the crisps set defined by $\underline{A} = \{x \in X | \mu_{\underline{A}} > 0\}$.*

Definition 2.3. *The core of a fuzzy set $\underline{A}$ is the crisp set of points $x \in X$ with $\mu_{\underline{A}} = 1$.*

Definition 2.4. *The boundary of a fuzzy set $\underline{A}$ are defined set of points $x \in X$ such that $0 < \mu_{\underline{A}} < 1$.*

It is evident that the boundary is defined as the region of the universal set containing elements that have non-zero membership but not complete membership. The Fig.1 illustrates the region.
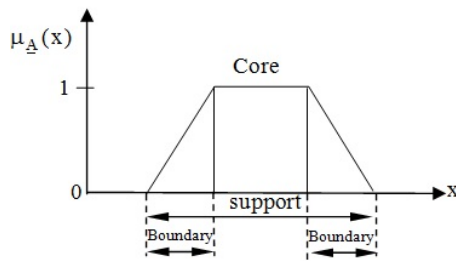

Figure 1

Definition 2.5. *A fuzzy set $\underline{A}$ is normal if and only there exists $x_i \in X$ such that $\mu_{\underline{A}}(x_i) = 1$.*

Definition 2.6. *A fuzzy set $\underline{A}$ is sub normal if $\mu_{\underline{A}} < 1$.*

Definition 2.7. *The α-cut of a fuzzy set $\underline{A}$ denoted by $[\underline{A}]_\alpha$ and is defined by $[\underline{A}]_\alpha = \{x \in X | \mu_{\underline{A}}(x) \geq \alpha\}$. If $\mu_{\underline{A}}(x) > \alpha$, then $[\underline{A}]_\alpha$ is called strong α-cut. It is clear that α-cut (strong α-cut) is a crisp set.*

Definition 2.8. *A fuzzy set $\underline{A}$ on X is convex if for any x_1, x_2 and $\lambda \in [0,1], \mu_{\underline{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\underline{A}}(x_1), \mu_{\underline{A}}(x_2)\}$.*

It is to be noted that a fuzzy set is convex if and only if its α–cut is convex.

Definition 2.9. *A fuzzy number is a fuzzy subset in universal set X which is both convex and normal.*

Definition 2.10. *A fuzzy number $\underline{A} = \{a, b, c, s\}$ is said to be a trapezoidal fuzzy number if its membership function is given*

by

$$
\mu_{\underline{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & q \leq x < b \\ 1, & b \leq x \leq c \\ \dfrac{(x-d)}{(c-d)}, & c < x \leq d \\ 0, & otherwise. \end{cases}
$$

Definition 2.11. *Let $\underline{A} = \{a^L, a^U, \alpha, \beta\}$ be the TrFN, where $(a^L - \alpha, a^U + \beta)$ is the support of $\underline{A}$ and $[a^L, a^U]$ is the core of $\underline{A}$.*

Arithmetic on Trapezoidal Fuzzy Numbers

Let $F(\mathbb{R})$ be set of all trapezoidal fuzzy numbers over the real line $\mathbb{R}$. The arithmetic operations on trapezoidal fuzzy numbers are defined as follows:

Let $\underline{a} = (a^L, a^U, \alpha, \beta)$ and $\underline{b} = (b^L, b^U, \gamma, \delta)\left(\frac{\pi}{2} - \theta\right)$ be two trapezoidal fuzzy numbers and $x \in \mathbb{R}$. We define

$$
x > 0, x \in \mathbb{R}; x\underline{a} = (xa^L, xa^U, x\alpha, x\beta),
$$
$$
x < 0, x \in \mathbb{R}; x\underline{a} = (xa^U, xa^L, -x\beta, -x\alpha),
$$
$$
\underline{a} + \underline{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \delta),
$$
$$
\underline{a} - \underline{b} = (a^L - b^L, a^U - b^U, \alpha + \gamma, \beta + \delta).
$$

Ranking Function

Ranking is one of the effective method for ordering fuzzy numbers. Various types of ranking function have been introduced and some have been used for solving linear programming problems with fuzzy parameters. An effective approach for ordering the element of $F(\mathbb{R})$ is to define a ranking function.

Let $\mathbb{R} : F(\mathbb{R}) \rightarrow \mathbb{R}$. We define order on $F(\mathbb{R})$ as follow:

(i) $\underline{a} \geq \underline{b}$ iff $\mathbb{R}(\underline{a}) \geq \mathbb{R}(\underline{b})$,

(ii) $\underline{a} > \underline{b}$ iff $\mathbb{R}(\underline{a}) > \mathbb{R}(\underline{b})$,

(iii) $\underline{a} = \underline{b}$ iff $\mathbb{R}(\underline{a}) = \mathbb{R}(\underline{b})$,

(iv) $\underline{a} \leq \underline{b}$ iff $\mathbb{R}(\underline{a}) \leq \mathbb{R}(\underline{b})$.

Here $\mathbb{R}$ is the ranking functions, such that $\mathbb{R}(k\underline{a} + \underline{b}) = k\mathbb{R}(\underline{a}) + \mathbb{R}(\underline{b})$.

Here, we introduce a linear ranking function that is similar to the ranking function adopted by Maleki [22]. For a trapezoidal fuzzy number $\underline{a} = (a^L, a^U, \alpha, \beta)$, we use ranking function as follows:

$$
\mathbb{R}(\underline{a}) = \int_0^1 (\inf \underline{a}_\alpha + \sup \underline{a}_\alpha) d\alpha
$$

which reduced to $\mathbb{R}(\underline{a}) = (a^L + a^U) + \frac{1}{2}(\beta - \alpha)$.

For any trapezoidal fuzzy numbers $\underline{a} = (a^L, a^U, \alpha, \beta)$ and $\underline{b} = (b^L, b^U, \gamma, \delta)$, we have $a \geq b$ if and only if $a^L + a^U + \frac{1}{2}(\beta - \alpha) \geq b^L + b^U + \frac{1}{2}(\delta - \gamma)$.

3. Fuzzy number integer programming problem

Fuzzy number integer programming problem is defined as Maximize $\underline{Z} = Cx$ subject to constraint $Ax \leq b, x \geq 0$ and integers.

4. Gomory's method

Gomory's cutting plane algorithm starts with solving fuzzy number linear programming problem by simplex method ignoring the restriction of integral values. In the optimum solution if all the variables have integer values, the current solution will be the desired optimum integer solution. Otherwise, we add a new constraint to the problem such that the new set of feasible solutions includes all the original feasible integer solution but does not include the optimum non-integer solution initially found. We then solve the revised problem using the simplex method and see if we can get an integer solution. If not we add another fractional cut and repeat the process until an integer solution is found. Since, we never eliminate any feasible integer solution from consideration when we add fractional cuts, the integer solution ultimately found must be optimum.

Algorithm:

Step 1: Let us consider FNIPP and convert the problem into maximization if it is in the minimization form. All the coefficients and constants should be integers.

Step 2: Ignoring the integer condition, find the optimum solution of the resulting FNLPP by simplex method.

Step 3: Test the integer condition of the optimum solution.

  (i) If all $X_{B_i} \geq 0$ and are integers, an optimum integer solution is obtained.

  (ii) If all $X_{B_i} \geq 0$ and at least one X_{B_i} is not an integer, then go to the next steps.

Step 4: Rewrite each X_{B_i} as $X_{B_i} = [X_{B_i}] + f_i$, where $[X_{B_i}]$ is the integral part of X_{B_i} and f_i is the positive fractional part of $X_{B_i}, 0 \leq f_i \leq 1$. Choose the largest fraction of X_{B_i}'s that is choose Max $\{f_i\}$. In case of a tie, select arbitrarily. Let Max $\{f_i\} = f_k$ corresponding to X_{B_k} (then k th row is called as a source row).

Step 5: Express each of the negative fractions if any, in the k th row (source row) of the optimum simplex Table as the sum of the negative integer and a non-negative fraction.

Step 6: Find the fractional cut constraint (Gomorian constraint or secondary constraint).

From the source row $\sum_{j=1}^{n} a_{k_j} x_j = X_{B_k}$; that is, $\sum_{j=1}^{n} ([a_{k_j}] + f_{k_j}) x_j = [X_{B_k}] + f_k$ in the form $\sum_{j=1}^{n} f_{k_j} x_j \geq f_k$ or, $-\sum_{j=1}^{n} f_{k_j} \leq -f_k$ or, $-\sum_{j=1}^{n} f_{k_j} + s_1 = -f_k$, where s_1 is the Gomorian slack.

Step 7: Add the fractional cut constraint obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step 8: Go to Step 3 and repeat the procedure until an optimum integer solution is obtained.

5. Numerical Examples

To illustrate the proposed algorithm the followings fuzzy number integer programming problems are solved.

Example 5.1. *Solve the fuzzy number integer programming problem:*

Max $\underline{Z} = (5, 8, 2, 5)X_1 + (6, 10, 2, 6)X_2$ subject to $2X_1 + 3X_2 \leq 6; 5X_1 + 4X_2 \leq 10; X_1, X_2 \geq 0$ and are integer.

Solution: Ignoring the integer restriction and applying simplex method for FNLPP the optimal simplex table is:

		$\underline{C}_j$	(5,8,2,5)	(6,10,2,6)	(0,0,0,0)	(0,0,0,0)
$\underline{C}_B$	Basis	X_B	x_1	x_2	x_3	x_4
(6,10,2,6)	X_2	10/7	0	1	5/7	-2/7
(5,8,2,5)	X_1	6/7	1	0	-4/7	3/7
	$\underline{Z}_j$	$\left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7}\right)$	(5,8,2,5)	(6,10,2,6)	$\left(\frac{10}{7}, \frac{18}{7}, \frac{2}{7}, \frac{10}{7}\right)$	$\left(\frac{3}{7}, \frac{4}{7}, \frac{2}{7}, \frac{3}{7}\right)$
		$\underline{Z}_j - \underline{C}_j$	(-3,3,7,7)	(-4,4,8,8)	$\left(\frac{10}{7}, \frac{18}{7}, \frac{2}{7}, \frac{10}{7}\right)$	$\left(\frac{3}{7}, \frac{4}{7}, \frac{2}{7}, \frac{3}{7}\right)$
$\Re(\underline{Z}_j) = \frac{267}{7}$		$\Re(\underline{Z}_j - \underline{C}_j)$	0	0	$\frac{32}{7}$	$\frac{15}{14}$

Since all $\mathbb{R}(\underline{Z}_j - \underline{C}_j) \geq 0$ the current solution is optimum. The optimum solution is $X_1 = \frac{6}{7}, X_2 = \frac{10}{7}$ and maximum $\max \underline{Z} = \left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7}\right), \mathbb{R}(\underline{Z}) = \frac{267}{7}$.

But the solution obtained is not an integer solution.

Construction of Gomory's constraint :

Now,

$$\frac{10}{7} = 1 + \frac{3}{7} \quad \left(f_1 = \frac{3}{7}\right)$$
$$\frac{6}{7} = 0 + \frac{6}{7} \quad \left(f_2 = \frac{6}{7}\right)$$

and maximum of $\left\{\frac{3}{7}, \frac{6}{7}\right\} = \frac{6}{7}$.

Hence we choose 2nd row (X_1-row) for constructing the Gomory's constraints.

Now X_1 equation is

$$0 + \frac{6}{7} = 1X_1 + 0X_2 + \left(-1 + \frac{3}{7}\right)X_3 + \frac{3}{7}X_4.$$

Hence, the Gomory's constraint is $-\frac{3}{7}X_3 - \frac{3}{7}X_4 + X_5 = -\frac{6}{7}$, where, X_5 is the Gomory's slack variable. Hence $X_5 = -\frac{6}{7}$. As the solution is infeasible, Let us apply dual simplex method after including the Gomory's constraint equation in the above table.

		$\underline{C}_j$	(5,8,2,5)	(6,10,2,6)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
$\underline{C}_B$	Basis	X_B	x_1	x_2	x_3	x_4	x_5
(6,10,2,6)	X_2	10/7	0	1	5/7	-2/7	0
(5,8,2,5)	X_1	6/7	1	0	-4/7	3/7	0
(0,0,0,0)	$\bar{x}_5$	-6/7	0	0	-3/7	-3/7	1
	$\underline{Z}_j$	$\left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7}\right)$	(5,8,2,5)	(6,10,2,6)	$\left(\frac{10}{7}, \frac{18}{7}, \frac{2}{7}, \frac{10}{7}\right)$	$\left(\frac{3}{7}, \frac{4}{7}, \frac{2}{7}, \frac{3}{7}\right)$	
		$\underline{Z}_j - \underline{C}_j$	(-3,3,7,7)	(-4,4,8,8)	$\left(\frac{10}{7}, \frac{18}{7}, \frac{2}{7}, \frac{10}{7}\right)$	$\left(\frac{3}{7}, \frac{4}{7}, \frac{2}{7}, \frac{3}{7}\right)$	
$\Re(\underline{Z}_j) = \frac{267}{7}$		$\Re(\underline{Z}_j - \underline{C}_j)$	0	0	$\frac{32}{7}$	$\frac{15}{14}$	

Since maximum of $\left\{\frac{36/7}{-3/7}, \frac{8/7}{-3/7}\right\} = \left\{-12, -\frac{8}{3}\right\} = -\frac{8}{3}$.

Corresponding to X_4:

Introduce x_4 and drop x_5 from the basis

$\underline{C}_B$	Basis	$\underline{C}_j$	(5,8,2,5)	(6,10,2,6)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
		X_B	x_1	x_2	x_3	x_4	x_5
(6,10,2,6)	X_2	2	0	1	1	0	-2/3
(5,8,2,5)	X_1	0	1	0	-1	0	1
(0,0,0,0)	x_4	2	0	0	1	1	-7/3
	$\underline{Z}_j$	(12,20,4,12)	(5,8,2,5)	(6,10,2,6)	(-2,5,7,8)	(0,0,0,0)	(-5/3,4,6,8)
		$\underline{Z}_j - \underline{C}_j$	(-3,3,7,7)	(-4,4,8,8)	(-2,5,7,8)	(0,0,0,0)	(-5/3,4,6,8)
$\mathfrak{R}(\underline{Z}_j)=36$		$\mathfrak{R}(\underline{Z}_j-\underline{C}_j)$	0	0	3.5	0	10/3

Since all $X_B \geq 0$ and integer and $\mathbb{R}(\underline{Z}_j - \underline{C}_j) \geq 0$ the current solution is optimum and the optimum integer solution is $X_1 = 0, X_2 = 2, X_3 = 0, X_4 = 2, X_5 = 0$ and Max $\underline{Z} = (12, 20, 4, 12), \mathbb{R}(\underline{Z}) = 36$.

Example 5.2. *Max* $\underline{Z} = (2,5,1,2)X_1 + (8,9,2,5)X_2$ *subject to* $X_1 + 2X_2 \leq 6; -X_1 + X_2 \leq 2; 2X_1 + X_2 \leq 6, X_1, X_2 \geq 0$ *and are integer.*

Solution: Ignoring the integer restriction and applying simplex method for FNLPP the optimal simplex table is: Since

$\underline{C}_B$	Basis	$\underline{C}_j$	(2,5,1,2)	(8,9,2,5)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
		X_B	x_1	x_2	x_3	x_4	x_5
(2,5,1,2)	X_1	2/3	1	0	1/3	-2/3	0
(8,9,2,5)	X_2	8/3	0	1	1/3	1/3	0
(0,0,0,0)	x_5	2	0	0	0	1	1
	$\underline{Z}_j$	$(\frac{68}{3},\frac{82}{3},\frac{18}{3},\frac{44}{3})$	(2,5,1,2)	(8,9,2,5)	$(\frac{10}{3},\frac{14}{3},1,\frac{7}{3})$	$(\frac{-2}{3},\frac{5}{3},2,\frac{7}{3})$	(0,0,0,0)
		$\underline{Z}_j - \underline{C}_j$	(-3,3,7,7)	(-1,1,7,7)	$(\frac{10}{3},\frac{14}{3},1,\frac{7}{3})$	$(\frac{-2}{3},\frac{5}{3},2,\frac{7}{3})$	(0,0,0,0)
$\mathfrak{R}(\underline{Z}_j)=\frac{163}{3}$		$\mathfrak{R}(\underline{Z}_j-\underline{C}_j)$	0	0	$\frac{26}{3}$	$\frac{7}{6}$	0

all $\mathbb{R}(\underline{Z}_j - \underline{C}_j) \geq 0$ the current solution is optimum. The optimum solution is $X_1 = \frac{2}{3}, X_2 = \frac{8}{3}$ and maximum $\max \underline{Z} = \left(\frac{68}{3}, \frac{82}{3}, \frac{18}{3}, \frac{44}{3}\right), \mathbb{R}(\underline{Z}_j) = \frac{163}{3}$.

But the solution obtained is not an integer solution.

Construction of Gomory's constraint :

Now,

$$\frac{2}{3} = 0 + \frac{2}{3} \quad \left(f_1 = \frac{2}{3}\right)$$
$$\frac{8}{3} = 2 + \frac{2}{3} \quad \left(f_2 = \frac{2}{3}\right)$$

and maximum of $\left\{\frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3}$.

Hence we choose first row (X_1-row) for constructing the Gomory's constraints.

Now X_1 equation is

$$0 + \frac{2}{3} = 1X_1 + 0X_2 + \frac{1}{3}X_3 + \left(-1 + \frac{1}{3}\right)X_4.$$

Hence, the Gomory's constraint is $-\frac{1}{3}X_3 - \frac{3}{7}X_4 + X_6 = -\frac{2}{3}$, where, X_6 is the Gomory's slack variable. Hence $X_6 = -\frac{2}{3}$. As the solution is infeasible, Let us apply dual simplex method after including the Gomory's constraint equation in the above table.

$\underline{C}_B$	Basis	$\underline{C}_j$	(2,5,1,2)	(8,9,2,5)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
		X_B	x_1	x_2	x_3	x_4	x_5	x_6
(2,5,1,2)	X_1	2/3	1	0	1/3	-2/3	0	0
(8,9,2,5)	X_2	8/3	0	1	1/3	1/3	0	0
(0,0,0,0)	x_5	2	0	0	0	1	1	0
(0,0,0,0)	$\vec{x}_6$	-2/3	0	0	-1/3	(-1/3)	0	1
	$\underline{Z}_j$	$(\frac{68}{3},\frac{82}{3},\frac{18}{3},\frac{44}{3})$	(2,5,1,2)	(8,9,2,5)	$(\frac{10}{3},\frac{14}{3},1,\frac{7}{3})$	$(\frac{-2}{3},\frac{5}{3},2,\frac{7}{3})$	(0,0,0,0)	(0,0,0,0)
		$\underline{Z}_j - \underline{c}_j$	(-3,3,7,7)	(-1,1,7,7)	$(\frac{10}{3},\frac{14}{3},1,\frac{7}{3})$	$(\frac{-2}{3},\frac{5}{3},2,\frac{7}{3})$	(0,0,0,0)	(0,0,0,0)
		$\mathfrak{R}(\underline{Z}_j-\underline{C}_j)$	0	0	$\frac{26}{3}$	$\frac{7}{6}$	0	0

Since maximum of $\left\{\frac{28/3}{-1/3}, \frac{4/3}{-1/3}\right\} = \{-28, -4\} = -4$. Corresponding to X_4:

Introducing x_4 and drop x_6 from the basis.

$\underline{C}_B$	Basis	$\underline{C}_j$	(2,5,1,2)	(8,9,2,5)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
		X_B	x_1	x_2	x_3	x_4	x_5	x_6
(2,5,1,2)	X_1	2	1	0	1	0	0	-2
(8,9,2,5)	X_2	2	0	1	0	0	0	3
(0,0,0,0)	x_5	0	0	0	-1	0	1	3
(0,0,0,0)	x_4	2	0	0	1	1	0	-3
	$\underline{Z}_j$	(20,28,6,14)	(2,5,1,2)	(8,9,2,5)	(2,5,1,2)	(0,0,0,0)	(0,0,0,0)	(14,23,10,17)
		$\underline{Z}_j - \underline{c}_j$	(-3,3,3,3)	(-1,1,7,7)	(2,5,1,2)	(0,0,0,0)	(0,0,0,0)	(14,23,10,17)
$\mathfrak{R}(\underline{Z}_j)=52$		$\mathfrak{R}(\underline{Z}_j-\underline{C}_j)$	0	0	7.5	0	0	40.5

Since all $X_B \geq 0$ and integer and $\mathbb{R}(\underline{Z}_j - \underline{C}_j) \geq 0$ the current solution is optimum and the optimum integer solution is $X_1 = 2, X_2 = 2$ and Max $\underline{Z} = (20, 28, 6, 14), \mathbb{R}(\underline{Z}) = 52$.

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