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Existence results for nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space

M. Mallika Arjunan[1]

Abstract

The main aim of this manuscript is to analyze the existence and uniqueness of PC-mild solution of nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space. Based on the Banach contraction principle, we develop the main results. An example is ultimately given for the theoretical results to be justified.

Keywords

Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification

34K30, 35R12, 26A33.

[1] *Department of Mathematics, Vel Tech High Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Avadi-600062, Tamil Nadu, India.*

*__Corresponding author__: [1]arjunphd07@yahoo.co.in

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1. Introduction

Differential systems with fractional order have acquired a great deal of consideration from mathematicians and researchers because of its broad application in different fields of science and designing. Truth be told, one may discover various applications in visco-versatility, seismology, electromagnetism, etc. More subtleties and applications can be found in the monograph by Kilbas et al. [5] and the papers of [2, 3, 6, 7].

Motivated by [1, 4], in this paper, we consider a class of nonlocal fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$
\begin{aligned}
{}^{C}D^{\alpha}x(t) &= f\left(t, x(t), K_1x(t), K_2x(t)\right), \\
&\qquad t \in \left(s_i, t_{i+1}\right], i = 0, 1, 2, \ldots, m \\
x(t) &= g_i(t, x(t)), \quad t \in \left(t_i, s_i\right], i = 1, 2, \ldots, m \\
x(0) + h(x) &= x_0,
\end{aligned}
\tag{1.1}
$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \leq 1, t \in [0, T]; x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C\left(\left(t_i, s_i\right] \times \mathbb{R}; \mathbb{R}\right)$, $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a nonlinear function, $h : PC(J, \mathbb{R}) \to \mathbb{R}$ and the functions K_1 and K_2 are defined by

$$
K_1x(t) = \int_0^t u(t, s, x(s))ds \quad \text{and} \quad K_2x(t) = \int_0^T \widetilde{u}(t, s, x(s))ds,
$$

$u, \widetilde{u} : \Delta \times \mathbb{R} \to \mathbb{R}$, where $\Delta = \{(x, s) : 0 \leq s \leq x \leq \tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.

2. Preliminaries

Let us set $J=[0,T], J_0=[0,t_1],\ J_1=(t_1,t_2],\ldots,\ J_{m-1}=(t_{m-1},t_m], J_m=(t_m,t_{m+1}]$ and introduce the space $PC(J,X):=\{u:J\to X \mid u\in C(J_k,X), k=0,1,2,\ldots,m,$ and there exist $u(t_k^+)$ and $u(t_k^-), k=1,2,\ldots,m,$ with $u(t_k^-)=u(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC}=\sup\{\|u(t)\|:t\in J\}$.

Let us recall the following well-known definitions [5].

Definition 2.1. *A real function $f(t)$ is said to be in the space $C_\alpha, \alpha\in\mathbb{R}$, if there exists a real number $p>\alpha$, such that $f(t)=t^p g(t)$, where $g\in C[0,\infty)$ and it is said to be in the space C_α^n if and only if $f^{(n)}\in C_\alpha, n\in\mathbb{N}$.*

Definition 2.2. *The Riemann-Liouville derivative of order $\alpha>0$ for a function $f\in C_\alpha^n, n\in\mathbb{N}$, is defined as*

$$\begin{aligned}D_t^\alpha f(t) &= D^n D^{\alpha-n} f(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1} f(s)ds, t>0, n-1<\alpha<n.\end{aligned}$$

Definition 2.3. *The Caputo fractional derivative of order $\alpha>0$ for a function $f\in C_\alpha^n, n\in\mathbb{N}$, is defined as*

$$\begin{aligned}CD_t^\alpha f(t) &= D^{\alpha-n} D^n f(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^n(s)ds, t>0, n-1<\alpha<n.\end{aligned}$$

Lemma 2.4. *Let $f:J\to\mathbb{R}$ be a continuous function. A function $x\in C(J,\mathbb{R})$ is a solution of the fractional integral equation*

$$x(t)=x_b-\frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} f(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$$

if and only if $x(t)$ is a solution of the following fractional Cauchy problem:

$$\begin{aligned}&CD^\alpha x(t)=f(t), \quad t\in J\\ &x(b)=x_b, \quad b>0\end{aligned}$$

In view of the Lemma 2.1, we define the PC-mild solution for the given system (1.1).

Definition 2.5. *A function $x\in PC(J,\mathbb{R})$ is a mild solution of the problem (1.1) if $x(0)+h(x)=x_0, x(t)=g_i(t,x(t)), t\in(t_i,s_i], i=1,2,\ldots,m$ and*

$$\begin{aligned}x(t)=&x_0-h(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\\ &f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \quad t\in[0,t_1]\end{aligned}$$

and

$$\begin{aligned}x(t)=g_i(s_i,x(s_i))&-\frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-t)^{\alpha-1}\\ &f(s,x(s),K_1(x(s)),K_2(x(s)))ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds,\end{aligned}$$

where, $t\in(s_i,t_{i+1}]$.

3. Existence and Uniqueness Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

To establish our results on the existence of solutions, we consider the following hypotheses:

(A1) Let $\Omega\subset \mathrm{Dom}(f)$ be an open subset of $J\times\mathbb{R}^3$. For each $(t,x_1,y_1,z_1)\in\Omega$ there is a neighbourhood $V_1\subset\Omega$ of (t,x_1,y_1,z_1) such that the nonlinear map

$$f:J\times\mathbb{R}^3\to\mathbb{R}$$

satisfies the following condition

$$\begin{aligned}&|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_z)|\\ &\quad\le L_f\left[|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right]\end{aligned}$$

$\forall (t,x_1,y_1,z_1), (t,x_2,y_2,z_2)\in V_1$ and

$$L_f=L_f(t,x_1,y_1,z_1)>0,$$

a constant.

(A2) The functions $u,\widetilde{u}:\Delta\times\mathbb{R}\to\mathbb{R}$ are continuous and there exist constants $L_u, L_{\widetilde{u}}>0$ such that

$$\left|\int_0^t [u(t,s,x(s))-u(t,s,y(s))]ds\right|\le L_u|x-y|,$$

for all, $x,y\in\mathbb{R}$; and

$$\left|\int_0^T [\widetilde{u}(t,s,x(s))-\widetilde{u}(t,s,y(s))]ds\right|\le L_{\widetilde{u}}|x-y|,$$

for all, $x,y\in\mathbb{R}$;.

(A3) For $i=1,2,\ldots,m$, the functions $g_i\in C((t_i,s_i]\times\mathbb{R};\mathbb{R})$ and there exists $L_{g_i}\in C(J,\mathbb{R}^+)$ such that

$$|g_i(t,x)-g_i(t,y)|\le L_{g_i}|x-y|$$

for all $x,y\in\mathbb{R}$ and $t\in(t_i,s_i]$.

(A4) $h:PC(J,\mathbb{R})\to\mathbb{R}$ is continuous and there exists a positive constant $L_h>0$ such that

$$|h(x)-h(y)|\le L_h\|x-y\|_{PC}, \quad \text{for all} \quad x,y\in PC(J,\mathbb{R}).$$

Theorem 3.1. *Assume that the hypotheses $(A1)-(A4)$ hold and*

$$\begin{aligned}\Lambda=\max\Bigg\{&\max_{1\le i\le m}\left\{L_{g_i}+\left(s_i^\alpha+t_{i+1}^\alpha\right)\frac{L_f(1+L_u+L_{\widetilde{u}})}{\Gamma(\alpha+1)}\right\},\\ &L_h+\frac{t_1^\alpha}{\Gamma(\alpha+1)}L_f(1+L_u+L_{\widetilde{u}})\Bigg\}<1\end{aligned}.$$

Then there exists a unique mild solution $x\in PC(J,\mathbb{R})$ for the problem (1.1).

Proof. Define the operator $\Upsilon : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}
\Upsilon x(t) &= x_0 - h(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \quad t\in[0,t_1] \\
\Upsilon x(t) &= g_i(t,x(t)), \quad t\in(t_i,s_i],\, i=1,2,\ldots,m, \\
\Upsilon x(t) &= g_i(s_i,x(s_i)) - \frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-s)^{\alpha-1} \\
&\quad f(s,x(s),K_1(x(s)),K_2(x(s)))ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \\
&\quad t\in(s_i,t_{i+1}],\, i=1,2,\ldots,m
\end{aligned}$$

From the assumptions, this definition is well-defined.

We prove that Υ is a contraction map on $PC(J,\mathbb{R})$. Let $x,y\in PC(J,R)$. For $t\in[0,t_1]$, we have

$$\begin{aligned}
&|\Upsilon x(t)-\Upsilon y(t)| \\
&|h(x)-h(y)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{|f(s,x(s),K_1(x(s)),K_2(x(s))) \\
&\quad -f(s,y(s),K_1(y(s)),K_2(y(s)))|\}ds \\
&\le \left[L_h + \frac{t_1^\alpha}{\Gamma(\alpha+1)}L_f(1+L_u+L_{\tilde{u}})\right]\|x-y\|_{PC}
\end{aligned}$$

and hence,

$$\|\Upsilon x-\Upsilon y\|_{C([0,t_1],\mathbb{R})} \le \left[L_h + \frac{t_1^\alpha}{\Gamma(\alpha+1)}L_f(1+L_u+L_{\tilde{u}})\right]\|x-y\|_{PC}.$$

For $t\in(t_i,s_i]$

$$|\Upsilon x(t)-\Upsilon y(t)| = |g_i(t,x(t))-g_i(t,y(t))| \le L_{g_i}\|x-y\|_{PC}.$$

In this case also

$$\|\Upsilon x-\Upsilon y\|_{C((t_i,s_i],\mathbb{R})} \le L_{g_i}\|x-y\|_{PC}.$$

For $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned}
&|\Upsilon x(t)-\Upsilon y(t)| \\
&= |g_i(s_i,x(s_i))-g_i(s_i,y(s_i))| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-s)^{\alpha-1}|f(s,x(s),K_1(x(s)),K_2(x(s))) \\
&- f(s,y(s),K_1(y(s)),K_2(y(s)))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),K_1(x(s)),K_2(x(s))) \\
&- f(s,y(s),K_1(y(s)),K_2(y(s)))|ds \\
&\le \left[L_{g_i} + \left(s_i^\alpha + t_{i+1}^\alpha\right)\frac{1}{\Gamma(\alpha+1)}L_f(1+L_u+L_{\tilde{u}})\right]\|x-y\|_{PC}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\Upsilon x-\Upsilon y\|_{C((s_i,t_{i+1}],\mathbb{R})} \\
&\le \left[L_{g_i} + \left(s_i^\alpha + t_{i+1}^\alpha\right)\frac{1}{\Gamma(\alpha+1)}L_f(1+L_u+L_{\tilde{u}})\right]\|x-y\|_{PC}.
\end{aligned}$$

Thus we observe that

$$|\Upsilon x-\Upsilon y||_{PC} \le \Lambda||x-y||_{PC},$$

which implies that Υ is a contraction and hence there exists a mild solution to the problem (1.1). □

4. Application

In this section we present an application of the result in Section 3.

Consider the following nonlocal fractional mixed type integro-differential equation with impulsive condition of the form

$$\begin{aligned}
CD^{\frac{1}{2}}x(t) &= \frac{|x(t)|}{(9+e^t)(1+|x(t)|)} + \frac{1}{9}\int_0^t e^{-\frac{1}{4}x(\sin s)}ds \\
&\quad + \frac{1}{9}\int_0^t e^{-\frac{1}{8}x(\sin s)}ds, \quad t\in(0,1]\cup(2,3] \quad (4.1)
\end{aligned}$$

$$x(t) = \frac{|x(t)|}{(9t+1)(1+|x(t)|)}, \quad t\in(1,2] \qquad (4.2)$$

$$x(0) = x_0 + \sum_{i=1}^{10} c_i x(p_i), \qquad (4.3)$$

where, c_i are constants with $\max\limits_{1\le i\le 10} c_i < 0.01$ and $0 < p_1 < p_2 < \ldots < p_{10} < 3$, $\alpha = \frac{1}{2}$, $J=[0,3]$, $0=s_0<t_1=1<s_1=2<t_2=3$.

Take

$$\begin{aligned}
&f(t,x(t),K_1(x(t)),K_2(x(t))) \\
&= \frac{|x(t)|}{(9+e^t)(1+|x(t)|)} + \frac{1}{9}\int_0^t e^{-\frac{1}{4}x(s)}ds + \frac{1}{9}\int_0^t e^{-\frac{1}{8}x(s)}ds, \\
&\quad t\in(0,1]\cup(2,3]
\end{aligned}$$

$$\begin{aligned}
K_1(x(t)) &= \frac{1}{9}\int_0^t e^{-\frac{1}{4}x(\sin s)}ds,\ t\in(0,1]\cup(2,3] \\
K_2(x(t)) &= \frac{1}{9}\int_0^t e^{-\frac{1}{8}x(\sin s)}ds,\ t\in(0,1]\cup(2,3] \\
g_1(t,x(t)) &= \frac{|x(t)|}{(9t+1)(1+|x(t)|)},\ t\in(1,2] \\
h(x) &= \sum_{i=1}^{10} c_i x(p_i).
\end{aligned}$$

For $x,y\in PC(J,\mathbb{R})$ and $t\in(1,2]$

$$|g_1(t,x)-g_1(t,y)| \le \frac{1}{10}|x(t)-y(t)| \le \frac{1}{10}\|x-y\|_{PC}.$$

For $x,y\in PC(J,\mathbb{R})$, and $t\in(0,1]\cup(2,3]$

$$\begin{aligned}
&|f(t,x(t),K_1(x(t)),K_2(x(t))) - f(t,y(t),K_1(y(t)),K_2(y(t)))| \\
&\le \frac{1}{10}|x(t)-y(t)| + \frac{1}{9}||K_1(x)-K_1(y)||_{PC} + \frac{1}{9}||K_2(x)-K_2(y)||_{PC} \\
&\le \frac{1}{9}\left[\|x-y\|_{PC} + ||K_1(x)-K_1(y)||_{PC} + \frac{1}{9}||K_2(x)-K_2(y)||_{PC}\right]
\end{aligned}$$

For $x,y\in PC(J,\mathbb{R})$ and $t\in(0,1]\cup(2,3]$

$$\|K_1(x)-K_1(y)\| \le \frac{1}{4}\|x(\sin t)-y(\sin t)\| \le \frac{1}{4}\|x-y\|_{PC};$$

$$\|K_2(x)-K_2(y)\| \le \frac{1}{8}\|x(\sin t)-y(\sin t)\| \le \frac{1}{8}\|x-y\|_{PC}$$

and

$$|h(x)-h(y)| < \frac{1}{10}||x-y||_{PC}.$$

We can verify that the assumptions (A1)-(A4) hold by putting $L_{g_1} = L_h = L_f = \frac{1}{10}, L_u = \frac{1}{4}, L_{\widetilde{u}} = \frac{1}{8}$, we have

$$\Lambda = \max\left\{\max_{1\le i\le m}\left\{L_{g_i} + \left(s_i^{\alpha} + t_{i+1}^{\alpha}\right)\frac{L_f\left(1+L_u+L_{\widetilde{u}}\right)}{\Gamma(\alpha+1)}\right\},\right.$$
$$\left.L_h + \frac{t_1^{\alpha}}{\Gamma(\alpha+1)}L_f\left(1+L_u+L_{\widetilde{u}}\right)\right\}$$
$$\le \max\left\{\frac{1}{10} + (\sqrt{2}+\sqrt{3})\frac{0.1(1+0.25+0.125)}{\Gamma(\frac{3}{2})}, \frac{1}{10}\right.$$
$$\left.+\frac{1}{\Gamma(\frac{3}{2})}(0.1)(1.375)\right\} = 0.588 < 1.$$

Thus, since all the assumptions in Theorem 3.1 are fulfilled, our results can be applied to the problem (4.1)-(4.3).

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