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Infinite horizon mean-field type forward backward stochastic delay differential game with Poisson jump processes

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Abstract
This paper analyzes the optimal control of mean field type forward-backward non-zero sum stochastic delay differential game with Poisson random measure over infinite time horizon. Further, infinite horizon version of stochastic maximum principle and necessary condition for optimality are established under the transversality conditions and the assumption of convex control domain. Finally, the Nash equilibrium for optimization problem in financial market is presented to illustrate the theoretical study.

Keywords
Infinite-horizon; Mean field; Nash equilibrium; Optimal control; Poisson jump processes; Stochastic delay differential game.

AMS Subject Classification
35B50, 91A15, 93E20.

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Contents

1. Introduction

Game theory arises in the situation in which two or more agents interact with each other. In particular, stochastic differential game problems are related to the context of dynamical system with stochastic nature see [3, 9]. In this manuscript optimal control of non-zero sum stochastic differential game problem is discussed. In classical stochastic optimal control problem, there is a single control $u(t)$ which corresponds to the single objective functional to be optimized. Rather, the stochastic optimal control of non-zero sum differential game have two controls namely $u_1(t)$, $u_2(t)$ and corresponding to two objective functional for each player, where as each player attempts to control the state of the system so as to achieve the desired goal. Moreover, optimal control of non-zero sum stochastic differential game is studied by Chen and Yu in [6], Deng et al. in [8], Wu and Shu in [21].

Economic and financial models disclose the jump type behavior see [7]. For instance, Lin et al. discussed the optimal portfolio selection problem of insurer who faces model uncertainty in jump-diffusion risk model under game theoretic approach. Stochastic optimal control of delay differential game problems are discussed in [18]. In order to combine delay and jump type behavior through optimal control of stochastic differential game approach, Pamen studied in [17]. Moreover, forward-backward stochastic differential equation (FBSDE) have natural occurrence in various fields such as financial market, optimal pricing, stochastic optimal control and recursive utility problem [5, 14]. For that, Wang et al. studied the forward-backward differential game problem with optimal investment and dividend problem of an insurer under model uncertainty in [19]. They applied the classical convex variation and adjoint techniques to derive necessary and sufficient condition for the prescribed system. For more details on optimal control problem for stochastic differential games of FBSDE, see Juan et al. in [12] and Øksendal in [16].

Further, natural system with large number of interacting particles are modeled by mean-field stochastic differential game see [4]. Mean field game problems are studied by many authors. In particular Wu and Liu studied the optimal control of mean field type zero sum stochastic differential game in [20]. Moreover, optimal control of mean field type non-zero sum game problem is studied by Hu et al. in [11]. Despite of numerous work have been reported on optimal control of mean field stochastic differential game, there is few of existing work in infinite horizon. Infinite horizon optimal control problems arise naturally in economics when dealing with dynamical models of optimal allocation of resources. Therefore the present work of this manuscript focused on optimal control of mean field type stochastic differential game with an infinite horizon time. For more details related to infinite horizon optimal control problems see [1, 2, 10, 13, 15] and references therein.

By the above motivation the authors consider the infinite horizon mean field type optimal control of non-zero sum stochastic delay differential game of two players with Poisson jump processes.

This paper is structured as follows: In Section 2, preliminaries, notations and formulation of the problems is provided. Section 3 of this paper contains the derivation of sufficient condition for optimality of the proposed problem under convexity assumption on control domain. Necessary conditions for optimal control problems are given in Section 4. In Section 5, application of the theoretical study is established by the optimization problem in financial market.

2. Preliminaries and Problem formulation

In this paper the following non zero sum game of infinite horizon mean field type optimal control of stochastic delay differential equation with Poisson jump processes is considered.

$$\begin{aligned} dX(t) &= b\left(t,\mathbf{X(t)},u_1(t),u_2(t)\right)dt \\ &\quad+\sigma\left(t,\mathbf{X(t)},u_1(t),u_2(t)\right)dW(t) \\ &\quad+\int_{\mathbb{R}_0} G(t,\mathbf{X(t)},u_1(t),u_2(t),\gamma)\tilde{N}(dt,d\gamma),\ t\in[0,\infty). \\ X(t) &= x_0\in\mathbb{R}; \qquad t\in[-\delta,0],\ \text{where } \delta>0 \end{aligned} \tag{2.1}$$

where, $\mathbf{X(t)}=\mathbf{X(t)}$ and $X(t)=X(t,\omega);t\geq 0,\omega\in\Omega$ is a state processes which is defined on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ with right continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and $\mathscr{F}(0)$ contains $\mathbb{P}$ null sets. Let $A(t)(:=X(t-\delta))$, $B(t)(:=\int_{t-\delta}^{t}e^{-\rho(t-r)}X(r)dr)$ are pointwise and distributed delay respectively, where $\rho\geq 0$. Expectations E denotes the average behavior of each players. Let $W(t)=W(t,\omega)$ be a one dimensional Brownian motion and $\tilde{N}(dt,d\gamma)=N(dt,d\gamma)-\nu(d\gamma)dt$ is a compensated Poisson random measure with $\gamma\in\mathbb{R}_0(:=\mathbb{R}-\{0\})$, where ν is the Lévy measure on the filtered probability space $(\Omega,\mathscr{F}_t,\mathbb{P})$. Moreover, $W(\cdot)$ and $N(\cdot)$ are independent processes. Let $X(t)$, $A(t)$, $B(t)$, $E[X(t)]$, $E[A(t)]$, $E[B(t)]$ be real valued $\mathscr{F}_t$ adapted processes on $[0,\infty)$. Moreover, $u_1(t)=u_1(t,\omega)$ and $u_2(t)=u_2(t,\omega)$ are strict control variables for Player I and Player II respectively with $E\left[\sup_{t\in[0.\infty)}|u_i(t)|^2\right]<\infty$, for $i=1,2$ holds. Here $u_1(t)$ and $u_2(t)$ are also $\mathscr{F}_t$ adapted process taking values in the convex subsets U_1 and U_2 of $\mathbb{R}$ respectively. The coefficients b, σ are real valued functions on $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U_1\times U_2\times\Omega$, and G is also a real valued functions on $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}_0\times U_1\times U_2\times\Omega$.
Assume that the following two subfiltrations

$$\varepsilon_t^{(1)}\subset\mathscr{F}_t,\quad \varepsilon_t^{(2)}\subset\mathscr{F}_t;\quad t\in[0,\infty),$$

representing the information available to Player I , Player II respectively. Let $\mathscr{U}_1\subset U_1$ denotes the set of admissible control processes for Player I with $\varepsilon_t^{(1)}$- predictable processes and $\mathscr{U}_2\subset U_2$ denotes the set of admissible control processes for Player II with $\varepsilon_t^{(2)}$- predictable processes.
Backward equation in the unknown real valued measurable processes $Y_i(t),Z_i(t),\theta_i(t,\gamma)$ corresponding to the forward system (2.1) is defined as follows:

$$\begin{aligned} dY_i(t) &= -g_i\left(t,\mathbf{X(t)},Y_i(t),Z_i(t),\theta_i(t,\gamma),u_1(t),u_2(t)\right)dt \\ &\quad+Z_i(t)dW(t)+\int_{\mathbb{R}_0}\theta_i(t,\gamma)\tilde{N}(dt,d\gamma), \qquad t\in[0,\infty). \end{aligned} \tag{2.2}$$

Here $i=1,2$. g_i is a function from $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathfrak{R}\times U_1\times U_2\times\Omega\to\mathbb{R}$, where $\mathfrak{R}$ is the set of all functions from $\mathbb{R}_0$ to $\mathbb{R}$. Another equivalent form of equation (2.2) is

$$\begin{aligned} Y_i(t) =& Y_i(T)-\int_t^T g_i(s,\mathbf{X(s)},Y_i(s),Z_i(s),\theta_i(s,\gamma), \\ & u_1(s),u_2(s))ds \\ & +\int_t^T Z_i(s)dW(s)+\int_t^T\int_{\mathbb{R}_0}\theta_i(t,\gamma)\tilde{N}(dt,d\gamma), \end{aligned}$$

the processes $Y_i(t)$, $Z_i(t)$, $\theta_i(t,\gamma)$ are solutions of the above equation, if it satisfies the following condition

$$E\left[\sup e^{ct}Y^2(t)+\int_0^\infty e^{ct}\left(Z^2(t)+\int_{\mathbb{R}_0}K^2(t,\gamma)\nu(d\gamma)\right)dt\right]<\infty.$$

The corresponding solution $X(t)$, $Y_i(t)$, $Z_i(t)$, $\theta_i(t)$ of system (2.1), (2.2) exists, if $E\left[\int_0^\infty|X(t)|^2dt\right]<\infty$ hold. Let us consider the cost functional corresponding to the system (2.1) and (2.2) of each player i as follows:

$$\begin{aligned} J_i(u_1,u_2) =& E\left[\int_0^\infty K_i\left(t,\mathbf{X(t)},Y_i(t),Z_i(t),\theta_i(t,\gamma),u_1(t),u_2(t)\right)dt\right] \\ & +E\left[h_i(Y_i(0))\right], \quad t\geq 0. \end{aligned} \tag{2.3}$$

where K_i is real valued function on $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathfrak{R}\times U_1\times U_2\times\Omega$, and h_i is also real valued function on $\mathbb{R}$ with the following assumption holds

$$E\left[\int_0^\infty\left\{\left|K_i\left(t,\mathbf{X(t)},u_1(t),u_2(t)\right)\right|\right\}dt\right]<\infty.$$

The Hamiltonian $H_i : [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathfrak{R} \times U_1 \times U_2 \times \mathbb{R}^3 \times \mathfrak{R} \to \mathbb{R}$ be defined for the system (2.1)-(2.3) as follows,

$$\begin{aligned}
&H_i(t,\mathbf{X(t)},Y_i(t),Z_i(t),\theta_i(t,\gamma),u_1(t),u_2(t),\lambda_i(t),\\
&\qquad \mathscr{P}_i(t),\mathscr{Q}_i(t),\mathscr{R}_i(t,\gamma))\\
=&K_i(t,\mathbf{X(t)},Y_i(t),Z_i(t),\theta_i(t,\gamma),u_1(t),u_2(t))\\
&+\lambda_i(t)g_i(t,\mathbf{X(t)},Y_i(t),Z_i(t),\theta_i(t,\gamma),u_1(t),u_2(t))\\
&+\mathscr{P}_i(t)b(t,\mathbf{X(t)},u_1(t),u_2(t))\\
&+\mathscr{Q}_i(t)\sigma(t,\mathbf{X(t)},u_1(t),u_2(t))\\
&+\int_{\mathbb{R}_0}\mathscr{R}_i(t,\gamma)G(t,\mathbf{X(t)},u_1(t),u_2(t)).
\end{aligned}$$

In order to simplify the notations the above equation can be written as

$$\begin{aligned}
H_i(t) =&K_i(t)+\lambda_i(t)g_i(t)+\mathscr{P}_i(t)b(t)+\mathscr{Q}_i(t)\sigma(t)\\
&+\int_{\mathbb{R}_0}\mathscr{R}_i(t,\gamma)G(t,\gamma)\nu(d\gamma). \qquad (2.4)
\end{aligned}$$

To establish the optimality condition for the system (2.1) and (2.2) with associated cost functional (2.3), we need to develop the following adjoint equations of adjoint processes $\lambda_i(t),\mathscr{P}_i(t),\mathscr{Q}_i(t),\mathscr{R}_i(t,\gamma)$ by using the Hamiltonian functional.

- ⋄ Forward stochastic differential equation in $\lambda_i(t)$ is,

$$\begin{aligned}
d\lambda_i(t) =&\frac{\partial H_i(t)}{\partial Y_i}dt+\frac{\partial H_i(t)}{\partial Z_i}dW(t)+\int_{\mathbb{R}_0}\frac{\partial H_i(t)}{\partial \theta_i}\tilde{N}(dt,d\gamma)\\
\lambda_i(0) =&h_i'(Y_i(0)) \qquad (2.5)
\end{aligned}$$

- ⋄ Backward stochastic differential equation in $\mathscr{P}_i(\cdot)$, $\mathscr{Q}_i(\cdot)$, $\mathscr{R}_i(\cdot)$ is,

$$\begin{aligned}
d\mathscr{P}_i(t) =&\mathscr{M}(t)dt+E[\mathscr{M}(t)]dt+\mathscr{Q}_i(t)dW(t)\\
&+\int_{\mathbb{R}_0}\mathscr{R}_i(t,\gamma)\tilde{N}(dt,d\gamma), \qquad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{M}(t) =&-\frac{\partial H_i(t)}{\partial X}-\frac{\partial H_i(t+\delta)}{\partial A}\\
&-e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H_i(r)}{\partial B}e^{-\rho r}dr\right), \qquad (2.7)\\
E[\mathscr{M}(t)] =&-E\left[\frac{\partial H_i(t)}{\partial \tilde{X}}\right]-E\left[\frac{\partial H_i(t+\delta)}{\partial \tilde{A}}\right]\\
&-e^{\rho t}\left(\int_t^{t+\delta}E\left[\frac{\partial H_i(r)}{\partial \tilde{B}}\right]e^{-\rho r}dr\right), \qquad (2.8)\\
\text{and}\quad & E\left[\int_0^\infty e^{\rho t}|\mathscr{P}_i(t)|^2dt\right]<\infty, \quad \text{for all } \rho\in\mathbb{R}.
\end{aligned}$$

here $E\left[\frac{\partial H_i(t)}{\partial \tilde{X}}\right]$, $E\left[\frac{\partial H_i(t+\delta)}{\partial \tilde{A}}\right]$, $E\left[\frac{\partial H_i(s)}{\partial \tilde{B}}\right]$ are partial derivatives of $H_i(t)$ with respective to $E[X(t)]$, $E[A(t)]$, $E[B(t)]$ respectively.

3. Sufficient Condition for Optimality

In this section, sufficient condition for optimality of the system (2.1)-(2.3) is established via the existence of Nash equilibrium pair $(\hat{u}_1,\hat{u}_2)\in\mathscr{U}_1\times\mathscr{U}_2$ such that

$$\begin{aligned}
J_1(u_1,\hat{u}_2)\le J_1(\hat{u}_1,\hat{u}_2) &\qquad \text{for all} \quad u_1\in\mathscr{U}_1.\\
J_2(\hat{u}_1,u_2)\le J_2(\hat{u}_1,\hat{u}_2) &\qquad \text{for all} \quad u_2\in\mathscr{U}_2.
\end{aligned}$$

In this problem no player has an incentive to deviate from his or her chosen strategy after considering an opponent's choice. Overall, an individual can receive no incremental benefit from changing actions and assuming other players remain constant in their strategies.

In order to prove Nash equilibrium for the proposed problem (2.1)-(2.3) the following hypotheses are needed:

($\mathscr{H}1$) Concavity:
The functions $x\mapsto h_i(x)$ and $x\mapsto H_i(x)$ are concave, for $t\in[0,\infty), i=1,2$.

($\mathscr{H}2$) Conditional maximum principle:

$$\begin{aligned}
&E\Big[H_i(t,\hat{\mathbf{X}}(t),\hat{Y}_i(t),\hat{Z}_i(t),\hat{\theta}_i(t,\gamma),\hat{u}_1(t),\hat{u}_2(t),\\
&\qquad\times\lambda_i(t),\hat{\mathscr{P}}_i(t),\hat{\mathscr{Q}}_i(t),\hat{\mathscr{R}}_i(t,\gamma))\Big|\mathscr{F}_t\Big]\\
&=\max_{u_1,u_2\in\mathscr{U}_1\times\mathscr{U}_2}E\Big[H_i(t,\hat{\mathbf{X}}(t),\hat{Y}_i(t),\hat{Z}_i(t),\hat{\theta}_i(t,\gamma),u_1(t),u_2(t),\\
&\qquad\times\lambda_i(t),\hat{\mathscr{P}}_i(t),\hat{\mathscr{Q}}_i(t),\hat{\mathscr{R}}_i(t,\gamma))\Big|\mathscr{F}_t\Big],
\end{aligned}$$

for $i=1,2$.

($\mathscr{H}3$) Transversality condition:

$$\begin{aligned}
&\lim_{T\to\infty}E\left[\hat{\lambda}_i(T)\left(Y_i(T)-\hat{Y}_i(T)\right)\right]\le 0 \text{ and}\\
&\lim_{T\to\infty}E\left[\hat{\mathscr{P}}_i(T)\left(X(T)-\hat{X}(T)\right)\right]\ge 0,
\end{aligned}$$

for $i=1,2$.

Theorem 3.1. *Let $\hat{X}(t)$, $\hat{Y}_i(t)$, $\hat{Z}_i(t)$, $\hat{\theta}_i(t)$ be solution of the system (2.1)-(2.3) which are corresponding to the admissible control $(\hat{u}_1,\hat{u}_2)\in\mathscr{U}_1\times\mathscr{U}_2$. Suppose the adjoint processes $\hat{\lambda}_i(t)$, $\hat{\mathscr{P}}_i(t)$, $\hat{\mathscr{Q}}_i(t)$, $\hat{\mathscr{R}}_i(t)$ which are satisfies the adjoint stochastic differential equation (2.5)-(2.8) and the hypothesis ($\mathscr{H}1$)-($\mathscr{H}3$) are holds, then $(\hat{u}_1,\hat{u}_2)$ is an optimal control for the system (2.1)-(2.3).*

Proof. Proof similar to Theorem 4.1 in [15]. □

4. Necessary Condition for Optimality

In this section necessary condition for optimality of the proposed problem is proved by the following hypotheses:

($\mathscr{H}4$) For all $t_0\in[0,\infty)$, $l>0$ and all bounded $\varepsilon_{t_0}^{(i)}$- measurable random variable $\alpha_i(\omega)$, the control process $\beta_i(t)$ defined by

$$\beta_i(t):=I_{[t_0,t_0+\ell)}(t)\alpha_i(\omega), \text{ which is belongs to } \mathscr{U}_i,$$

($\mathscr{H}5$) For all $u_i \in \mathscr{U}_i$ and all bounded $\beta_i \in \mathscr{U}_i$, there exists $\delta_i > 0$ such that the control

$$\tilde{u}_i(t) := u_i + s\beta_i(t), \quad t \in [0,\infty),$$

for all $s \in (-\delta_i, \delta_i), i = 1,2.$

($\mathscr{H}6$) $\lim_{T\to\infty} E\left[\hat{\mathscr{P}}_i(T)\mathscr{A}_i(T)\right] = 0$ and $\lim_{T\to\infty} E\left[\hat{\lambda}_i(T)\phi_i(T)\right] = 0.$

Theorem 4.1. *Assume that the hypotheses ($\mathscr{H}4$) – ($\mathscr{H}6$) are holds. Let $(\hat{u}_1, \hat{u}_2) \in \mathscr{U}_1 \times \mathscr{U}_2$ be the optimal control and $\hat{X}(t)$, $\hat{Y}_i(t)$, $\hat{Z}_i(t)$, $\hat{\theta}_i(t)$ are corresponding solutions of the system (2.1)-(2.3). If there exist a unique adjoint processes $\hat{\lambda}_i(t)$, $\hat{\mathscr{P}}_i(t)$, $\hat{\mathscr{Q}}_i(t)$, $\hat{\mathscr{R}}_i(t)$ which solves the adjoint stochastic differential equation (2.5)-(2.6), then the following conditions are equivalent:*

($\mathscr{S}1$) *For all bounded $\beta_1 \in \mathscr{U}_1, \beta_2 \in \mathscr{U}_2$, we have*

$$\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = \frac{d}{ds}J_2^{(u_1,u_2+s\beta_2)}(t)\Big|_{s=0} = 0.$$

($\mathscr{S}2$) *For all $t \in [0,\infty)$ we have*

$$E\left[\frac{\partial H_1}{\partial u_1}\left(\hat{X}(t),\, \hat{Y}_1(t), \hat{Z}_1(t), \hat{\theta}_1(t,\gamma), u_1(t), \hat{u}_2(t), \hat{\mathscr{P}}_1(t), \hat{\mathscr{Q}}_1(t), \hat{\mathscr{R}}_1(t)\right)\Big|\mathscr{F}_t\right]_{u_1=\hat{u}_1(t)} = 0,$$

and

$$E\left[\frac{\partial H_2}{\partial u_2}\left(\hat{X}(t),\, \hat{Y}_2(t), \hat{Z}_2(t), \hat{\theta}_2(t,\gamma), \hat{u}_1(t), u_2(t), \hat{\mathscr{P}}_2(t), \hat{\mathscr{Q}}_2(t), \hat{\mathscr{R}}_2(t)\right)\Big|\mathscr{F}_t\right]_{u_2=\hat{u}_2(t)} = 0,$$

Proof. Let us define the following derivative processes,

$$\begin{aligned}
&\frac{d}{ds}X^{u_1+s\beta_1,u_2}(t)|_{s=0} = \mathscr{A}_1(t),\\
&\frac{d}{ds}X^{u_1,u_2+s\beta_2}(t)|_{s=0} = \mathscr{A}_2(t),\\
&\frac{d}{ds}A^{u_1+s\beta_1,u_2}(t)|_{s=0} = \mathscr{A}_1(t-\delta),\\
&\frac{d}{ds}A^{u_1,u_2+s\beta_2}(t)|_{s=0} = \mathscr{A}_2(t-\delta),\\
&\frac{d}{ds}B^{u_1+s\beta_1,u_2}(t)\Big|_{s=0} = \int_{t-\delta}^{t} e^{-\lambda(t-s)}\mathscr{A}_1(s)ds,\\
&\frac{d}{ds}B^{u_1,u_2+s\beta_2}(t)\Big|_{s=0} = \int_{t-\delta}^{t} e^{-\lambda(t-s)}\mathscr{A}_2(s)ds,\\
&\frac{d}{ds}Y_1^{u_1+s\beta_1,u_2}(t)|_{s=0} = \phi_1(t),\\
&\frac{d}{ds}Y_2^{u_1,u_2+s\beta_2}(t)|_{s=0} = \phi_2(t),\\
&\frac{d}{ds}Z_1^{u_1+s\beta_1}(t),u_2|_{s=0} = \psi_1(t),\\
&\frac{d}{ds}Z_2^{u_1,u_2+s\beta_2}(t)|_{s=0} = \psi_2(t),\\
&\frac{d}{ds}\theta_1^{u_1+s\beta_1}(t),u_2|_{s=0} = \mathscr{D}_1(t,\gamma),\\
&\frac{d}{ds}\theta_2^{u_1,u_2+s\beta_2}(t)|_{s=0} = \mathscr{D}_2(t,\gamma).
\end{aligned}$$

$$\begin{aligned}
d\mathscr{A}_1(t) = &\left\{\frac{\partial b(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial b(t)}{\partial A}\mathscr{A}_1(t-\delta)\right.\\
&+ \frac{\partial b(t)}{\partial B}\int_{t-\delta}^{t} e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial b(t)}{\partial u_1}\beta_1(t)\\
&+E\left[\frac{\partial b(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial b(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right]\\
&\left.+E\left[\frac{\partial b(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^{t} e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\right\}dt\\
&+\left\{\frac{\partial \sigma(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial \sigma(t)}{\partial A}\mathscr{A}_1(t-\delta)\right.\\
&+ \frac{\partial \sigma(t)}{\partial B}\int_{t-\delta}^{t} e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial \sigma(t)}{\partial u_1}\beta_1(t)\\
&+E\left[\frac{\partial \sigma(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial \sigma(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right]\\
&\left.+E\left[\frac{\partial \sigma(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^{t} e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\right\}dW(t)\\
&+\int_{\mathbb{R}_0}\left\{\frac{\partial G(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial G(t)}{\partial A}\mathscr{A}_1(t-\delta)\right.\\
&+ \frac{\partial G(t)}{\partial B}\int_{t-\delta}^{t} e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial G(t)}{\partial u_1}\beta_1(t)\\
&+E\left[\frac{\partial G(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial G(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right]\\
&\left.+E\left[\frac{\partial G(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^{t} e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\right\}\tilde{N}(dt,d\gamma).
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
d\phi_1(t) = &-\left\{\frac{\partial g_1(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial g_1(t)}{\partial A}\mathscr{A}_1(t-\delta)\right.\\
&+ \frac{\partial g_1(t)}{\partial B}\int_{t-\delta}^{t} e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial g_1(t)}{\partial u_1}\beta_1(t)\\
&+E\left[\frac{\partial g_1(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial g_1(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right]\\
&+E\left[\frac{\partial g_1(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^{t} e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\\
&\left.+\frac{\partial g_1(t)}{\partial Y_1}\phi_1(t) + \frac{\partial g_1(t)}{\partial Z_1}\psi_1(t) + \frac{\partial g_1(t)}{\partial \theta_1}\mathscr{D}_1(t,\gamma)\right\}dt\\
&+\psi_1(t)dW(t) + \int_{\mathbb{R}_0}\mathscr{D}_1(t,\gamma)\tilde{N}(dt,d\gamma).
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}|_{s=0} = &E\left[\int_0^\infty\left\{\frac{\partial K_1(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial K_1(t)}{\partial A}\mathscr{A}_1(t-\delta)\right.\right.\\
&+ \frac{\partial K_1(t)}{\partial B}\int_{t-\delta}^{t} e^{-\rho(t-r)}\mathscr{A}_1(r)dr\\
&+E\left[\frac{\partial K_1(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial K_1(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right]\\
&+E\left[\frac{\partial g_1(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^{t} e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\\
&+\frac{\partial K_1(t)}{\partial u_1}\beta_1(t) + \frac{\partial K_1(t)}{\partial Y_1}\phi_1(t) + \frac{\partial K_1(t)}{\partial Z_1}\psi_1(t)\\
&\left.\left.+\frac{\partial K_1(t)}{\partial \theta_1}\mathscr{D}_1(t,\gamma)\right\}dt + h_1'(Y_1(0))\phi_1(0)\right].
\end{aligned} \tag{4.3}$$

By Hamiltonian in (2.4) we have

$$K_1(t) = H_1(t) - \lambda_1(t)g_1(t) - \hat{\mathscr{P}}_1(t)b(t) - \hat{\mathscr{Q}}_1(t)\sigma(t) - \int_{\mathbb{R}_0} \hat{\mathscr{R}}_1(t)G(t,\gamma)\nu(d\gamma),$$

and

$$\frac{\partial K_1(t)}{\partial X} = \frac{\partial H_1(t)}{\partial X} - \lambda_1(t)\frac{\partial g_1(t)}{\partial X} - \hat{\mathscr{P}}_1(t)\frac{\partial b(t)}{\partial X} - \hat{\mathscr{Q}}_1(t)\frac{\partial \sigma(t)}{\partial X} - \int_{\mathbb{R}_0} \hat{\mathscr{R}}_1(t)\frac{\partial G(t,\gamma)}{\partial X}, \quad (4.4)$$

and also for $\frac{\partial K_1(t)}{\partial A}$, $\frac{\partial K_1(t)}{\partial B}$, $E\left[\frac{\partial K_1(t)}{\partial \tilde{X}}\right]$, $E\left[\frac{\partial K_1(t)}{\partial \tilde{A}}\right]$, $E\left[\frac{\partial K_1(t)}{\partial \tilde{B}}\right]$, $\frac{\partial K_1(t)}{\partial u_1}$, $\frac{\partial K_1(t)}{\partial u_2}$, $\frac{\partial K_1(t)}{\partial Y_1}$, $\frac{\partial K_1(t)}{\partial Z_1}$, $\frac{\partial K_1(t)}{\partial \theta_1}$. Substituting the above equation in (4.3), Applying the Itô formula to the processes $\hat{\lambda}_1(t)\phi_1(t)$ on $[0,T]$ and using (2.5), (4.2) then taking limit as $T\to\infty$ which implies that

$$\begin{aligned}
E\left[\hat{\lambda}_1(0)\phi_1(0)\right] &= \lim_{T\to\infty} E\left[\hat{\lambda}_1(T)\phi_1(T)\right] \\
&- E\Bigg[-\int_0^\infty \hat{\lambda}_1(t)\Bigg\{\frac{\partial g_1(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial g_1(t)}{\partial A}\mathscr{A}_1(t-\delta) \\
&+ \frac{\partial g_1(t)}{\partial B}\int_{t-\delta}^t e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial g_1(t)}{\partial u_1}\beta_1(t) \\
&+ E\left[\frac{\partial g_1(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial g_1(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right] \\
&+ E\left[\frac{\partial g_1(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^t e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr \\
&+ \frac{\partial g_1(t)}{\partial Y_1}\phi_1(t) + \frac{\partial g_1(t)}{\partial Z_1}\psi_1(t) + \frac{\partial g_1(t)}{\partial \theta_1}\mathscr{D}_1(t,\gamma)\Bigg\}dt \\
&+ \int_0^\infty \phi_1(t)\frac{\partial H_1(t)}{\partial Y_1}dt + \int_0^\infty \psi_1(t)\frac{\partial H_1(t)}{\partial Z_1}dt \\
&+ \int_0^\infty\int_{\mathbb{R}_0}\mathscr{D}_1(t,\gamma)\frac{\partial H_1(t)}{\partial \theta_1}\nu(d\gamma)dt\Bigg].
\end{aligned}$$

Since $h_1'(\hat{Y}_1(0)) = \hat{\lambda}_1(0)$, which implies that $E\left[h_1'(\hat{Y}_1(0))\phi_1(0)\right] = E\left[\hat{\lambda}_1(0)\phi_1(0)\right]$, and using $(\mathscr{H}6)$ the above inequality can be written as

$$\begin{aligned}
E\left[h_1'(\hat{Y}_1(0))\phi_1(0)\right] &= E\left[\hat{\lambda}_1(0)\phi_1(0)\right] \\
&- E\Bigg[-\int_0^\infty \hat{\lambda}_1(t)\Bigg\{\frac{\partial g_1(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial g_1(t)}{\partial A}\mathscr{A}_1(t-\delta) \\
&+ \frac{\partial g_1(t)}{\partial B}\int_{t-\delta}^t e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial g_1(t)}{\partial u_1}\beta_1(t) \\
&+ E\left[\frac{\partial g_1(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial g_1(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right] \\
&+ E\left[\frac{\partial g_1(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^t e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr \\
&+ \frac{\partial g_1(t)}{\partial Y_1}\phi_1(t) + \frac{\partial g_1(t)}{\partial Z_1}\psi_1(t) + \frac{\partial g_1(t)}{\partial \theta_1}\mathscr{D}_1(t,\gamma)\Bigg\}dt \\
&+ \int_0^\infty \phi_1(t)\frac{\partial H_1(t)}{\partial Y_1}dt \\
&+ \int_0^\infty \psi_1(t)\frac{\partial H_1(t)}{\partial Z_1}dt + \int_0^\infty\int_{\mathbb{R}_0}\mathscr{D}_1(t,\gamma)\frac{\partial H_1(t)}{\partial \theta_1(t)}\nu(d\gamma)dt\Bigg].
\end{aligned} \quad (4.5)$$

Applying the Itô formula to the processes $\hat{\mathscr{P}}_1(t)\mathscr{A}_1(t)$ on $[0,T]$ and using (2.6), (4.1), then taking limit as $T\to\infty$ implies that

$$\begin{aligned}
&\lim_{T\to\infty} E[\hat{\mathscr{P}}_1(T)\mathscr{A}_1(T)] \\
&= \int_0^\infty \mathscr{P}_1(t)\Bigg\{\frac{\partial b(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial b(t)}{\partial A}\mathscr{A}_1(t-\delta) \\
&+ \frac{\partial b(t)}{\partial B}\int_{t-\delta}^t e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial b(t)}{\partial u_1}\beta_1(t) \\
&+ E\left[\frac{\partial b(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial b(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right] \\
&+ E\left[\frac{\partial b(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^t e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\Bigg\}dt \\
&+ \int_0^\infty \mathscr{A}_1(t)\Bigg\{-\frac{\partial H_i(t)}{\partial X} - \frac{\partial H_i(t+\delta)}{\partial A} \\
&- e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H_i(r)}{\partial B}e^{-\rho r}dr\right) \\
&- E\left[\frac{\partial H_i(t)}{\partial \tilde{X}}\right] - E\left[\frac{\partial H_i(t+\delta)}{\partial \tilde{A}}\right] \\
&- e^{\rho t}\left(\int_t^{t+\delta}E\left[\frac{\partial H_i(r)}{\partial \tilde{B}}\right]e^{-\rho r}dr\right)\Bigg\}dt \\
&+ \int_0^\infty \mathscr{Q}_1(t)\Bigg\{\frac{\partial \sigma(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial \sigma(t)}{\partial A}\mathscr{A}_1(t-\delta) \\
&+ \frac{\partial \sigma(t)}{\partial B}\int_{t-\delta}^t e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial \sigma(t)}{\partial u_1}\beta_1(t) \\
&+ E\left[\frac{\partial \sigma(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial \sigma(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right] \\
&+ E\left[\frac{\partial \sigma(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^t e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\Bigg\}dt \\
&+ \int_0^\infty \mathscr{R}_1(t,\gamma)\Bigg\{\frac{\partial G(t)}{\partial X}\mathscr{A}_1(t) + \frac{\partial G(t)}{\partial A}\mathscr{A}_1(t-\delta) \\
&+ \frac{\partial G(t)}{\partial B}\int_{t-\delta}^t e^{-\rho(t-r)}\mathscr{A}_1(r)dr + \frac{\partial G(t)}{\partial u_1}\beta_1(t) \\
&+ E\left[\frac{\partial G(t)}{\partial \tilde{X}}\right]E\left[\mathscr{A}_1(t)\right] + E\left[\frac{\partial G(t)}{\partial \tilde{A}}\right]E\left[\mathscr{A}_1(t-\delta)\right] \\
&+ E\left[\frac{\partial G(t)}{\partial \tilde{B}}\right]\int_{t-\delta}^t e^{-\rho(t-r)}E[\mathscr{A}_1(r)]dr\Bigg\}\nu(d\gamma)dt.
\end{aligned} \quad (4.6)$$

Substitute (4.5), (4.6) and (4.4) in (4.3) and using $(\mathscr{H}6)$,

$$\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = E\left[\int_0^\infty \frac{\partial H_1(t)}{\partial u_1}\beta_1(t)dt\right].$$

If $\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = 0$, then

$$E\left[\int_0^\infty \frac{\partial H_1(t)}{\partial u_1}\beta_1(t)dt\right] = 0.$$

For all bounded $\beta_1 \in U_1$, then this holds in particular for β_1 of the form

$$\beta_1(t) = \alpha_1(\omega)I_{[s,s+h]}(t),$$

where $\alpha_1(\omega)$ is bounded and $\varepsilon_{t_0}^{(1)}$- measurable, $s \geq t_0$. Then we get

$$E\left[\int_s^{s+h} \frac{\partial H_1(t)}{\partial u_1} dt \alpha_1\right] = 0. \tag{4.7}$$

Differentiating (4.7) with respect to h at $h = 0$ gives

$$E\left[\frac{\partial H_1(s)}{\partial u_1}\alpha_1\right] = 0. \tag{4.8}$$

Since the condition (4.8) holds for all $s \geq t_0$ and all $\alpha_1(\omega)$ is bounded and $\varepsilon_{t_0}^{(1)}$- measurable random variable, we conclude that

$$E\left[\frac{\partial H_1(t_0)}{\partial u_1}\Big|\varepsilon_{t_0}^{(1)}\right] = 0, \quad \text{for all } t_0 \in [0,\infty).$$

Similar argument for Player II, one can get

$$E\left[\frac{\partial H_2(t_0)}{\partial u_2}\Big|\varepsilon_{t_0}^{(2)}\right] = 0, \quad \text{for all } t_0 \in [0,\infty),$$

with the following condition,

$$\frac{d}{ds}J_2^{(u_1,u_2+s\beta_2)}(t)\Big|_{s=0} = 0 \quad \text{for all bounded } \beta_2 \in U_2.$$

This shows that $(\mathscr{S}1) \Rightarrow (\mathscr{S}2)$. By reversing the above argument one can prove $(\mathscr{S}2) \Rightarrow (\mathscr{S}1)$. □

5. Example

In this section the authors presented a optimization problem in financial market. Consider the two retail investors as players in stock market. In this problem each player maximizes their own profit. One player's profit does not imply that another player's loss. So this problem is a two player non cooperative stochastic differential game. Our aim is to establish the Nash-equilibrium, that is to find the optimal control for this game problem.

Let us consider the dynamical system corresponding to the optimization problem as follows:

$$\begin{aligned} dD(t) =& (\alpha_1 D(t) + \alpha_2 D(t-\delta) + \alpha_3 E[D(t)] + \alpha_4 C_1(t) \\ &+ \alpha_5 C_2(t))dt + (\alpha_6 C_1(t) + \alpha_7 C_2(t))\,dW(t) \\ &+ \int_{\mathbb{R}_0} (\alpha_8 C_1(t) + \alpha_9 C_2(t))\tilde{N}(dt,d\gamma), \quad t \geq 0, \\ D(t) =& D_0 \in \mathbb{R}, \quad t \in [-\delta, 0]. \end{aligned} \tag{5.1}$$

Here α_j, $j = 1,2,\ldots,9$ are constants, $D(t) \in \mathbb{R}$ is a demand rate and E refers to the average behavior of demand rate. $D(t-\delta) \in \mathbb{R}$ is the delayed demand rate by δ time, where the goods are received at t time. $C_1(t) \in \mathscr{U}_1 \subset U_1$ denotes stock price controlled by the first retailer and $C_2(t) \in \mathscr{U}_2 \subset U_2$ denotes the stock price controlled by the second retailer.

Maximizing the cost functional for each retailer is given below:

$$J_i = E\left[\int_0^\infty \Big\{(C_2(t) - S_i)D(t) - (1-S_i)C_1(t)\Big\}dt\right] < \infty,, \tag{5.2}$$

for all $t \geq 0$, for $i = 1,2$, where S_i denotes the fixed salvage price for retailer i. Then the corresponding backward stochastic differential equations for the problem (5.1), (5.2) as follows:

$$dY_i(t) = -(a_i Y_i(t) + b_i D(t) + c_i \ln C_1(t) + d_i \ln C_2(t))\,dt \tag{5.3}$$

$$+ Z_i(t)dW(t) + \int_{\mathbb{R}_0} \theta_i(t,\gamma)\tilde{N}(dt,d\gamma), \tag{5.4}$$

where a_i, b_i, c_i, d_i for $i = 1,2$ are constants. The Hamiltonian for the system (5.1)-(5.3) is defined as:

$$\begin{aligned} H_i(t) =& (C_2(t) - S_i)D(t) - (1-S_i)C_1(t) - \lambda_i(t)(a_i Y_i(t) \\ &+ b_i D(t) + b_i \ln C_1(t) + d_i \ln C_2(t)) + \mathscr{P}_i(t)(\alpha_1 D(t) \\ &+ \alpha_2 D(t-\delta) + \alpha_3 E[D(t)] + \alpha_4 C_1(t) + \alpha_5 C_2(t)) \\ &+ \mathscr{Q}_i(t)(\alpha_6 C_1(t) + \alpha_7 C_2(t)) \\ &+ \int_{\mathbb{R}_0} \mathscr{R}_i(t)(\alpha_8 C_1(t) + \alpha_9 C_2(t))\,\nu(d\gamma)dt, \end{aligned} \tag{5.5}$$

and assume that H_i's are concave. Using (2.5) and (2.8), the pair of forward and backward stochastic differential equations with the adjoint processes $\lambda_i(t)$, $\mathscr{P}_i(t)$, $\mathscr{Q}_i(t)$, $\mathscr{R}_i(t)$ are written as follows:

$$d\lambda_i(t) = -a_i\lambda_i(t)dt \tag{5.6}$$

$$\begin{aligned} d\mathscr{P}_i(t) =& [(S_i - C_2(t)) + \lambda_i(t)b_i - \mathscr{P}_i(t)(\alpha_1 + \alpha_2 + \alpha_3)] \\ &+ \mathscr{Q}_i(t)dW(t) + \int_{\mathbb{R}_0} \mathscr{R}_i(t)\tilde{N}(dt,d\gamma) \end{aligned} \tag{5.7}$$

with

$$E\left[\int_0^\infty e^{\rho t}|\mathscr{P}_i(t)|^2 dt\right] < \infty \quad \text{for all } \rho \in \mathbb{R}.$$

In order to find the optimal control $\hat{C}_1(t)$ and $\hat{C}_2(t)$ the following conditions are hold:

$$J_1(\hat{C}_1(t), \hat{C}_2(t)) = \sup_{C_1(t)} J_1(C_1(t), \hat{C}_2(t));$$

$$J_2(\hat{C}_1(t), \hat{C}_2(t)) = \sup_{C_2(t)} J_2(\hat{C}_1(t), C_2(t)).$$

Differentiating (5.5) with respect to their control variables which give the first order conditions as follows:

$$\begin{aligned} \frac{\partial H_1}{\partial C_1(t)} =& S_1 - 1 - \frac{c_1\lambda_1(t)}{C_1(t)} + \alpha_4\mathscr{P}_1(t) + \alpha_6\mathscr{Q}_1(t) \\ &+ \int_{\mathbb{R}_0} \mathscr{R}_1(t)\alpha_8\nu(d\gamma)dt = 0, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \frac{\partial H_2}{\partial C_2(t)} =& D(t) - \frac{d_2\lambda_2(t)}{C_2(t)} + \alpha_5\mathscr{P}_2(t) + \alpha_7\mathscr{Q}_2(t) \\ &+ \int_{\mathbb{R}_0} \mathscr{R}_2(t)\alpha_9\nu(d\gamma)dt = 0. \end{aligned} \tag{5.9}$$

The solution of the forward equation (5.6) is written as:

$$\lambda_i(t) = e^{-a_i t}. \tag{5.10}$$

Take a definite value of time $t = T$ and from equation (5.10), $\lim_{T\to\infty} \lambda_i(T) = 0$, which satisfies the transversality condition $(\mathscr{H}3)$. Similarly, let us take $\mathscr{P}_i(t)$, holds the hypothesis $(\mathscr{H}3)$. Also by the assumption that the concavity of H_i's, the hypotheses $(\mathscr{H}1)$, $(\mathscr{H}2)$ are holds. Thus the proposed model in this example satisfies $(\mathscr{H}1)$ - $(\mathscr{H}3)$. Then by using Theorem 3.1 of Section 3, one can conclude that there exist $\hat{C}_1(t)$ and $\hat{C}_2(t)$ which are the optimal control.

The required optimal control $\hat{C}_1(t)$ and $\hat{C}_2(t)$ for the system (5.1)-(5.2) are given by the equations (5.8) and (5.9) as follows:

$$\hat{C}_1(t) = \frac{c_1\lambda_1(t)}{S_1 - 1 + \alpha_4\mathscr{P}_1(t) + \alpha_6\mathscr{Q}_1(t) + \int_{\mathbb{R}_0}\mathscr{R}_1(t)\alpha_8\nu(d\gamma)dt},$$
$$\hat{C}_2(t) = \frac{d_2\lambda_2(t)}{D(t) + \alpha_5\mathscr{P}_2(t) + \alpha_7\mathscr{Q}_2(t) + \int_{\mathbb{R}_0}\mathscr{R}_2(t)\alpha_9\nu(d\gamma)dt}.$$

where the adjoint processes $\lambda_i(t)$, $\mathscr{P}_i(t)$, $\mathscr{Q}_i(t)$, $\mathscr{R}_i(t)$ for $i = 1, 2$ are the solutions of the adjoint differential equations (5.6)-(5.7).

6. Conclusion

In this paper, optimal control of mean field type forward-backward stochastic delay differential game problem has been discussed through infinite horizon. In particular, sufficient and necessary conditions for the optimality are derived by using transversality condition and convex control domain. The applicability of the developed theoretical study is illustrated through an example of optimization problem in financial market.

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