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Mild solution for fractional mixed type integro-differential equations with non-instantaneous impulses through sectorial operator

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Abstract
The main aim of this manuscript is to analyze the existence of PC-mild solution of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space through sectorial operator. Based on the general Banach contraction principle, we develop the main results. An example is ultimately given for the theoretical results to be justified.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K30, 35R12, 26A33.

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1. Introduction

There are a few fields of examination where the subject of fractional differential systems have as of late arose as a significant tool to demonstrate genuine issues. It has extraordinary applications in a few discipline and different fields of science for example, physical science, polymer rheology, thermodynamics, biophysics, blood flow phenomena, and control theory hypothesis, and so forth. For additional subtleties, one can see [4, 6, 7] and references in that.

The differential conditions with impulsive impacts have been showed up as in normal depiction development measures. The impulsive effect can be appeared in several biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated system etc. See the cited papers [1–3, 6, 7] for more detail of this topic.

Motivated by [1–3], in this paper we consider a class of fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$\begin{aligned} {}^cD^\alpha x(t) &= Ax(t) + J^{1-\alpha} f\left(t, x(t), G_1x(t), G_2x(t)\right), \\ &\qquad t \in \left(s_i, t_{i+1}\right], i = 0,1,2,\ldots,m \\ x(t) &= g_i(t, x(t)), \quad t \in \left(t_i, s_i\right], i = 1,2,\ldots,m \qquad (1.1) \\ x(0) &= x_0, \end{aligned}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $0 < \alpha \leq 1$, $J^{1-\alpha}$ is Riemann-Liouville fractional integral operator and $J = [0,T]$ is operational interval. The map $A : D(A) \subset X \to X$ is a closed linear sectorial operator defined on a Banach space $(X, \|\cdot\|), x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C\left(\left(t_i, s_i\right] \times X; X\right)$, $f : [0,T] \times X^3 \to X$ is a nonlinear function and the functions G_1 and G_2 are defined by

$$G_1x(t) = \int_0^t h(t,s,x(s))ds \quad \text{and} \quad G_2x(t) = \int_0^T \widetilde{h}(t,s,x(s))ds,$$

$h, \widetilde{h} : \Delta \times X \to X$, where $\Delta = \{(x,s) : 0 \leq s \leq x \leq \tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.

2. Preliminaries

Let us set $J=[0,T], J_0=[0,t_1], J_1=(t_1,t_2],\ldots, J_{m-1}=(t_{m-1},t_m], J_m=(t_m,t_{m+1}]$ and introduce the space $PC(J,X):=\{u:J\to X \mid u\in C(J_k,X), k=0,1,2,\ldots,m,$ and there exist $u(t_k^+)$ and $u(t_k^-), k=1,2,\ldots,m,$ with $u(t_k^-)=u(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC}=\sup\{\|u(t)\|: t\in J\}$.

Let us recall the following definition of mild solutions for fractional evolution equations involving the Caputo fractional derivative.

Definition 2.1. *[4] Caputo's derivative of order $\alpha>0$ with lower limit a, for a function $f:[a,\infty)\to\mathbb{R}$ is defined as*

$$ {}_a^C D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds=aJ_t^{n-\alpha}f^{(n)}(t)$$

where $a\geq 0, n\in N$. The Laplace transform of the Caputo derivative of order $\alpha>0$ is given as

$$L\left\{{}_0^C D_t^\alpha f(t);\lambda\right\}=\lambda^\alpha\hat{f}(\lambda)-\sum_{k=0}^{n-1}\lambda^{\alpha-k-1}f^k(0);\quad n-1<\alpha\leq n.$$

Definition 2.2. *[2] A closed and linear operator A is said to be sectorial if there are constants $\omega\in R, \theta\in\left[\frac{\pi}{2},\pi\right], M_A>0$, such that the following two conditions are satisfied:*

(1) $\sum_{(\theta,\omega)}=\{\lambda\in C:\lambda\neq\omega, |\arg(\lambda-\omega)|<\theta\}\subset\rho(A)$

(2) $\|R(\lambda,A)\|_{L(X)}\leq\frac{M}{|\lambda-\omega|}, \lambda\in\sum_{(\theta,\omega)}$,

where X is the complex Banach space with norm denoted $\|\cdot\|_X$.

Lemma 2.3. *[2] Let f satisfies the uniform Holder condition with exponent $\beta\in(0,1]$ and A is a sectorial operator. Consider the fractional equations of order $0<\alpha<1$*

$$ {}_a^C D_t^\alpha x(t)=Ax(t)+J^{1-\alpha}f(t), \tag{2.1}$$

$t\in J=[a,T],\quad a\geq 0,\quad x(a)=x_0$. Then a function $x(t)\in C([a,T],X)$ is the solution of the equation (2.1) if it satisfies the following integral equation

$$x(t)=Q_\alpha(t-a)x_0+\int_a^t Q_\alpha(t-s)f(s)ds$$

where $Q_\alpha(t)$ is solution operator generated by A defined as

$$Q_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}d\lambda$$

Γ is a suitable path lying on $\sum_{\theta,\omega}$.

Remark 2.4. *If $A\in\mathscr{A}^\alpha(\theta_0,\omega_0)$, then strongly continuous $\|Q_\alpha(t)\|\leq M_A e^{\omega t}$. Let $\widetilde{M_Q}:=\sup_{0\leq t\leq T}\|Q_\alpha(t)\|_{L(X)}$. So we have $\|Q_\alpha(t)\|_{L(X)}\leq\widetilde{M_Q}$.*

Now, we recall the following important Lemma which is very useful to prove our main result.

Lemma 2.5. *[5] Let $0<\rho<1, \gamma>0$,*

$$S=\rho^n+D_n^1\rho^{n-1}\gamma+\frac{D_n^2\rho^{n-2}\gamma^2}{2!}+\cdots+\frac{\gamma^n}{n!},\quad n\in\mathbb{N}.$$

Then, for all constant $0<\xi<1$ and all real number $s>1$, we get

$$S\leq O\left(\frac{\xi^n}{\sqrt{n}}\right)+O\left(\frac{1}{n^s}\right)=O\left(\frac{1}{n^s}\right),\quad n\to+\infty.$$

Definition 2.6. *A function $x\in PC(J,X)$ is said to be a PC-mild solution of problem (1.1) if it satisfies the following relation:*

$$x(t)=\begin{cases} Q_\alpha(t)x_0+\int_0^t Q_\alpha(t-s)f(s,x(s),G_1x(s),G_2x(s))ds,\ t\in[0,t_1]\\ g_i(t,x(t)),\ t\in(t_i,s_i]\\ Q_\alpha(t-s_i)g_i(s_i,x(s_i))\\ \quad+\int_{s_i}^t Q_\alpha(t-s)f(s,x(s),G_1x(s),G_2x(s))ds,\ t\in(s_i,t_{i+1}]\end{cases}$$

for all $i=1,2,\ldots,m$.

3. Existence Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under general Banach contraction principle fixed point theorem.

From Definition 2.3, we define an operator $\Upsilon: PC(J,X)\to PC(J,X)$ as $(\Upsilon x)(t)=(\Upsilon_1 x)(t)+(\Upsilon_2 x)(t)$, where

$$(\Upsilon_1 x)(t)=\begin{cases} Q_\alpha(t)x_0, & t\in[0,t_1]\\ g_i(t,x(t)), & t\in(t_i,s_i]\\ Q_\alpha(t-s_i)g_i(s_i,x(s_i)), & t\in(s_i,t_{i+1}],\end{cases} \tag{3.1}$$

and

$$(\Upsilon_2 x)(t)=\begin{cases} \int_0^t Q_\alpha(t-s)f(s,x(s),G_1x(s),G_2x(s))ds, & t\in[0,t_1]\\ 0, & t\in(t_i,s_i]\\ \int_{s_i}^t Q_\alpha(t-s)f(s,x(s),G_1x(s),G_2x(s))ds, & t\in(s_i,t_{i+1}].\end{cases} \tag{3.2}$$

To prove our first existence result we introduce the following assumptions:

(H(f)) The function $f \in C(J \times X^3; X)$ and there exist positive constants $L_{f_k} \in L^{\frac{1}{T}}(J, \mathbb{R}^+)\,(k = 1,2,3)$ such that

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \leq L_{f_1}(t)\|x_1 - y_1\| + L_{f_2}(t)\|x_2 - y_2\| + L_{f_3}(t)\|x_3 - y_3\|$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3) \in X$ and every $t \in J$.

$(H(h,\ \widetilde{h}))$ The functions $h, \widetilde{h} : \Delta \times X \to X$ are continuous and there exist constants $L_h, L_{\widetilde{h}} > 0$ such that

$$\left\| \int_0^t [h(t,s,x(s)) - h(t,s,y(s))]ds \right\| \leq L_h \|x - y\|,$$

for all, $x, y \in X$ and

$$\left\| \int_0^T [\widetilde{h}(t,s,x(s)) - \widetilde{h}(t,s,y(s))]ds \right\| \leq L_{\widetilde{h}} \|x - y\|,$$

for all, $x, y \in X$;

(H(g)) For $i = 1,2,\ldots,m$, the functions $g_i \in C([t_i,s_i] \times X; X)$ and there exists $L_{g_i} \in C(J, \mathbb{R}^+)$ such that

$$\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i}\|x - y\|$$

for all $x, y \in X$ and $t \in [t_i, s_i]$.

Theorem 3.1. *If hypotheses* H(f), $H(k,\widetilde{k})$ *and* H(g) *hold and* $0 \leq \Lambda < 1 \left(\Lambda = \max\left\{L_{g_i}, \widetilde{M_Q}L_{g_i}\right\}\right)$, *then problem (1.1) has a unique PC-mild solution* $x^* \in PC(J,X)$.

Proof. For any $x, y \in PC(J,X)$, by (3.1) we sustain

$$\|(\Upsilon_1 x)(t) - (\Upsilon_1 y)(t)\| \leq \begin{cases} 0, & t \in [0,t_1] \\ \Lambda\|x-y\|_{PC}, & t \in (t_i,s_i],\, i = 1,2,\ldots,m, \\ \Lambda\|x-y\|_{PC}, & t \in (s_i,t_{i+1}],\, i = 1,2,\ldots,m, \end{cases} \tag{3.3}$$

which means

$$\|(\Upsilon_1 x)(t) - (\Upsilon_1 y)(t)\| \leq \Lambda\|x-y\|_{PC},$$

where $t \in [0,t_1] \cup (t_i,s_i] \cup (s_i,t_{i+1}],\, i = 1,2,\ldots,m$. Then we obtain

$$\left\|\left(\Upsilon_1^2 x\right)(t) - \left(\Upsilon_1^2 y\right)(t)\right\| \leq \Lambda^2\|x-y\|_{PC},$$

where $t \in [0,t_1] \cup (t_i,s_i] \cup (s_i,t_{i+1}],\, i = 1,2,\ldots,m$. It is clear that, we have

$$\|(\Upsilon_1^n x)(t) - (\Upsilon_1^n y)(t)\| \leq \Lambda^n\|x-y\|_{PC}, \tag{3.4}$$

where $t \in [0,t_1] \cup (t_i,s_i] \cup (s_i,t_{i+1}],\, i = 1,2,\ldots,m$.

For any real number $0 < \varepsilon < 1$, there exists a continuous function $\phi(s)$ such that $\int_0^T |\ \ell(s) - \phi(s)\ |\ ds < \varepsilon$, where $\ell(s) = \widetilde{M_Q}[L_{f_1}(s) + L_{f_2}(s)L_h + L_{f_3}(s)L_{\widetilde{h}}]$ is a Lebesgue integrable function. For any $t \in [0,t_1], x, y \in PC(J,X)$ and by (3.2), we obtain

$$\begin{aligned}
&\|(\Upsilon_2 x)(t) - (\Upsilon_2 y)(t))\| \\
&\leq \int_0^t \|Q_\alpha(t-s)\|\|f(s,x(s),G_1x(s),G_2x(s)) \\
&\quad - f(s,y(s),G_1y(s),G_2y(s))\|ds \\
&\leq \widetilde{M_Q}\int_0^t [L_{f_1}(s) + L_{f_2}(s)L_h + L_{f_3}(s)L_{\widetilde{h}}]\|x(s) - y(s)\|ds \\
&\leq \int_0^t \ell(s)ds\|x-y\|_{PC} \\
&\leq \left(\int_0^t |\ell(s) - \phi(s)|ds + \int_0^t |\phi(s)|ds\right)\|x-y\|_{PC} \\
&\leq (\varepsilon + \lambda t)\|x-y\|_{PC} \\
&= \left(D_1^0\varepsilon^1 + D_1^1\frac{(\lambda t)^1}{1!}\right)\|x-y\|_{PC},
\end{aligned}$$

where $\max_{t\in J}|\phi(t)| = \lambda$.

Assume that, for any natural number k, we get

$$\begin{aligned}
&\|(\Upsilon_2^k x)(t) - (\Upsilon_2^k y)(t))\| \\
&\leq \left(D_k^0\varepsilon^k + D_k^1\varepsilon^{k-1}\frac{(\lambda t)^1}{1!} + \cdots + D_k^k\varepsilon^{k-k}\frac{(\lambda t)^k}{k!}\right)\|x-y\|_{PC}.
\end{aligned}$$

From the above inequality and the formula $D_{k+1}^m = D_k^m + D_k^{m-1}$, we obtain

$$\begin{aligned}
&\|(\Upsilon_2^{k+1}x)(t) - (\Upsilon_2^{k+2}y)(t))\| \\
&\leq \widetilde{M_Q}\int_0^t [L_{f_1}(s) + L_{f_2}(s)L_h + L_{f_3}(s)L_{\widetilde{h}}]\|(\Upsilon_2^k x)(s) - (\Upsilon_2^k)y(s)\|ds \\
&= \int_0^t \ell(s)\|(\Upsilon_2^k x)(s) - (\Upsilon_2^k)y(s)\|ds \\
&\leq \left(\int_0^t |\ell(s) - \phi(s)|\left(D_k^0\varepsilon^k + D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!} + \cdots + D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds\right) \\
&\|x-y\|_{PC} + \left(\int_0^t |\phi(s)|\left(D_k^0\varepsilon^k + D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!} + \cdots + D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds\right) \\
&\quad \|x-y\|_{PC} \\
&\leq \varepsilon\left(D_k^0\varepsilon^k + D_k^1\varepsilon^{k-1}\frac{(\lambda t)^1}{1!} + \cdots + D_k^k\varepsilon^{k-k}\frac{(\lambda t)^k}{k!}\right)\|x-y\|_{PC} \\
&\quad + \lambda\int_0^t \left(D_k^0\varepsilon^k + D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!} + \cdots + D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds\|x-y\|_{PC} \\
&\leq \left(D_{k+1}^0\varepsilon^{k+1} + D_{k+1}^1\varepsilon^k\frac{(\lambda t)^1}{1!} + \cdots + D_{k+1}^{k+1}\varepsilon^{(k+1)-(k+1)}\frac{(\lambda t)^{k+1}}{(k+1)!}\right)\|x-y\|_{PC}.
\end{aligned}$$

By mathematical methods of induction, for any natural number n, we get

$$\begin{aligned}
&\|\Upsilon_2^n x - \Upsilon_2^n y\|_{PC} \\
&\leq \left(D_n^0\varepsilon^n + D_n^1\varepsilon^{n-1}\frac{\zeta^1}{1!} + \cdots + D_n^n\varepsilon^{n-n}\frac{\zeta^n}{n!}\right)\|x-y\|_{PC},
\end{aligned}$$

where $\zeta = \lambda T$. By Lemma 2.2, we have

$$\|\Upsilon_2^n x - \Upsilon_2^n y\|_{PC} \leq \left[O\left(\frac{\eta^n}{\sqrt{n}}\right) + O\left(\frac{1}{h^\mu}\right)\right] \|x-y\|_{PC} = O\left(\frac{1}{n^\mu}\right)\|x-y\|_{PC}, \quad (n \to +\infty), \tag{3.5}$$

where $0 < \eta < 1, \mu > 1$. It is easy to see that the above equation (3.5) holds for $t \in (s_i, t_{i+1}], i = 1, 2, \dots, m$. By (3.4) and (3.5), we obtain

$$\|\Upsilon^n x - \Upsilon^n y\|_{PC} \leq \left(\Lambda^n + O\left(\frac{1}{n^\mu}\right)\right)\|x-y\|_{PC}, \quad \forall n > n_0.$$

Thus, for any fixed constant $\mu > 1$, we can find a positive integer n_0 such that, for any $n > n_0$, we get $0 < \Delta^n + \frac{1}{n^\mu} < 1$. Therefore, for any $x, y \in PC(J, X)$, we have

$$\|\Upsilon^n x - \Upsilon^n y\|_{PC} \leq \left(\Lambda^n + \frac{1}{n^\mu}\right)\|x-y\|_{PC} \leq \|x-y\|_{PC}, \quad \forall n > n_0.$$

By the general Banach contraction mapping principle, we get that the operator Υ has a unique fixed point $x^* \in PC(J, X)$, which means that problem (1.1) has a unique PC-mild solution. □

4. Application

In this section, we are going to present an example to validate the results established in Theorem 3.1. Consider the fractional partial integro-differential equation of the form

$$
\begin{aligned}
{}^c D^\alpha x(t,z) &= \frac{\partial^2 x}{\partial z^2} + J^{1-\alpha} P\left(t, x(t,z), \int_0^t h(t,s,x(s,z))ds, \int_0^T \widetilde{h}(t,s,x(s,z))ds\right), \\
&\qquad a.e. (t,z) \in \cup_{i=1}^m (s_i, t_{i+1}] \times [0,\pi] && (4.1)\\
x(t,0) &= x(t,\pi) = 0,\ t \in [0,T] && (4.2)\\
x(0,z) &= x_0(z),\ z \in [0,\pi] && (4.3)\\
x(t,z) &= G_i(t, x(t,z)), \quad t \in (t_i, s_i],\ z \in [0,\pi], \\
&\qquad i = 1, 2, \dots, N, && (4.4)
\end{aligned}
$$

where $0 = t_0 = s_0 < t_1 \leq s_1 \cdots < t_m \leq s_m < t_{m+1} = T, P \in C\left([0,T] \times \mathbb{R}^3, \mathbb{R}\right)$ and $G_i \in C\left((t_i, s_i] \times \mathbb{R}, \mathbb{R}\right)$ for all $i = 1, 2, \dots, m$.

Let $X = L^2([0,\pi])$. Define an operator $A : D(A) \subseteq X \to X$ by $Ax = x''$ with $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$ in X. Moreover, the subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator $\{Q_\alpha(t)\}_{t\geq 0}$ such that $\|Q_\alpha(t)\|_{L(X)} \leq \widetilde{Q_S}$ for $t \in [0,1]$.

Set

$$
\begin{aligned}
x(t)z &= x(t,z), J^{1-\alpha} f(t, x(t), G_1 x(t), G_2 x(t)) z \\
&= J^{1-\alpha} P\left(t, x(t,z), \int_0^t h(t,s,x(s,z))ds, \int_0^T \widetilde{h}(t,s,x(s,z))ds\right) \\
g_i(t, x(t))z &= G_i(t, x(t,z)).
\end{aligned}
$$

Thus with this set-up, equations (4.1)-(4.4) can be written in the abstract form for the system (1.1).

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