



# Infinite horizon optimal control of mean-field type stochastic partial differential equation with Poisson jumps

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## Abstract

The aim of this paper is to investigate the optimal control of infinite horizon mean-field type stochastic partial differential equation with Poisson jumps. In contrast to finite horizon case, optimality conditions are established through transversality condition. Further, the stochastic maximum principle for optimality is examined under convexity assumption on the control domain, which guarantees the existence of optimal control to concerned system. The necessary condition for optimality is also established. Finally, the theoretical study is discussed through an example of stochastic optimal harvesting problem.

## Keywords

Infinite-horizon optimal control, Mean field theory, Stochastic maximum principle, Stochastic partial differential equation, Poisson jump processes.

## AMS Subject Classification

35B50, 35R60, 93E20.

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## 1. Introduction

Stochastic partial differential equation is defined as

$$dY(t, x) = \{LY(t, x) + b(t, x, Y(t, x), u(t, x))\} dt + \sigma(t, x, Y(t, x), u(t, x))dW(t), \quad (1.1)$$

with the initial condition  $Y(0, x) = y_0(x)$ ,  $x \in \mathcal{G}$ , here  $\mathcal{G}$  is compact subset of  $\mathbb{R}^n$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. The function  $Y : [0, \infty) \times \mathcal{G} \rightarrow \mathbb{R}$  represents the state of the system (1.1). Coefficients  $b, \sigma$  are real valued functions on  $[0, \infty) \times \mathcal{G} \times \mathbb{R} \times \mathcal{U} \times \Omega$  and  $L$  is a partial differential operator with respect to  $x$ .

The process  $W(t) = W(t, \omega)$ ;  $t \geq 0$ ,  $\omega \in \Omega$  is the one dimensional Brownian motion. The control domain  $\mathcal{U}$  is

a nonempty convex subset of  $\mathbb{R}^k$ . The corresponding cost functional of system (1.1) for a finite time  $T > 0$  is as follows:

$$J(u) = E \left[ \int_0^T \left( \int_{\mathcal{G}} f(t, x, Y(t, x), u(t, x)) dx \right) dt \right], \quad (1.2)$$

where  $f : [0, \infty) \times \mathcal{G} \times \mathbb{R} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ .

As an increasing applications of optimal control of stochastic partial differential equation model (1.1)-(1.2), there are numerous works have been reported in literature [8, 10, 14, 15]. In order to resolve a problem of optimal harvesting from a system described by a stochastic reaction-diffusion equation, Ok-sendal [14] proposed a sufficient maximum principle for the optimal control of system modeled through the quasi-linear stochastic heat equation. Further, the existence of stochastic partial differential equation with Poisson jump processes and its optimal control have been discussed by various authors [5, 11, 13]. In particular, Pomen et al. [11] studied the optimal control of stochastic partial differential equation with Poisson jumps through Malliavin calculus approach.

Meanwhile, mean-field partial differential equation models of large number of stochastic systems with average effect

of interaction have been studied in literature [2],[17],[6], [16]. A maximum principle for optimal control of mean-field type stochastic systems have established in [7], [12]. However there is no existing work on optimal control of mean-field stochastic partial differential equation with Poisson jumps models. Hence, there is a need for studying these models with the wide applications in mechanics, quantum physics, diffusion model of heating/cooling loads, metal-superconductor transition. Further, optimal control problem of finding the behavior of admissible trajectories are not always having finite time, it may infinite time. Moreover, optimal control of infinite horizon stochastic system have been studied in [1, 3, 4, 9]. These studies motivated the research direction to construct the system of infinite horizon mean-field type stochastic partial differential equation and to investigate their optimal control.

Let us define the system of infinite horizon mean-field type stochastic partial differential equation with Poisson jump processes as

$$\begin{aligned} dY(t,x) &= \{LY(t,x) + b(t,x,Y(t,x),E[Y(t,x)],u(t,x))\} dt \\ &+ \sigma(t,x,Y(t,x),E[Y(t,x)],u(t,x))dW(t) \\ &+ \int_{\mathbb{R}_0} \theta(t,x,Y(t,x),E[Y(t,x)],u(t,x),a)\tilde{N}(dt,da), \\ Y(0,x) &= y_0(x), \quad x \in \mathcal{G}, \end{aligned} \tag{1.3}$$

where,  $(t,x) \in [0,\infty) \times \mathcal{G}$ ,  $Y(t,x)$  and  $E[Y(t,x)]$  are state processes, the process  $E[Y(t,x)]$  is a function from  $[0,\infty) \times \mathcal{G}$  to  $\mathbb{R}$ , where expectation  $E$  denotes the average behavior of the state process. Coefficients  $b, \sigma$  are real valued functions on  $[0,\infty) \times \mathcal{G} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega$ ,  $\theta$  is also real valued function on  $[0,\infty) \times \mathcal{G} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega$  and  $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ .  $\tilde{N}(dt,da) = N(dt,da) - \nu(da)dt$  is an independent compensated Poisson random measure, where  $\nu$  is the Levy measure of  $N$ . Assume that  $\{\mathcal{F}_t, t \geq 0\}$  is  $\mathbb{P}$ - augmentation of the natural filtration associated with Brownian motion and Poisson random measure which is defined from the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

An admissible control process  $u(t,x) = u(t,x,\omega)$  is defined as  $\mathcal{F}_t$ -predictable process with values in  $\mathcal{U}$  such that,

$$E \left[ \int_0^\infty |u(t,x)|^2 dt \right] < \infty.$$

The set of all admissible control processes  $u(t,x)$  is denoted as  $\mathcal{A} \subset \mathcal{U}$ . The cost functional associated with system (1.3) has the following form

$$J(u) = E \left[ \int_0^\infty \left( \int_{\mathcal{G}} f(t,x,Y(t,x),E[Y(t,x)],u(t,x)) dx \right) dt \right], \tag{1.4}$$

where  $f: [0,\infty) \times \mathcal{G} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$  and  $f$  satisfying

$$E \left[ \int_0^\infty |f(t,x,Y(t,x),E[Y(t,x)],u(t,x))| dt \right] < \infty, \quad \forall u \in \mathcal{A}.$$

The aim of our optimal control problem (1.3)-(1.4) is to examine the existence of optimal control  $u^*(t,x) \in \mathcal{A}$  such

that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

Let us define the Hamiltonian  $H: [0,\infty) \times \mathcal{G} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \rightarrow \mathbb{R}$  as follows:

$$\begin{aligned} H(t,x,Y(t,x),E[Y(t,x)],u(t,x),\mathcal{P}(t,x),\mathcal{Q}(t,x),\mathcal{R}(t,x,a)) \\ = f(t,x,Y(t,x),E[Y(t,x)],u(t,x)) \\ + \mathcal{P}(t,x)b(t,x,Y(t,x),E[Y(t,x)],u(t,x)) \\ + \mathcal{Q}(t,x)\sigma(t,x,Y(t,x),E[Y(t,x)],u(t,x)) \\ + \int_{\mathbb{R}_0} \mathcal{R}(t,x,a)\theta(t,x,Y(t,x),E[Y(t,x)],u(t,x),a)\nu(da). \end{aligned} \tag{1.5}$$

Henceforth the following shorthand notation is used for  $H$  defined in (1.5):

$$\begin{aligned} H(t,x) &= f(t,x) + \mathcal{P}(t,x)b(t,x) + \mathcal{Q}(t,x)\sigma(t,x) \\ &+ \int_{\mathbb{R}_0} \mathcal{R}(t,x,a)\theta(t,x)\nu(da), \end{aligned}$$

where  $\mathcal{R}$  is set of all functions form  $\mathbb{R}_0$  to  $\mathbb{R}$ . For our convenience, let us denote  $y = Y(t,x)$ ,  $\bar{y} = E[Y(t,x)]$  in  $\frac{\partial H(t)}{\partial y}$  and  $\frac{\partial H(t)}{\partial \bar{y}}$  respectively. The adjoint forward stochastic differential equation for (1.3)-(1.4) is of the following form:

$$\begin{aligned} d\mathcal{P}(t,x) &= - \left\{ \frac{\partial H(t)}{\partial y} + E \left[ \frac{\partial H(t)}{\partial \bar{y}} \right] + L' \mathcal{P}(t,x) \right\} dt + \mathcal{Q}(t,x)dW(t) \\ &+ \int_{\mathbb{R}_0} \mathcal{R}(t,x,a)\tilde{N}(dt,da); \quad 0 \leq t \leq \infty, x \in \mathcal{G}, \\ \mathcal{P}(0,x) &= 0, \end{aligned} \tag{1.6}$$

here  $L'$  denotes the adjoint operator of  $L$ , which is defined as  $(L'\phi_1, \phi_2) = (\phi_1, L\phi_2)$ , for all  $\phi_1, \phi_2 \in C_0^\infty(\mathcal{G})$ , where the set of all infinitely many times differentiable functions with compact support in  $\mathcal{G}$  is denoted by  $C_0^\infty(\mathcal{G})$ . Also  $(\phi_1, \phi_2) = \int_{\mathcal{G}} \phi_1(x)\phi_2(x)dx$  denotes the inner product in  $\mathcal{L}^2(\mathcal{G})$ .

This paper is organized as follows: Section 2 reveals the sufficient maximum principle for optimality of infinite horizon mean-field type stochastic partial differential equation. Section 3 presents the necessary condition for optimality of infinite horizon mean-field type stochastic partial differential equation. Application of the theoretical results are illustrated in Section 4.

## 2. Sufficient maximum principle for optimality

This section narrates the stochastic maximum principle for infinite horizon mean-field type stochastic partial differential equation through Ito formula, convexity assumptions on the control domain and concavity condition on Hamiltonian function.

**Theorem 2.1.** *Let  $\hat{u} \in \mathcal{A}$ , the corresponding state processe  $\hat{Y}(t,x)$  and the adjoint processes  $\hat{\mathcal{P}}(t)$ ,  $\hat{\mathcal{Q}}(t)$ ,  $\hat{\mathcal{R}}(t)$  are assumed to satisfy the adjoint forward stochastic differential equation (1.6). If the following assumptions holds,*



( $\mathcal{A}_1$ ) *Concavity*  
*The Hamiltonian*

$$(Y(t,x), E[Y(t,x)], u(t,x)) \mapsto H(t,x, Y(t,x), E[Y(t,x)], u(t,x), \hat{\mathcal{P}}(t,x), \hat{\mathcal{Q}}(t,x), \hat{\mathcal{R}}(t,x,a)),$$

is concave for all  $(t,x) \in [0, \infty) \times \mathcal{G}$ .

( $\mathcal{A}_2$ ) *Conditional maximum principle*

$$E \left[ H(t,x, \hat{Y}(t,x), E[\hat{Y}(t,x)], \hat{u}(t,x), \hat{\mathcal{P}}(t,x), \hat{\mathcal{Q}}(t,x), \hat{\mathcal{R}}(t,x,a)) \middle| \mathcal{F}_t \right] \\ = \max_{u \in \mathcal{U}} E \left[ H(t,x, \hat{Y}(t,x), E[\hat{Y}(t,x)], u(t,x), \hat{\mathcal{P}}(t,x), \hat{\mathcal{Q}}(t,x), \hat{\mathcal{R}}(t,x,a)) \middle| \mathcal{F}_t \right],$$

where  $\mathcal{F}_t$  for all  $(t,x) \in [0, \infty) \times \mathcal{G}$  be a given filtration, representing the information available to the controller at time  $t$ .

( $\mathcal{A}_3$ ) *Transversality condition*

$$\lim_{T \rightarrow \infty} E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T,x) (Y(T,x) - \hat{Y}(T,x)) dx \right] \geq 0.$$

then  $\hat{u}$  is the optimal control for the system (1.3)-(1.4),

$$\text{i.e., } J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u).$$

*Proof.* In order to show  $\hat{u}$  is an optimal control for the system (1.3)-(1.4), it is enough to prove that  $J(u) - J(\hat{u}) \leq 0$ . Let

$$\mathcal{J} = J(u) - J(\hat{u}) = E \int_0^\infty \left( \int_{\mathcal{G}} (f(t,x) - \hat{f}(t,x)) dx \right) dt, \quad (2.1)$$

by the definition of  $H$  gives,

$$f(t,x) - \hat{f}(t,x) = (H(t,x) - \hat{H}(t,x)) - \hat{\mathcal{P}}(t,x)(b(t,x) - \hat{b}(t,x)) - \hat{\mathcal{Q}}(t,x)(\sigma(t,x) - \hat{\sigma}(t,x)) - \int_{\mathbb{R}_0} \hat{\mathcal{R}}(t,x,a)(\theta(t,x,a) - \hat{\theta}(t,x,a)) \nu(da). \quad (2.2)$$

Since  $H$  is concave, that is

$$H(t,x) - \hat{H}(t,x) \leq \frac{\partial H(t,x)}{\partial y} (Y(t,x) - \hat{Y}(t,x)) + E \left[ \frac{\partial H(t,x)}{\partial \bar{y}} \right] E[(Y(t,x) - \hat{Y}(t,x))] + \frac{\partial H(t,x)}{\partial u} (u(t,x) - \hat{u}(t,x)), \quad (2.3)$$

using (2.2), (2.3) in (2.1), it can be written as

$$\mathcal{J} = E \int_0^\infty \left( \int_{\mathcal{G}} \left\{ \frac{\partial H(t,x)}{\partial y} (Y(t,x) - \hat{Y}(t,x)) + E \left[ \frac{\partial H(t,x)}{\partial \bar{y}} \right] E[(Y(t,x) - \hat{Y}(t,x))] + \frac{\partial H(t,x)}{\partial u} (u(t,x) - \hat{u}(t,x)) - \hat{\mathcal{P}}(t,x)(b(t,x) - \hat{b}(t,x)) - \hat{\mathcal{Q}}(t,x)(\sigma(t,x) - \hat{\sigma}(t,x)) - \int_{\mathbb{R}_0} \hat{\mathcal{R}}(t,x,a)(\theta(t,x,a) - \hat{\theta}(t,x,a)) \right\} dx \right) dt. \quad (2.4)$$

Take  $\int_{\mathcal{G}} \hat{\mathcal{P}}(t,x)(Y(t,x) - \hat{Y}(t,x)) dx$ , using Ito formula for the processes  $\hat{\mathcal{P}}(t,x)(Y(t,x) - \hat{Y}(t,x))$  on  $[0, T]$ , the equations (1.3), (1.6) and applying expectation, one can get the expression of the following form,

$$E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T,x) (Y(T,x) - \hat{Y}(T,x)) dx \right] \\ = E \left[ \int_{\mathcal{G}} \left\{ \hat{\mathcal{P}}(0,x) (Y(0,x) - \hat{Y}(0,x)) + \int_0^T \hat{\mathcal{P}}(t,x) d(Y(t,x) - \hat{Y}(t,x)) + \int_0^T d\hat{\mathcal{P}}(t,x) (Y(t,x) - \hat{Y}(t,x)) + \int_0^T \hat{\mathcal{Q}}(t,x) (\sigma(t,x) - \hat{\sigma}(t,x)) dt + \int_0^T \int_{\mathbb{R}_0} \hat{\mathcal{R}}(t,a) (\theta(t,x,a) - \hat{\theta}(t,x,a)) \nu(da) dt \right\} dx \right], \quad (2.5)$$

here

$$\int_0^T \hat{\mathcal{P}}(t,x) d(Y(t,x) - \hat{Y}(t,x)) \\ = \int_0^T \hat{\mathcal{P}}(t,x) \{ L(Y(t,x) - \hat{Y}(t,x)) + (b(t,x) - \hat{b}(t,x)) \} dt, \quad (2.6)$$

$$\int_0^T d\hat{\mathcal{P}}(t,x) (Y(t,x) - \hat{Y}(t,x)) \\ = - \int_0^T \left( \frac{\partial \hat{H}(t)}{\partial y} + E \left[ \frac{\partial \hat{H}(t)}{\partial \bar{y}} \right] + L' \hat{\mathcal{P}}(t,x) \right) (Y(t,x) - \hat{Y}(t,x)) dt. \quad (2.7)$$

Substitute the equations (2.6), (2.7) in (2.5) and taking limit as  $T \rightarrow \infty$  gives

$$\lim_{T \rightarrow \infty} E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T,x) (Y(T,x) - \hat{Y}(T,x)) dx \right] \\ = E \left[ \int_{\mathcal{G}} \left\{ \int_0^\infty \hat{\mathcal{P}}(t,x) L(Y(t,x) - \hat{Y}(t,x)) dt + \int_0^\infty \hat{\mathcal{P}}(t,x) (b(t,x) - \hat{b}(t,x)) dt - \int_0^\infty \frac{\partial \hat{H}(t)}{\partial y} (Y(t,x) - \hat{Y}(t,x)) dt - \int_0^\infty E \left[ \frac{\partial \hat{H}(t)}{\partial \bar{y}} \right] (E[Y(t,x)] - E[\hat{Y}(t,x)]) dt - \int_0^\infty (Y(t,x) - \hat{Y}(t,x)) L' \hat{\mathcal{P}}(t,x) dt + \int_0^\infty \hat{\mathcal{Q}}(t,x) (\sigma(t,x) - \hat{\sigma}(t,x)) dt + \int_0^\infty \int_{\mathbb{R}_0} \hat{\mathcal{R}}(t,x,a) (\theta(t,x,a) - \hat{\theta}(t,x,a)) \nu(da) dt \right\} dx \right]. \quad (2.8)$$

Adding & subtracting  $\int_{\mathcal{G}} \int_0^\infty \frac{\partial \hat{H}(t,x)}{\partial u} (u(t,x) - \hat{u}(t,x)) dt$  in equation (2.8), using assumption ( $\mathcal{A}_3$ ) and (2.4) which gives (2.8) becomes

$$0 \leq -J + E \left[ \int_0^\infty \left\{ \int_{\mathcal{G}} \left( \hat{\mathcal{P}}(t,x) L(Y(t,x) - \hat{Y}(t,x)) - (Y(t,x) - \hat{Y}(t,x)) L' \hat{\mathcal{P}}(t,x) \right) dx \right\} dt \right] + E \left[ \int_{\mathcal{G}} \left\{ \int_0^\infty \left[ \frac{\partial \hat{H}(t,x)}{\partial u} \middle| \mathcal{F}_t \right] (u(t,x) - \hat{u}(t,x)) dt \right\} dx \right].$$



By the first Green formula, there exist first order boundary differential operator  $\mathbb{A}_1, \mathbb{A}_2$  such that

$$\int_{\mathcal{G}} \left\{ \hat{\mathcal{P}}(t,x)L(Y(t,x) - \hat{Y}(t,x)) - (Y(t,x) - \hat{Y}(t,x))L' \mathcal{P}(t,x) \right\} dx = \int_{\partial \mathcal{G}} \left\{ \hat{\mathcal{P}}(t,x)\mathbb{A}_1(Y(t,x) - \hat{Y}(t,x)) - (Y(t,x) - \hat{Y}(t,x))\mathbb{A}_2 \mathcal{P}(t,x) \right\} dS,$$

where the integral on the right is the surface integral over  $\partial \mathcal{G}$ , we have

$$\int_{\mathcal{G}} \left\{ \hat{\mathcal{P}}(t,x)L(Y(t,x) - \hat{Y}(t,x)) - (Y(t,x) - \hat{Y}(t,x))L' \mathcal{P}(t,x) \right\} dx = 0, \tag{2.9}$$

where, for all  $t \in (0, \infty)$ . Substitute (2.9) in equation (2) which becomes,

$$\mathcal{J} \leq E \left[ \int_{\mathcal{G}} \left\{ \int_0^\infty \left[ \frac{\partial \hat{H}(t,x)}{\partial u} \Big|_{\mathcal{F}_t} \right] (u(t,x) - \hat{u}(t,x)) dt \right\} dx \right], \leq 0.$$

Thus  $\mathcal{J} = J(u) - J(\hat{u}) \leq 0$ , since  $u \in \mathcal{A}$  is arbitrary.  $\square$

### 3. Necessary condition for optimality

This section presents the necessary condition for the optimal control problem (1.3)-(1.4). The following assumptions will make the proof is easier one.

**Assumptions:**

- (A4) For all  $t_0 \in (0, \infty), h > 0$  and all bounded  $\varepsilon_{t_0}$  - measurable random variables  $\alpha$ , the control processes  $\beta(t, x)$  defined by  $\beta(t, x) = \alpha I_{[t_0, t_0+h)}(t) \in \mathcal{A}$ , where  $I$  is an indicator function. Also  $\Theta(s) = J(\hat{u} + s\omega)$  is maximal at  $s = 0$ . i.e.,

$$\Theta'(s) = \frac{d}{ds} (J(\hat{u} + s\omega)) = 0.$$

- (A5) For all  $u, \beta \in \mathcal{A}$  with  $\beta$  is bounded, there exists  $\varepsilon > 0$  such that,  $u(t, x) + s\beta(t, x) \in \mathcal{A}, \forall s \in (-\varepsilon, \varepsilon), t \in [0, \infty)$ , and the following derivative processes are exist,

$$\xi(t, x) = \xi^\beta(t, x) = \frac{d}{ds} Y^{u+s\beta}(t, x),$$

$$L\xi(t, x) = \xi^\beta(t, x) = \frac{d}{ds} LY^{u+s\beta}(t, x).$$

- (A6)  $\lim_{T \rightarrow \infty} E [\hat{\mathcal{P}}(T, x)\xi(T, x)] = 0$ .

**Theorem 3.1.** Suppose the assumptions (A4) – (A6) holds, then there exist a unique adjoint processes  $\mathcal{P}(\cdot), \mathcal{Q}(\cdot), \mathcal{R}(\cdot)$  is the solution of the adjoint backward stochastic differential equation (1.6) such that  $\hat{u}(t)$  is the optimal control for the system (1.3)-(1.4) then we have

$$E \left[ \frac{\partial H}{\partial u}(t, x, \hat{Y}(t, x), E[\hat{Y}(t, x)], u(t, x), \hat{\mathcal{P}}(t, x), \hat{\mathcal{Q}}(t, x), \hat{\mathcal{R}}(t, x, a)) \Big| \mathcal{F}_t \Big]_{u=\hat{u}} = 0.$$

*Proof.* Let  $\xi(t, x)$  follows the following stochastic partial differential equation

$$d\xi(t, x) = \left( L\xi(t, x) + \frac{\partial b(t, x)}{\partial y} \xi(t, x) + E \left[ \frac{\partial b(t, x)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial b(t, x)}{\partial u} \beta(t, x) \right) dt + \left( \frac{\partial \sigma(t, x)}{\partial y} \xi(t, x) + E \left[ \frac{\partial \sigma(t, x)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial \sigma(t, x)}{\partial u} \beta(t, x) \right) dW(t) + \int_{\mathbb{R}_0} \left( \frac{\partial \theta(t, x, a)}{\partial y} \xi(t, x) + E \left[ \frac{\partial \theta(t, x, a)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial \theta(t, x, a)}{\partial u} \beta(t, x) \right) \tilde{N}(dt, da),$$

$$\xi(0, x) = 0. \tag{3.1}$$

By using the assumption (A4),

$$0 = \frac{d}{ds} J(u + s\beta) \Big|_{s=0}, = E \left[ \int_0^\infty \left\{ \int_{\mathcal{G}} \left( \frac{\partial f(t, x)}{\partial y} \xi(t, x) + E \left[ \frac{\partial f(t, x)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial f(t, x)}{\partial u} \beta(t, x) \right) dx \right\} dt \right], \tag{3.2}$$

from the Hamiltonian in (1.5),

$$\frac{\partial f(t, x)}{\partial y} = \frac{\partial H(t, x)}{\partial y} - \hat{\mathcal{P}}(t, x) \frac{\partial b(t, x)}{\partial y} - \mathcal{Q}(t, x) \frac{\partial \sigma(t, x)}{\partial y} - \int_{\mathbb{R}_0} \mathcal{R}(t, x, a) \frac{\partial \theta(t, x, a)}{\partial y} \nu(da), \tag{3.3}$$

and for  $\frac{\partial f(t, x)}{\partial u}, E \left[ \frac{\partial f(t, x)}{\partial \bar{y}} \right]$ .

Now taking  $\int_{\mathcal{G}} \hat{\mathcal{P}}(T, x)\xi(T, x)$ , apply Ito's formula for the process  $\hat{\mathcal{P}}(T, x)\xi(T, x)$  on  $[0, T]$ , the equations (1.6), (3.1) and taking expectation which gives,

$$E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T, x)\xi(T, x) \right] = E \left[ \int_{\mathcal{G}} \left\{ \hat{\mathcal{P}}(0, x)\xi(0, x) + \int_0^T \hat{\mathcal{P}}(t, x)d\xi(t, x) dt + \int_0^T \xi(t, x)d\hat{\mathcal{P}}(t, x) dt + \int_0^T \mathcal{Q}(t, x) \left( \frac{\partial \sigma(t, x)}{\partial y} \xi(t, x) + E \left[ \frac{\partial \sigma(t, x)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial \sigma(t, x)}{\partial u} \beta(t, x) \right) dt + \int_0^T \int_{\mathbb{R}_0} \mathcal{R}(t, x, a) \left( \frac{\partial \theta(t, x, a)}{\partial y} \xi(t, x) + E \left[ \frac{\partial \theta(t, x, a)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial \theta(t, x, a)}{\partial u} \beta(t, x) \right) \nu(da) \right\} dx \right], \tag{3.4}$$

here

$$\int_0^T \hat{\mathcal{P}}(t, x)d\xi(t, x) dt = \int_0^T \hat{\mathcal{P}}(t, x) \left( L\xi(t, x) + \frac{\partial b(t, x)}{\partial y} \xi(t, x) + E \left[ \frac{\partial b(t, x)}{\partial \bar{y}} \right] E[\xi(t, x)] + \frac{\partial b(t, x)}{\partial u} \beta(t, x) \right) dt, \tag{3.5}$$



$$\int_0^T \xi(t,x)d\hat{\mathcal{P}}(t,x)dt = \int_0^T \hat{\xi}(t,x) \left( -\frac{\partial H(t)}{\partial y} - E \left( \frac{\partial H(t)}{\partial \bar{y}} \right) - L' \mathcal{P}(t,x) \right) dt. \quad (3.6)$$

Using the equations (3.5), (3.6) in (3.4) and taking limit as  $T \rightarrow \infty$  which becomes

$$\begin{aligned} & \lim_{T \rightarrow \infty} E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T,x)\xi(T,x) \right] \\ &= E \left[ \int_{\mathcal{G}} \left\{ \int_0^\infty \hat{\mathcal{P}}(t,x) \left( L\xi(t,x) + \frac{\partial b(t,x)}{\partial y} \xi(t,x) \right. \right. \right. \\ & \quad + E \left[ \frac{\partial b(t,x)}{\partial \bar{y}} \right] E[\xi(t,x)] + \frac{\partial b(t,x)}{\partial u} \beta(t,x) \left. \left. \left. \right) dt \right. \right. \\ & \quad + \int_0^\infty \hat{\xi}(t,x) \left( -\frac{\partial H(t)}{\partial y} - E \left( \frac{\partial H(t)}{\partial \bar{y}} \right) - L' \mathcal{P}(t,x) \right) dt \\ & \quad + \int_0^\infty \mathcal{Q}(t,x) \left( \frac{\partial \sigma(t,x)}{\partial y} \xi(t,x) + E \left[ \frac{\partial \sigma(t,x)}{\partial \bar{y}} \right] E[\xi(t,x)] \right. \\ & \quad + \frac{\partial \sigma(t,x)}{\partial u} \beta(t,x) \left. \right) dt + \int_0^\infty \int_{\mathbb{R}_0} \mathcal{R}(t,x,a) \left( \frac{\partial \theta(t,x,a)}{\partial y} \xi(t,x) \right. \\ & \quad \left. \left. + E \left[ \frac{\partial \theta(t,x,a)}{\partial \bar{y}} \right] E[\xi(t,x)] + \frac{\partial \theta(t,x,a)}{\partial u} \beta(t,x) \right) v(da) \right\} dx \right] \\ &= E \int_0^\infty \left\{ \int_{\mathcal{G}} \left( \mathcal{P}(t,x)L\xi(t,x) - \xi(t,x)L' \mathcal{P}(t,x) \right) dx \right\} dt \\ & \quad + \int_{\mathcal{G}} \left( \int_0^\infty \left\{ \left( -\frac{\partial H(t,x)}{\partial y} + \hat{\mathcal{P}}(t,x) \frac{\partial b(t,x)}{\partial y} \right. \right. \right. \\ & \quad \left. \left. \left. + \mathcal{Q}(t,x) \frac{\partial \sigma(t,x)}{\partial y} + \int_{\mathbb{R}_0} \mathcal{R}(t,x,a) \frac{\partial \theta(t,x,a)}{\partial y} v(da) \right) \xi(t,x) \right. \right. \\ & \quad \left. \left. + \left( -E \left[ \frac{\partial H(t,x)}{\partial \bar{y}} \right] + \hat{\mathcal{P}}(t,x) E \left[ \frac{\partial b(t,x)}{\partial \bar{y}} \right] \right) \right. \right. \\ & \quad \left. \left. + \mathcal{Q}(t,x) E \left[ \frac{\partial \sigma(t,x)}{\partial \bar{y}} \right] + \int_{\mathbb{R}_0} \mathcal{R}(t,x,a) E \left[ \frac{\partial \theta(t,x,a)}{\partial \bar{y}} \right] v(da) \right) \right. \right. \\ & \quad \left. \left. \times E[\xi(t,x)] + \left( \hat{\mathcal{P}}(t,x) \frac{\partial b(t,x)}{\partial u} + \mathcal{Q}(t,x) \frac{\partial \sigma(t,x)}{\partial u} \right. \right. \right. \\ & \quad \left. \left. \left. + \int_{\mathbb{R}_0} \mathcal{R}(t,x,a) \frac{\partial \theta(t,x,a)}{\partial u} v(da) \right) \beta(t,x) \right\} dt \right) dx \\ &= E \int_0^\infty \left\{ \int_{\mathcal{G}} \left( \mathcal{P}(t,x)L\xi(t,x) - \xi(t,x)L' \mathcal{P}(t,x) \right) dx \right\} dt \\ & \quad - \int_{\mathcal{G}} \left( \int_0^\infty \left\{ \frac{\partial f(t,x)}{\partial y} \xi(t,x) + E \left[ \frac{\partial f(t,x)}{\partial \bar{y}} \right] E[\xi(t,x)] \right. \right. \\ & \quad \left. \left. + \frac{\partial f(t,x)}{\partial u} \beta(t,x) \right\} dt \right) dx + \int_{\mathcal{G}} \left( \int_0^\infty \frac{\partial H(t,x)}{\partial u} \beta(t,x) dt \right) dx. \end{aligned} \quad (3.7)$$

Substitute the equation (3.2) in (3.7) which gives

$$\begin{aligned} & \lim_{T \rightarrow \infty} E \left[ \int_{\mathcal{G}} \hat{\mathcal{P}}(T,x)\xi(T,x) \right] \\ &= E \int_0^\infty \left\{ \int_{\mathcal{G}} \left( \mathcal{P}(t,x)L\xi(t,x) - \xi(t,x)L' \mathcal{P}(t,x) \right) dx \right\} dt \\ & \quad + \int_{\mathcal{G}} \left( \int_0^\infty \frac{\partial H(t,x)}{\partial u} \beta(t,x) dt \right) dx, \end{aligned} \quad (3.8)$$

using (2.9) in (3.8) one can get the following equality

$$\lim_{T \rightarrow \infty} E[\hat{\mathcal{P}}(T,x)\xi(T,x)] = \int_{\mathcal{G}} \left( \int_0^\infty \frac{\partial H(t,x)}{\partial u} \beta(t,x) dt \right) dx.$$

By applying the assumption (A6) gives

$$\int_{\mathcal{G}} \left( \int_0^\infty \frac{\partial H(t,x)}{\partial u} \beta(t,x) dt \right) dx = 0.$$

Now apply  $\beta(t) = \alpha I_{[s,s+h)}(t)$ , where  $\alpha$  is bounded and  $\xi_{t_0}$ -measurable,  $s \geq t_0$ ,

$$E \left[ \int_{\mathcal{G}} \left( \int_s^{s+h} \frac{\partial H(s,x)}{\partial u} ds \right) dx \alpha \right] = 0.$$

Differentiating with respect to  $h$  at  $h = 0$ ,

$$E \left[ \int_{\mathcal{G}} \frac{\partial H(s,x)}{\partial u} dx \alpha \right] = 0.$$

Since this holds for all  $s \geq 0$  and all  $\alpha$ , gives

$$E \left[ \frac{\partial H(t_0,x)}{\partial u} \Big| \xi_{t_0} \right] = 0,$$

which completes the proof.  $\square$

## 4. Example

Consider the following mean-field type optimal harvesting problem given by a stochastic reaction-diffusion equation

$$\begin{aligned} dY(t,x) &= \left[ \frac{1}{2} \Delta Y(t,x) + a(t,x)Y(t,x) + b(t,x)E[Y(t,x)] \right. \\ & \quad \left. - u(t,x) \right] dt + c(t,x)dW(t) + \int_{\mathbb{R}_0} \theta(t,x,a)\tilde{N}(dt, da), \\ Y(0,x) &= y_0(x), x \in \mathcal{G}. \end{aligned} \quad (4.1)$$

here

$$\Delta Y(t,x) = \sum_{i=1}^n \frac{\partial^2 Y(t,x)}{\partial x_i^2},$$

where  $u(t,x)$  be the harvesting rate at  $(t,x)$  acting as a control variable of the system (4.1). The coefficients  $a, b, c$  are real valued functions on  $[0, \infty) \times \mathcal{G}$  and  $\theta$  is also a real valued function on  $[0, \infty) \times \mathcal{G} \times \mathbb{R}_0$ . The problem is to maximize total expected utility of the consumption. The corresponding utility functional of system (4.1) as in the following form

$$J(u) = E \left[ \int_0^\infty \left( \int_{\mathcal{G}} \frac{u^\gamma(t,x)}{\gamma} dx \right) dt \right], \quad (4.2)$$

where  $\gamma \in (0, 1)$ . For this problem Hamiltonian in (1.5) is defined as follows

$$\begin{aligned} & H(t,x,Y(t,x), E[Y(t,x)], u(t,x), \mathcal{P}(t,x), \mathcal{Q}(t,x), \mathcal{R}(t,x,a)) \\ &= \frac{u^\gamma}{\gamma} + (a(t,x)Y(t,x) + b(t,x)E[Y(t,x)] \\ & \quad - u(t,x)) \mathcal{P}(t,x) + c(t,x)\mathcal{Q}(t,x) \\ & \quad + \int_{\mathbb{R}_0} \theta(t,x,a)\mathcal{R}(t,x,a)v(da), \end{aligned} \quad (4.3)$$

and assume that  $H$  is concave. The adjoint equation (1.6) becomes,

$$\begin{aligned} d\mathcal{P}(t,x) &= -[a(t,x)\mathcal{P}(t,x) + b(t,x)E[\mathcal{P}(t,x)]] dt \\ & \quad + \mathcal{Q}(t,x)dW(t) + \int_{\mathbb{R}_0} \mathcal{R}(t,x,a)\tilde{N}(dt, da), \\ \mathcal{P}(0,x) &= 0, \end{aligned} \quad (4.4)$$



also assume that this adjoint variable  $\mathcal{P}(t, x)$  satisfies the transversality condition given in  $(\mathcal{A}_3)$ . Differentiating (4.3) with respect to the control variable and equating zero which gives the following:

$$\frac{\partial H}{\partial u} = u^{\gamma-1}(t, x) - \mathcal{P}(t, x) = 0. \quad (4.5)$$

Since  $H$  is concave, the assumption  $(\mathcal{A}_1)$  hold, also  $H$  satisfies the assumption  $(\mathcal{A}_2)$ , then by using Theorem 2.1 of Section 2, one can conclude that there exist  $u^*(t, x)$  which is the optimal control.

The required optimal control  $u^*(t, x)$  for the system (4.1), (4.2) is given by the equation (4.5) as follows:

$$u^*(t, x) = (\mathcal{P}(t, x))^{\frac{1}{\gamma-1}}, \quad (4.6)$$

where  $\mathcal{P}(t, x)$  is the solution of the adjoint equation (4.4). In order to solve the equation (4.4), conjecture the adjoint process to get the adjoint process  $\mathcal{P}(t, x)$  as the following form.

$$\mathcal{P}(t, x) = \Phi_1(t, x)Y(t, x) + \Phi_2(t, x)E[Y(t, x)] + \Phi_3(t, x),$$

where  $\Phi_1(\cdot, \cdot)$ ,  $\Phi_2(\cdot, \cdot)$ ,  $\Phi_3(\cdot, \cdot)$  are deterministic differential functions, one can obtain the values of these coefficients by applying similar procedure given in Section 5 in [7].

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