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Double fuzzy contra-continuous multifunctions

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Abstract

This paper is devoted to the concepts of double fuzzy upper and double fuzzy lower contra-continuous multifunctions. Several characterizations and properties of this multifunctions are established in fuzzy topological spaces.

Keywords

(r,s)-fuzzy open sets, double fuzzy contra-continuous multifunctions.

AMS Subject Classification

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Contents

1. Introduction

The notion of fuzzy set was introduced by L.A. Zadeh in the year 1965. Since than it has been applied in almost all the branches of science and technology, where set theory and mathematical logic play an important role. This newly introduced concept opened lot of scope and directions for investigations in many ways in all the branches for research. Recently fuzzy set theory has been applied and fuzzy topological spaces have been studied by Kubiak [10] and Sostak [18] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang [5] and Goguen [9]). Berge [4] introduced the concept multimapping $F:(X,\tau)\to(Y,\sigma)$ where X and Y are topological spaces. After Chang introduced the concept of fuzzy topology [5], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (e.g. see [2, 3, 12–14]). Tsiporkova et. al., [19, 20] introduced the continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [5]. On the other hand, as a generalization of fuzzy topological spaces Samanta and Mondal [17], introduced the concept of intuitionistic gradation of openness. In 2005, the term intuitionistic is ended by Garcia and Rodabaugh [8]. They proved that the term intuitionistic is unsuitable in mathematics and applications and they replaced it by double. In this paper, we introduce and study the concepts of double fuzzy upper and double fuzzy lower contra-continuous multifunctions. Several characterizations and properties of this multifunctions are established in double fuzzy topological spaces.

2. Preliminaries

Throughout this paper, Let X be a non-empty set, I the unit interval [0, 1], $I_0=(0,1]$ and $I_1=[0,1)$. The family of all fuzzy sets on X is denoted by I^X. By $\bar{0}$ and $\bar{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $\lambda\in I^X$, $\bar{1}-\lambda$ denotes its complement.

Definition 2.1 ([1]). *Let $F:(X,\tau)\to(Y,\sigma)$, then F is called a fuzzy multifunction if, and only if $F(x)\in L^Y$ for each $x\in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y)=G_F(x,y)$ for any $(x,y)\in X\times Y$. The domain of F, denoted by $dom(F)$ and the range of F, denoted by $rng(F)$ for any $x\in X$ and $y\in Y$, are defined by: $dom(F)(x)=\bigvee_{y\in Y}G_F(x,y)$ and $rng(F)(x)=\bigvee_{x\in X}G_F(x,y)$.*

Definition 2.2 ([1]). *Let $F:(X,\tau)\to(Y,\sigma)$ be a fuzzy multifunction. Then F is called:*

1. *normalized if, and only if for each $x\in X$, there exists $y_0\in Y$ such that $G_F(x,y_0)=\bar{1}$.*
2. *a crisp if, and only if $G_F(x,y_0)=\bar{1}$ for each $x\in X$ and $y\in Y$.*

Definition 2.3 ([1]). *Let $F:(X,\tau)\to(Y,\sigma)$ be a fuzzy multifunction. Then*

1. *the lower inverse of* $\mu \in L^Y$ *is an L-fuzzy set* $F^l(\mu) \in L^X$ *defined by* $F^l(\mu)(x) = \bigvee_{y \in Y} (G_F(x,y) \wedge \mu(y))$.
2. *the upper inverse of* $\mu \in L^Y$ *is an L-fuzzy set* $F^u(\mu) \in L^X$ *defined by* $F^u(\mu)(x) = \bigwedge_{y \in Y} (G^c_F(x,y) \vee \mu(y))$.

Definition 2.4 ([1]). *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a fuzzy multifunction. Then*

1. $F(\lambda_1) \leq F(\lambda_2)$ *if* $\lambda_1 \leq \lambda_2$.
2. $F^l(\mu_1) \leq F^l(\mu_2)$ *and* $F^u(\mu_1) \leq F^u(\mu_2)$ *if* $\mu_1 \leq \mu_2$.
3. $F^l(\mu^c) \leq (F^u(\mu))^c$.
4. $F^u(\mu^c) \leq (F^l(\mu))^c$.
5. $F(F^u(\mu)) \leq \mu$ *if F is a crisp.*
6. $F^u(F(\lambda)) \geq \lambda$ *if F is a crisp.*

Definition 2.5 ([1]). *Let* $F : (X, \tau) \to (Y, \sigma)$ *and* $H : (Y, \sigma) \to (Z, \eta)$ *be two fuzzy multifunctions. Then the composition* $H \circ F$ *is defined by* $((H \circ F)(x))(z) = \bigvee_{y \in Y} (G_F(x,y) \wedge G_H(y,z))$.

Theorem 2.6 ([1]). *Let* $F : (X, \tau) \to (Y, \sigma)$ *and* $H : (Y, \sigma) \to (Z, \eta)$ *be two fuzzy multifunctions. Then we have the following*

1. $H \circ F = F(H)$.
2. $(H \circ F)^u = F^u(H^u)$.
3. $(H \circ F)^l = F^l(H^l)$.

Definition 2.7. *A fuzzy point* x_t *in X is a fuzzy set taking value* $t \in I_0$ *at x and zero elsewhere,* $x_t \in \lambda$ *if and only if* $t \leq \lambda(x)$.

Definition 2.8 ([6, 17]). *A double fuzzy topology on X is a pair of maps* $\tau, \tau^\star : I^X \to I$, *which satisfies the following properties:*

1. $\tau(\lambda) \leq \bar{1} - \tau^\star(\lambda)$ *for each* $\lambda \in I^X$.
2. $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ *and* $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ *for each* $\lambda_1, \lambda_2 \in I^X$.
3. $\tau(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} \tau(\lambda_i)$ *and* $\tau^\star(\bigvee_{i \in \Gamma} \lambda_i) \leq \bigvee_{i \in \Gamma} \tau^\star(\lambda_i)$ *for each* $\lambda_i \in I^X, i \in \Gamma$.

The triplet $(X, \tau, \tau^\star)$ *is called a double fuzzy topological space.*

Definition 2.9 ([6, 17]). *A fuzzy set* λ *is called an* (r,s)*-fuzzy open if* $\tau(\lambda, r, s) \geq r$ *and* $\tau^\star(\lambda, r, s) \leq s$, λ *is called an* (r,s)*-fuzzy closed if, and only if* $\bar{1} - \lambda$ *is an* (r,s)*-fuzzy open set.*

Theorem 2.10 ([7, 11]). *Let* $(X, \tau, \tau^\star)$ *be a double fuzzy topological space. Then double fuzzy closure operator and double fuzzy interior operator of* $\lambda \in I^X$ *are defined by*

$$C_{\tau,\tau^\star}(\lambda, r, s) = \bigwedge \{\mu \in I^X \mid \lambda \leq \mu, \tau(\bar{1} - \mu) \geq r, \tau^\star(\bar{1} - \mu) \leq s\},$$

$$I_{\tau,\tau^\star}(\lambda, r, s) = \bigvee \{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq s\},$$

where $r \in I_0$ *and* $s \in I_1$ *such that* $r + s \leq 1$.

Definition 2.11 ([15]). *Let* $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ *be a double fuzzy multifunction and* $r \in I_0$, $s \in I_1$. *Then F is called:*

1. *double fuzzy upper semi-continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$ *and* $\sigma(\mu) \geq r$, $\sigma^\star(\mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(F) \leq F^u(\mu)$.
2. *double fuzzy lower semi-continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$ *and* $\sigma^\star(\mu) \geq r$, $\sigma(\mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \leq F^l(\mu)$.
3. *double fuzzy upper* $(lower)$ *semi-continuous if it is double fuzzy upper* $(lower)$ *semi-continuous at every fuzzy point* $x_t \in dom(F)$.

Definition 2.12 ([16]). *Let* $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ *be a double fuzzy multifunction and* $r \in I_0$, $s \in I_1$. *Then F is called:*

1. *double fuzzy upper weakly continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$ *with* $\sigma(\mu) \geq r$ *and* $\sigma^\star(\mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(F) \leq F^u(C_{\sigma,\sigma^\star}(\mu, r, s))$.
2. *double fuzzy lower weakly continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$, *with* $\sigma(\mu) \geq r$ *and* $\sigma^\star(\mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \leq F^l(C_{\sigma,\sigma^\star}(\mu, r, s))$.
3. *double fuzzy upper (lower) weakly continuous if it is double fuzzy upper (lower,s) weakly continuous at every* $x_t \in dom(F)$.

3. Double fuzzy contra-continuous multifunctions

Definition 3.1. *A double fuzzy multifunction* $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, $r \in L_0$, $s \in I_1$ *is said to be*

1. *double fuzzy upper contra-continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\bar{1} - \mu) \geq r$ *and* $\sigma^\star(\bar{1} - \mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(F) \leq F^u(\mu)$.
2. *double fuzzy lower contra-continuous at a fuzzy point* $x_t \in dom(F)$ *if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$ *and* $\sigma(\bar{1} - \mu) \geq r$ *and* $\sigma^\star(\bar{1} - \mu) \leq s$, *there exists an* (r,s)*-fuzzy open set* $\lambda \in I^X$ *and* $x_t \in \lambda$ *such that* $\lambda \leq F^l(\mu)$.
3. *double fuzzy upper (lower) contra-continuous if it is double fuzzy upper (lower) contra-continuous at every fuzzy point* $x_t \in dom(F)$.

Proposition 3.2. *If F is normalized, then* $F : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ *is a double fuzzy upper contra-continuous multifunction at a fuzzy point* $x_t \in dom(F)$ *if, and only if* $x_t \in F^u(\mu)$ *for each* $\mu \in I^Y$ *and* $\sigma(\bar{1} - \mu) \geq r$ *and* $\sigma^\star(\bar{1} - \mu) \leq s$, *there exists an* $\lambda \in I^X$, $\tau(\lambda) \geq r$, $\tau^\star(\lambda) \leq s$ *and* $x_t \in \lambda$ *such that* $\lambda \leq F^u(\mu)$.

Theorem 3.3. *For a double fuzzy multifunction* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$*, the following statements are equivalent, where* $r\in I_0$ *and* $s\in I_1$*:*

1. *F is double fuzzy lower contra-continuous.*
2. $F^l(\mu)$ *is* (r,s)*-fuzzy open if* $\sigma(\overline{1}-\mu)\geq r$ *and* $\sigma^{\star}(\overline{1}-\mu)\leq s$*.*
3. $F^u(\mu)$ *is* (r,s)*-fuzzy closed if* $\sigma(\mu)\geq r$ *and* $\sigma^{\star}(\mu)\leq s$*.*

Proof. (1) $\Leftrightarrow$ (2): Let $x_t\in dom(F)$, $\mu\in I^Y$, $\sigma(\overline{1}-\mu)\geq r$, $\sigma^{\star}(\overline{1}-\mu)\leq s$ and $x_t\in F^l(\mu)$. Then there exists an (r,s)-fuzzy open set $\lambda\in I^X$ and $x_t\in\lambda$ such that $\lambda\leq F^l(\mu)$ and hence $x_t\in I_{\tau,\tau^{\star}}(F^l(\mu),r,s)$. Therefore, we obtain $F^l(\mu)=I_{\tau,\tau^{\star}}(F^l(\mu),r,s)$. Thus, $F^l(\mu)$ is (r,s)-fuzzy open. Let $x_t\in dom(F)$, $\mu\in I^Y$, $\sigma(\overline{1}-\mu)\geq r$ and $\sigma^{\star}(\overline{1}-\mu)\leq s$ with $x_t\in F^l(\mu)$. Then by (2), $F^l(\mu)=\lambda$ (say) is (r,s)-fuzzy open. Then there exists an (r,s)-fuzzy open set $\lambda\in I^X$ and $x_t\in\lambda$ such that $\lambda\leq F^l(\mu)$. Thus, F is double fuzzy lower contra-continuous.
(2) $\Leftrightarrow$ (3): Let $\mu\in I^Y$, $\sigma(\overline{1}-\mu)\geq r$ and $\sigma^{\star}(\overline{1}-\mu)\leq s$. Then by (1), $F^l(\overline{1}-\mu)=\overline{1}-F^u(\mu)$ is (r,s)-fuzzy open. Then $F^u(\mu)$ is (r,s)-fuzzy closed in $(X,\tau,\tau^{\star})$. The converse is clear. □

Theorem 3.4. *Let* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$ *be a double fuzzy multifunction and* $\mu\in I^Y$*, then the following are equivalent:*

1. *F is double fuzzy lower contra-continuous.*
2. $F^u(\mu)$ *is* (r,s)*-fuzzy open if* $\sigma(\overline{1}-\mu)\geq r$ *and* $\sigma^{\star}(\overline{1}-\mu)\leq s$*.*
3. $F^l(\mu)$ *is* (r,s)*-fuzzy closed if* $\sigma(\mu)\geq r$ *and* $\sigma^{\star}(\mu)\leq s$*.*

Proof. Simillar to the proof of Theorem 3.3. □

Definition 3.5. *Let* $(X,\tau,\tau^{\star})$ *be a double fuzzy topological space. For each* $\lambda\in I^X$, $r\in I_0$*, and* $s\in I_1$ *an operator* $K_{\tau,\tau^{\star}}:I^X\times I_0\to I^X$ *is defined as* $K_{\tau,\tau^{\star}}(\lambda,r,s)=\wedge\{\mu:\mu\geq\lambda,\tau(\mu)\geq r,\tau^{\star}(\mu)\leq s\}$*.*

Lemma 3.6. *Let* $(X,\tau,\tau^{\star})$ *be a double fuzzy topological space. If* $\mu\in I^X$*,* $\tau(\mu)\geq r$ *and* $\tau^{\star}(\mu)\leq s$*, then* $\mu=K_{\tau,\tau^{\star}}(\lambda,r,s)$*.*

Proof. Clear. □

Theorem 3.7. *Let* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$ *be a double fuzzy multifunction. If* $C_{\tau,\tau^{\star}}(F^l(\mu,r,s))\geq F^l(K_{\sigma,\sigma^{\star}}(\mu,r,s))$ *for any* $\mu\in I^Y$*, then F is double fuzzy upper contra-continuous.*

Proof. Suppose that $C_{\tau,\tau^{\star}}(F^l(\mu,r,s))\geq F^l(K_{\sigma,\sigma^{\star}}(\mu,r,s))$ for any $\mu\in I^Y$. Let $\rho\in I^Y$, $\sigma(\rho)\geq r$ and $\sigma^{\star}(\rho)\leq s$. By Lemma 3.6, we have $C_{\tau,\tau^{\star}}(F^l(\rho,r,s))\leq F^l(K_{\sigma,\sigma^{\star}}(\rho,r,s))=F^l(\rho)$. This implies that $C_{\tau,\tau^{\star}}(F^l(\rho,r,s))=\rho$ and hence $F^l(\rho)$ is (r,s)-closed in $(X,\tau,\tau^{\star})$. Thus, by Theorem 3.3, F is double fuzzy upper contra-continuous. □

Theorem 3.8. *Let* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$ *be a double fuzzy multifunction. If* $C_{\tau,\tau^{\star}}(F^u(\mu,r,s))\geq F^u(K_{\sigma,\sigma^{\star}}(\mu,r,s))$ *for any* $\mu\in I^Y$*, then F is double fuzzy lower contra-continuous.*

Proof. Suppose that $C_{\tau,\tau^{\star}}(F^u(\mu,r,s))\geq F^u(K_{\sigma,\sigma^{\star}}(\mu,r,s))$ for any $\mu\in I^Y$. Let $\rho\in I^Y$, $\sigma(\rho)\geq r$ and $\sigma^{\star}(\rho)\leq s$. By Lemma 3.6, we have $C_{\tau,\tau^{\star}}(F^u(\rho,r,s))\leq F^u(K_{\sigma,\sigma^{\star}}(\rho,r,s))=F^u(\rho)$. This implies that $C_{\tau,\tau^{\star}}(F^u(\rho,r,s))=\rho$ and hence $F^u(\rho)$ is (r,s)-fuzzy closed in I^X. Thus, by Theorem 3.4, F is double fuzzy lower contra-continuous. □

The following example shows that the notions double fuzzy lower contra-continuity and double fuzzy lower semi-continuity are independent.

Example 3.9. *Let* $X=\{x_1,x_2\}$*,* $Y=\{y_1,y_2,y_3\}$ *and a double fuzzy multifunction* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$ *defined by* $G_F(x_1,y_1)=0.1$*,*
$G_F(x_1,y_2)=\overline{1},G_F(x_1,y_3)=\overline{0},G_F(x_2,y_1)=0.5,G_F(x_2,y_2)=\overline{0}$ *and* $G_F(x_2,y_3)=\overline{1}$*. Define a double fuzzy topology* $\tau,\tau^{\star}:I^X\to I$ *as follows:*

$$\tau(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda\in\{\overline{0.5},\overline{0.6}\}\\ 0 & \text{otherwise}\end{cases}\qquad \tau^{\star}(\lambda)=\begin{cases}0 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda\in\{\overline{0.5},\overline{0.6}\}\\ 1 & \text{otherwise.}\end{cases}$$

Define a double fuzzy topology $\sigma,\sigma^{\star}:I^Y\to I$ *as follows:*

$$\sigma(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda=\overline{0.5}\\ \frac{1}{3} & \text{if } \lambda=\overline{0.4}\\ 0 & \text{otherwise}\end{cases}\qquad \sigma^{\star}(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda=\overline{0.5}\\ \frac{2}{3} & \text{if } \lambda=\overline{0.4}\\ 0 & \text{otherwise.}\end{cases}$$

Let $r=\frac{1}{3}$ *and* $s=\frac{2}{3}$*. Then the double fuzzy multifunction* $F:(X,\tau,\tau^{\star})\to(Y,\sigma,\sigma^{\star})$ *is double fuzzy lower contra-continuous but not double fuzzy lower semi-continuous.*

Example 3.10. *Let* $X=\{x_1,x_2\}$*,* $Y=\{y_1,y_2,y_3\}$ *and* $F:X\to Y$ *be a double fuzzy nultifunction defined by* $G_F(x_1,y_1)=0.1,G_F(x_1,y_2)=\overline{1},G_F(x_1,y_3)=\overline{0},G_F(x_2,y_1)=0.5,G_F(x_2,y_2)=\overline{0}$ *and* $G_F(x_2,y_3)=\overline{1}$*.Define a double fuzzy topology* $\tau:I^X\to I$ *and* $\tau^{\star}:I^Y\to I$ *as follows:*

$$\tau(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda\in\{\overline{0.4},\overline{0.5}\}\\ 0 & \text{otherwise}\end{cases}\qquad \tau^{\star}(\lambda)=\begin{cases}0 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda\in\{\overline{0.4},\overline{0.5}\}\\ 1 & \text{otherwise.}\end{cases}$$

Define a double fuzzy topology $\sigma:I^X\to I$ *and* $\sigma^{\star}:I^Y\to I$ *as follows:*

$$\sigma(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda=\overline{0.5}\\ \frac{1}{3} & \text{if } \lambda=\overline{0.4}\\ 0 & \text{otherwise}\end{cases}\qquad \sigma^{\star}(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0}\text{ or }\overline{1}\\ \frac{1}{2} & \text{if } \lambda=\overline{0.5}\\ \frac{2}{3} & \text{if } \lambda=\overline{0.4}\\ 0 & \text{otherwise.}\end{cases}$$

Let $r = \frac{1}{3}$ and $s = \frac{2}{3}$. Then the double fuzzy multifunction $F : (X, \tau, \tau^{\star}) \to (Y, \sigma, \sigma^{\star})$ is double fuzzy lower semi-continuous but not double fuzzy lower contra-continuous.

Theorem 3.11. *Let $F : (X, \tau, \tau^{\star}) \to (Y, \sigma, \sigma^{\star})$ and $H : (Y, \sigma, \sigma^{\star}) \to (Z, \eta, \eta^{\star})$ be double fuzzy multifunctions. If F is double fuzzy lower contra-continuous and H is double fuzzy lower semi-continuous, then $H \circ F$ is double fuzzy lower contra-continuous.*

Proof. Let $\lambda \in I^Z$, $\eta(\overline{1} - \lambda) \geq r$ and $\eta^{\star}(\overline{1} - \lambda) \leq s$. Then from Theorem 3.4 (2), we have $(H \circ F)^l(\lambda) = F^l(H^l(\lambda))$ and $F^l(H^l(\lambda))$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$ with $\tau(H^l(\overline{1} - \lambda)) \geq r$ and $\tau^{\star}(H^l(\overline{1} - \lambda)) \leq s$. Thus, $H \circ F$ is double fuzzy lower contra-continuous. □

Theorem 3.12. *Let $F : (X, \tau, \tau^{\star}) \to (Y, \sigma, \sigma^{\star})$ and $H : (Y, \sigma, \sigma^{\star}) \to (Z, \eta, \eta^{\star})$ be double fuzzy multifunctions. If F and H are normalized, F is double fuzzy upper contra-continuous and H is double fuzzy upper semi-continuous, then $H \circ F$ is double fuzzy upper contra-continuous.*

Proof. Simillar to the proof of Theorem 3.11. □

Theorem 3.13. *Let $\{F_i : i \in \Gamma\}$ be a family of double fuzzy lower contra-continuous between two double fuzzy topological spaces $(X, \tau, \tau^{\star})$ and (Y, σ, σ'). Then $\bigcup_{i \in \Gamma} F_i$ is double fuzzy lower contra-continuous.*

Proof. Let $\mu \in I^Y$, $\sigma(\overline{1} - \mu) \geq r$, $\sigma^{\star}(\overline{1} - \mu) \leq s$. Since $\{F_i : i \in \Gamma\}$ is a family of double fuzzy lower contra-continuous multifunctions, $F_i^l(\mu)$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$. Then we have, $(\bigcup_{i \in \Gamma} F_i)^l(\mu) = \bigvee_{i \in \Gamma} F_i^l(\mu)$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$. Hence $\bigcup_{i \in \Gamma} F_i$ is a double fuzzy lower contra-continuous multifunction. □

Theorem 3.14. *If F_1 and F_2 are normalized double fuzzy upper contra-continuous multifunctions between two double fuzzy topological spaces $(X, \tau, \tau^{\star})$ and $(Y, \sigma, \sigma^{\star})$, then $F_1 \cup F_2$ is a double fuzzy upper contra-continuous multifunction.*

Proof. Let $\mu \in I^Y$, $\sigma(\overline{1} - \mu) \geq r$, $\sigma^{\star}(\overline{1} - \mu) \leq s$. Since F_1 and F_2 are normalized double fuzzy upper contra-continuous multifunctions, $F_1^u(\mu)$ and $F_2^u(\mu)$ are (r,s)-fuzzy open sets in $(X, \tau, \tau^{\star})$. Since $(F_1 \cup F_2)^u(\mu) = F_1^u(\mu) \vee F_2^u(\mu)$ is an (r,s)-fuzzy open set, $(F_1 \cup F_2)^u(\mu)$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$. Hence $F_1 \cup F_2$ is a double fuzzy upper contra-continuous multifunction. □

Definition 3.15. *A double fuzzy set λ in a double fuzzy topological space $(X, \tau, \tau^{\star})$ is called (r,s)-fuzzy compact if every family $\{\mu : \tau(\mu) \geq r, \tau^{\star}(\mu) \leq s, \mu \in I^X\}$, where $r \in I_0$, $s \in I_1$ covering λ has a finite subcover.*

Definition 3.16. *A double fuzzy set λ in a double fuzzy topological space $(X, \tau, \tau^{\star})$ is called (r,s)-fuzzy closed-compact if every family $\{\mu : \tau(\overline{1} - \mu) \geq r, \tau^{\star}(\overline{1} - \mu) \leq s, \mu \in I^X\}$, where $r \in I_0$, $s \in I_1$ covering λ has a finite subcover.*

Theorem 3.17. *Let $F : (X, \tau, \tau^{\star}) \to (Y, \sigma, \sigma^{\star})$ be a crisp double fuzzy upper contra-continuous multifunction. Suppose that $F(x_t)$ is (r,s)-fuzzy closed-compact for each $x_t \in dom(F)$. If λ is (r,s)-fuzzy compact in $(X, \tau, \tau^{\star})$, then $F(\lambda)$ is (r,s)-fuzzy closed-compact in $(Y, \sigma, \sigma^{\star})$.*

Proof. Let λ be (r,s)-fuzzy compact set in $(X, \tau, \tau^{\star})$ and $\{\gamma_i : \sigma(\overline{1} - \gamma_i) \geq r, \sigma^{\star}(\overline{1} - \gamma_i) \leq s, i \in \Gamma\}$ be a family covering of $F(\lambda)$, that is, $F(\lambda) \leq \bigvee_{i \in \Gamma} \gamma_i$. Since $\lambda = \bigvee_{x_t \in \lambda} x_t$, we have $F(\lambda) = F(\bigvee_{x_t \in \lambda} x_t) = \bigvee_{x_t \in \lambda} F(x_t) \leq \bigvee_{i \in \Gamma} \gamma_i$. It follows that for each $x_t \in \lambda$, $F(x_t) \leq \bigvee_{i \in \Gamma} \gamma_i$. Since $F(x_t)$ is (r,s)-fuzzy closed-compact, there exists a finite subset Γ_{x_t} of Γ such that $F(x_t) \leq \bigvee_{n \in \Gamma_{x_t}} \gamma_n = \gamma_{x_t}$. Then we have $x_t \leq F^u(F(x_t)) \leq F^u(\gamma_{x_t})$ and $\lambda = \bigvee_{x_t \in \lambda} x_t \leq \bigvee_{x_t \in \lambda} F^u(F(x_t)) \leq F^u(\gamma_{x_t})$. Since $\sigma(\gamma_{x_t}) \geq r$ and $\sigma^{\star}(\gamma_{x_t}) \leq s$, then we have $F^u(\gamma_{x_t})$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$. Hence $\{F^u(\gamma_{x_t}) : F^u(\gamma_{x_t})$ is (r,s)-fuzzy open, $x_t \in \lambda\}$ is a family covering the λ. Since λ is (r,s)-fuzzy compact, then there exists a finite index set N such that $\lambda \leq \bigvee_{n \in N} F^u(\gamma_{x_{t_n}})$. Then we have $F(\lambda) \leq F(\bigvee_{n \in N} F^u(\gamma_{x_{t_n}})) = \bigvee_{n \in N} F(F^u(\gamma_{x_{t_n}})) \leq \bigvee_{n \in N} \gamma_{x_{t_n}}$. Hence $F(\lambda)$ is (r,s)-fuzzy closed-compact in $(Y, \sigma, \sigma^{\star})$.. □

Theorem 3.18. *Let $F : (X, \tau, \tau^{\star}) \to (Y, \sigma, \sigma^{\star})$ be a double fuzzy multifunction. If F is double fuzzy lower contra-continuous, then it is double fuzzy lower weakly continuous.*

Proof. Let $x_t \in dom(F)$, $\mu \in I^Y$, $\sigma(\mu) \geq r$, $\sigma^{\star}(\mu) \leq s$ and $x_t \in F^l(\mu)$. Since F is double fuzzy lower contra-continuous, $\overline{1} - C_{\sigma,\sigma^{\star}}(\mu, r, s)$ is (r,s)-fuzzy open in $(X, \tau, \tau^{\star})$ and $x_t \in F^l(C_{\sigma,\sigma^{\star}}(\mu, r, s))$. Then there exists an (r,s)-fuzzy open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_{\sigma,\sigma^{\star}}(\mu, r, s))$. Hence F is a double fuzzy lower weakly continuous multifunction. □

Theorem 3.19. *Let $(X, \tau, \tau^{\star})$ and $(X_i, \tau_i, \tau_i^{\star})$ be double fuzzy topological spaces $(i \in I)$. If a double fuzzy multifunction $F : (X, \tau, \tau^{\star}) \to \prod_{i \in I}(X_i, \tau_i, \tau_i^{\star})$ is double fuzzy lower contra-continuous (where $\prod_{i \in I}(X_i, \tau_i, \tau_i^{\star})$ is the product space), then $P_i \circ F$ is double fuzzy lower contra-continuous for each $i \in I$, where $P_i : \prod_{i \in I}(X_i, \tau_i, \tau_i^{\star}) \to (X_i, \tau_i, \tau_i^{\star})$ is the projection multifunction which is defined by $P_i((x_i)) = \{x_i\}$ for each $i \in I$.*

Proof. Let $\mu_{i_0} \in I^{X_i}$, $\tau_i(\overline{1} - \mu_{i_0}) \geq r$ and $\tau_i^{\star}(\overline{1} - \mu_{i_0}) \leq s$. Then $(P_{i_0} \circ F)^l(\mu_{i_0}) = F^l(P_{i_0}^l(\mu_{i_0})) = F^l(\mu_{i_0} \times \prod_{i \neq i_0} X_i)$. Since F is double fuzzy lower contra-continuous and $\tau_i(\overline{1} - \mu_{i_0} \times \prod_{i \neq i_0} X_i) \geq r$ and $\tau_i^{\star}(\overline{1} - \mu_{i_0} \times \prod_{i \neq i_0} X_i) \leq s$, $F^l(\mu_{i_0} \times \prod_{i \neq i_0} X_i)$ is an (r,s)-fuzzy open set in $(X, \tau, \tau^{\star})$. Then $P_i \circ F$ is a double fuzzy lower contra-continuous multifunction. □

Theorem 3.20. *Let $(X, \tau, \tau^{\star})$ and $(X_i, \tau_i, \tau_i^{\star})$ be double fuzzy topological spaces $(i \in I)$. If a double fuzzy multifunction*

$F:(X,\tau,\tau^{\star})\rightarrow\prod_{i\in I}(X_i,\tau_i,\tau_i^{\star})$ is double fuzzy upper contra-continuous and normalized, then $P_i\circ F$ is a double fuzzy upper contra-continuous multifunction for each $i\in I$.

Proof. Simillar to the proof of Theorem 3.19. □

Theorem 3.21. *Let $(X_i,\tau_i,\tau_i^{\star})$, $(Y_i,\sigma_i,\sigma_i^{\star})$ be double fuzzy topological spaces and $F_i:(X_i,\tau_i,\tau_i^{\star})\rightarrow(Y_i,\sigma_i,\sigma_i^{\star})$ be a double fuzzy multifunction for each $i\in I$. Suppose that*

$$F:\prod_{i\in I}(X_i,\tau_i,\tau_i^{\star})\rightarrow\prod_{i\in I}(Y_i,\sigma_i,\sigma_i^{\star})$$

is defined by $F((x_i))=\prod_{i\in I}F_i(x_i)$. If F is double fuzzy lower contra-continuous, then F_i is double fuzzy lower contra-continuous for each $i\in I$.

Proof. Let $\mu_i\in I^{Y_i}$, $\sigma_i(\overline{1}-\mu_i)\geq r$ and $\sigma_i^{\star}(\overline{1}-\mu_i)\leq s$. Then $\sigma_i(\overline{1}-\mu_i\times\prod_{i\neq j}Y_j)\geq r$ and $\sigma_i^{\star}(\overline{1}-\mu_i\times\prod_{i\neq j}Y_j)\leq s$. Since F is double fuzzy lower semi-continuous, $F^l(\mu_i\times\prod_{i\neq j}X_j)$ is (r,s)-fuzzy open in $\prod_{i\in I}(X_i,\tau_i,\tau_i^{\star})$. Consequently, we obtain that $F_i^l(\mu_i)$ is (r,s)-fuzzy open in $(X_i,\tau_i,\tau_i^{\star})$ for all $i\in I$. Thus, F_i is a double fuzzy lower contra-continuous multifunction. □

Theorem 3.22. *Let $(X_i,\tau_i,\tau_i^{\star})$, $(Y_i,\sigma_i,\sigma_i^{\star})$ be double fuzzy topological spaces and $F_i:(X_i,\tau_i,\tau_i^{\star})\rightarrow(Y_i,\sigma_i,\sigma_i^{\star})$ be a double fuzzy multifunction for each $i\in I$. Suppose that*

$$F:\prod_{i\in I}(X_i,\tau_i,\tau_i^{\star})\rightarrow\prod_{i\in I}(Y_i,\sigma_i,\sigma_i^{\star})$$

is defined by $F((x_i))=\prod_{i\in I}F_i(x_i)$. If F is a double fuzzy upper contra-continuous multifunction, then F_i is double fuzzy upper contra-continuous for each $i\in I$.

Proof. Simillar to the proof of Theorem 3.21. □

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