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Existence and uniqueness of fractional mixed type integro-differential equations with non-instantaneous impulses through sectorial operator

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Abstract
The key purpose of this paper is to examine the existence and uniqueness of PC-mild solution of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space through sectorial operator. Based on the Banach contraction principle, we develop the main results. At the end, an example is offered to explain theoretical outcomes.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K30, 35R12, 26A33.

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Contents

1. Introduction

There are many areas of study in which the question of fractional differential equations has recently arisen as a significant method for modeling real-world problems. It has great uses in a variety of sciences and diverse fields of study, such as mechanics, polymer rheology, frequent variance of thermodynamics, blood flow phenomenon, etc. For additional information, see [1, 2] and the references in it.

Motivated by [1, 3, 4], in this paper we consider a class of fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$\begin{aligned} {}^cD^{\alpha}x(t) &= Ax(t) + J^{1-\alpha} f\left(t, x(t), K_1x(t), K_2x(t)\right), \\ & \qquad t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m \\ x(t) &= g_i(t, x(t)), \quad t \in (t_i, s_i], i = 1, 2, \dots, m \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \leq 1, J^{1-\alpha}$ is Riemann-Liouville fractional integral operator and $J = [0, T]$ is operational interval. The map $A : D(A) \subset X \to X$ is a closed linear sectorial operator defined on a Banach space $(X, \|\cdot\|), x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \dots < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C\left((t_i, s_i] \times X; X\right)$, $f : [0, T] \times X^3 \to X$ is a nonlinear function and the functions K_1 and K_2 are defined by

$$K_1x(t) = \int_0^t u(t, s, x(s))ds \quad \text{and} \quad K_2x(t) = \int_0^T \widetilde{u}(t, s, x(s))ds,$$

$u, \widetilde{u} : \Delta \times X \to X$, where $\Delta = \{(x, s) : 0 \leq s \leq x \leq \tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.

2. Preliminaries

Let us set $J=[0,T], J_0=[0,t_1],\ J_1=(t_1,t_2],\ldots,\ J_{m-1}=(t_{m-1},t_m],J_m=(t_m,t_{m+1}]$ and introduce the space $PC(J,X):=\{u:J\to X \mid u\in C(J_k,X), k=0,1,2,\ldots,m,$ and there exist $u(t_k^+)$ and $u(t_k^-), k=1,2,\ldots,m,$ with $u(t_k^-)=u(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC}=\sup\{\|u(t)\|: t\in J\}$.

Let us recall the following definition of mild solutions for fractional evolution equations involving the Caputo fractional derivative.

Definition 2.1. *[5] Caputo's derivative of order $\alpha>0$ with lower limit a, for a function $f:[a,\infty)\to\mathbb{R}$ is defined as*

$$ {}_a^C D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = aJ_t^{n-\alpha} f^{(n)}(t) $$

where $a\geq 0, n\in N$. The Laplace transform of the Caputo derivative of order $\alpha>0$ is given as

$$ L\left\{ {}_0^C D_t^\alpha f(t);\lambda \right\} = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0); \quad n-1<\alpha\leq n. $$

Definition 2.2. *[3] A closed and linear operator A is said to be sectorial if there are constants $\omega\in R, \theta\in\left[\frac{\pi}{2},\pi\right], M_A>0$, such that the following two conditions are satisfied:*

(1) $\quad \sum_{(\theta,\omega)}=\{\lambda\in C:\lambda\neq\omega, |\arg(\lambda-\omega)|<\theta\}\subset\rho(A)$

(2) $\quad \|R(\lambda,A)\|_{L(X)}\leq\frac{M}{|\lambda-\omega|}, \lambda\in\sum_{(\theta,\omega)}$,

where X is the complex Banach space with norm denoted $\|\cdot\|_X$.

Lemma 2.3. *[3] Let f satisfies the uniform Holder condition with exponent $\beta\in(0,1]$ and A is a sectorial operator. Consider the fractional equations of order $0<\alpha<1$*

$$ {}_a^C D_t^\alpha x(t)=Ax(t)+J^{1-\alpha}f(t), \quad t\in J=[a,T],. \tag{2.1} $$

$a\geq 0,\ x(a)=x_0$. Then a function $x(t)\in C([a,T],X)$ is the solution of the equation (2.1) if it satisfies the following integral equation

$$ x(t)=Q_\alpha(t-a)x_0+\int_a^t Q_\alpha(t-s)f(s)ds $$

where $Q_\alpha(t)$ is solution operator generated by A defined as

$$ Q_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha I-A\right)^{-1}d\lambda $$

Γ is a suitable path lying on $\sum_{\theta,\omega}$.

Remark 2.4. *If $A\in\mathscr{A}^\alpha(\theta_0,\omega_0)$, then strongly continuous $\|Q_\alpha(t)\|\leq M_A e^{\omega t}$. Let $\widetilde{M_Q}:=\sup_{0\leq t\leq T}\|Q_\alpha(t)\|_{L(X)}$. So we have $\|Q_\alpha(t)\|_{L(X)}\leq\widetilde{M_Q}$.*

Now, we recall the following important Lemma which is very useful to prove our main result.

Definition 2.5. *A function $x\in PC(J,X)$ is said to be a PC-mild solution of problem (1.1) if it satisfies the following relation:*

$$ x(t)=\begin{cases} Q_\alpha(t)x_0+\int_0^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds, \\ t\in[0,t_1] \\ g_i(t,x(t)), t\in(t_i,s_i] \\ Q_\alpha(t-s_i)\,g_i(s_i,x(s_i)) \\ +\int_{s_i}^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds, \\ t\in(s_i,t_{i+1}] \end{cases} $$

for all $i=1,2,\ldots,m$.

3. Existence Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

From Definition 2.3, we define an operator $\Upsilon: PC(J,X)\to PC(J,X)$ as where

$$ (\Upsilon x)(t) \tag{3.1} $$

$$ =\begin{cases} Q_\alpha(t)x_0+\int_0^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds, \\ t\in[0,t_1] \\ g_i(t,x(t)), t\in(t_i,s_i] \\ Q_\alpha(t-s_i)\,g_i(s_i,x(s_i)) \\ +\int_{s_i}^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds, \\ t\in(s_i,t_{i+1}]. \end{cases} \tag{3.2} $$

To prove our first existence result we introduce the following assumptions:

(H(f)) The function $f\in C(J\times X^3;X)$ and there exist positive constants $L_{f_k}>0 (k=1,2,3)$ such that

$$ \|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)\| \\ \leq L_{f_1}\|x_1-y_1\|+L_{f_2}\|x_2-y_2\|+L_{f_3}\|x_3-y_3\| $$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3)\in X$ and every $t\in J$.

$(H(u,\ \widetilde{u}))$ The functions $u,\widetilde{u}:\Delta\times X\to X$ are continuous and there exist constants $L_u,L_{\widetilde{u}}>0$ such that

$$ \left\|\int_0^t [u(t,s,x(s))-u(t,s,y(s))]ds\right\|\leq L_u\|x-y\|, $$

for all, $x,y\in X$; and

$$ \left\|\int_0^T [\widetilde{u}(t,s,x(s))-\widetilde{u}(t,s,y(s))]ds\right\|\leq L_{\widetilde{u}}\|x-y\|, $$

for all, $x,y\in X$;

(H(g)) For $i = 1,2,\ldots,m$, the functions $g_i \in C([t_i,s_i]\times X;X)$ and there exists $L_{g_i} \in C(J,\mathbb{R}^+)$ such that

$$\|g_i(t,x) - g_i(t,y)\| \le L_{g_i}\|x-y\|$$

for all $x,y \in X$ and $t \in [t_i,s_i]$.

Theorem 3.1. *Expect that the hypotheses* H(f)$,H(u,\widetilde{u})$ *and* H(g) *hold, then problem (1.1) has a unique PC-mild solution* $x \in X$ *provided*

$$\Lambda = \max_{1\le i\le m}\Big[\widetilde{M_Q}L_{g_i} + \widetilde{M_Q}(t_{i+1}-s_i)[L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}],\ \widetilde{M_Q}t_1[L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}]\Big] < 1. \quad (3.3)$$

Proof. For any $x,y \in PC(J,X)$, by (3.1) and $t \in [0,t_1]$, we obtain

$$\begin{aligned}
&\|\Upsilon x(t) - \Upsilon y(t)\| \\
&= \Big\|\int_0^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds \\
&- \int_0^t Q_\alpha(t-s)f(s,y(s),K_1y(s),K_2y(s))ds\Big\| \\
&\le \int_0^t \|Q_\alpha(t-s)\|[\|f(s,x(s),K_1x(s),K_2x(s)) \\
&- f(s,y(s),K_1y(s),K_2y(s))\|]ds \\
&\le \widetilde{M_Q}\int_0^t [L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}]\|x-y\|_{PC}ds \\
&\le \widetilde{M_Q}t_1[L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}]\|x-y\|_{PC}.
\end{aligned}$$

For $t \in (t_i,s_i]$, we have

$$\|\Upsilon x(t) - \Upsilon y(t)\| \le L_{g_i}\|x-y\|_{PC}.$$

For $t \in (s_i,t_{i+1}]$, we get

$$\begin{aligned}
&\|\Upsilon x(t) - \Upsilon y(t)\| \\
&\le \Big\|Q_\alpha(t-s_i)g_i(s_i,x(s_i)) + \int_{s_i}^t Q_\alpha(t-s)f(s,x(s),K_1x(s),K_2x(s))ds \\
&\quad - Q_\alpha(t-s_i)g_i(s_i,y(s_i)) + \int_{s_i}^t Q_\alpha(t-s)f(s,y(s),K_1y(s),K_2y(s))ds\Big\| \\
&\le \widetilde{M_Q}L_{g_i}\|x-y\|_{PC} + \int_{s_i}^t \|Q_\alpha(t-s)\|[\|f(s,x(s),K_1x(s),K_2x(s)) \\
&- f(s,y(s),K_1y(s),K_2y(s))\|]ds \\
&\le \widetilde{M_Q}L_{g_i}\|x-y\|_{PC} + \widetilde{M_Q}(t_{i+1}-s_i)[L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}]\|x-y\|_{PC} \\
&\le \Big[\widetilde{M_Q}L_{g_i} + \widetilde{M_Q}(t_{i+1}-s_i)[L_{f_1}+L_{f_2}L_u+L_{f_3}L_{\widetilde{u}}]\Big]\|x-y\|_{PC}.
\end{aligned}$$

Thus

$$\|\Upsilon x - \Upsilon y\| \le \Lambda\|x-y\|_{PC}.$$

Therefore, the operator Υ is a contraction on $PC(J,X)$ and there exists a unique mild solution of the system (1.1).

□

4. Application

In this section, we are going to present an example to validate the results established in Theorem 3.1. Consider the fractional partial integro-differential equation of the form

$$\begin{aligned}
{}^cD^\alpha x(t,z) &= \frac{\partial^2 x}{\partial z^2} + J^{1-\alpha}P\Big(t,x(t,z), \\
&\int_0^t u(t,s,x(s,z))ds, \int_0^T \widetilde{u}(t,s,x(s,z))ds\Big), \\
&a.e. (t,z) \in \cup_{i=1}^m (s_i,t_{i+1}] \times [0,\pi] \qquad (4.1)
\end{aligned}$$

$$x(t,0) = x(t,\pi) = 0,\ t \in [0,T] \qquad (4.2)$$

$$x(0,z) = x_0(z),\ z \in [0,\pi] \qquad (4.3)$$

$$x(t,z) = G_i(t,x(t,z)), \quad t \in (t_i,s_i],\ z \in [0,\pi],\ i = 1,2,\ldots,N, \qquad (4.4)$$

where $0 = t_0 = s_0 < t_1 \le s_1 \cdots < t_m \le s_m < t_{m+1} = T, P \in C([0,T]\times\mathbb{R}^3,\mathbb{R})$ and $G_i \in C((t_i,s_i]\times\mathbb{R},\mathbb{R})$ for all $i = 1,2,\ldots,m$.

Let $X = L^2([0,\pi])$. Define an operator $A : D(A) \subseteq X \to X$ by $Ax = x''$ with $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t) : t \ge 0\}$ in X. Moreover, the subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator $\{Q_\alpha(t)\}_{t\ge 0}$ such that $\|Q_\alpha(t)\|_{L(X)} \le \widetilde{Q_S}$ for $t \in [0,1]$.

Set

$$\begin{aligned}
x(t)z &= x(t,z), J^{1-\alpha}f(t,x(t),K_1x(t),K_2x(t))z \\
&= J^{1-\alpha}P\Big(t,x(t,z), \int_0^t u(t,s,x(s,z))ds, \\
&\int_0^T \widetilde{u}(t,s,x(s,z))ds\Big) \\
g_i(t,x(t))z &= G_i(t,x(t,z)).
\end{aligned}$$

Thus with this set-up, equations (4.1)-(4.4) can be written in the abstract form for the system (1.1).

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