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Multi-derivative hybrid methods for integration of general second order differential equations

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Abstract

In this study, new multi-derivative hybrid methods for the integration of general second order initial value problems of ordinary differential equations are considered. Linear multistep formula was used in the development of the methods taking Taylor series as the basis function. The unknown parameters were solved by the systematic reduction of simultaneous nonlinear equations. Due to the lapses in number of equations compared to the number of unknowns, we make $\beta_0 = 0$ as a free parameter. Analysis of the resulting methods shows that they are zero stable, consistent and convergent. Numerical examples are given to demonstrate and compare the efficiency of the methods for the stepnumbers $k = 1$ and $k = 2$ respectively. The results shows a better performance on existing methods.

Keywords

Initial value problems, Ordinary differential equation, Linear Multistep Method, Taylor series.

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1. Introduction

In physical and life sciences, there are always difficulties in taking decisions about some phenomenons. These phenomenons can be represented in form of mathematical equations; most atimes, we arrive at the form of the equations below

$$y'' = f(x, y, y'), y'(x_0) = y'_0, y(x_0) = y_0 \tag{1.1}$$

$$\beta'' = \alpha(\delta, \beta, \beta'), \beta'(\delta_0) = \beta'_0, \beta(\delta_0) = \beta_0 \tag{1.2}$$

The equations above are the mathematical representations of real life events, which may have several analytical methods of solutions. Many of the nonlinear differential equations arising from such equations above may not be solved analytically, hence the need for numerical methods of solutions. Linear multistep methods provide an approximate solution of (1.1) (Lambert 1973)

Some authors have attempted to solve this form of equation using linear multistep method (Awoyemi 1999) where a class of continuous method for the general solution of second order ordinary differential equations, collocation method was used and the resulting schemes for $k = 1$, $k = 2$ and $k = 3$ was found to be consistent and zero stable.

(Kayode and Ademiluyi 2001); where a k-step second derivative hybrid method was developed to solve both stiff and non stiff initial value problems of ordinary differential equation. The resulting schemes were found to be A-stable, consistent and convergent.

Meanwhile, in a bid to improve the existing methods, some authors have introduced the hybrid term (Kayode and Adeyeye 2013); where a Two-point two-step hybrid method for the direct solution of second order ordinary differential equation

was introduced. A predictor-corrector mode of solution was used and the introduction of the hybrid terms makes the generated results zero stable, consistent and normalized.
However, some few authors who have worked in this field have their solutions being improved over the years. It is observed that the order of accuracy of the methods in the papers of these authors are low. Thus this paper develops methods with better performance over existing methods.

2. Derivation Method

The linear multistep method modified by kayode and Ademuluyi for the solution of first order ODEs was improved to provide a numerical solution to second order ODEs. The general linear multistep method considered as a basis function is of the form below

$$y_{n+k} = y_{n+k-1} + hy'_n + h^2 \sum_{j=0}^{k} \beta_j y''_{n+j} + h^2 \beta_r y''_{n+k-r} + h^3 \rho y'''_{n+k} \tag{2.1}$$

The equation above was used to generate a truncation error. Taylor series expansion was used to expand each term of the basis function. The values of the unknown parameters were obtained at k-steps. The truncation error developed from equation above has the form

$$\begin{aligned} L[y(x),h] =& y_{n+k} - y_{n+k-1} - hy'_n - h^2 \sum_{j=0}^{k} \beta_j y''_{n+j} \\ & - h^2 \beta_r y''_{n+k-r} - h^3 \rho y'''_{n+k} \end{aligned} \tag{2.2}$$

There is need to develop a suitable method to solve for the unknown parameters β_j, β_r, r and ρ. Meanwhile, 'r' is the hybrid point i.e. $0 \le r \le 1$. In order to do this, expand each term above by Taylors series expansion as follows

$$y_{n+k} = \sum_{s=0}^{k} \frac{h^s y^s_{xn} k^s}{s!} \tag{2.3}$$

$$y_{n+k-1} = \sum_{s=0}^{k} \frac{h^s y^s_{xn} (k-1)^s}{s!} \tag{2.4}$$

$$y'_n \tag{2.5}$$

$$y''_{n+j} = \sum_{s=2}^{k} \frac{h^{s-2} y^s_{xn} j^{s-2}}{(s-2)!} \tag{2.6}$$

$$y''_{n+k-r} = \sum_{s=2}^{k} \frac{h^{s-2} y^s_{xn} (k-r)^{s-2}}{(s-2)!} \tag{2.7}$$

$$y'''_{n+k} = \sum_{s=3}^{k} \frac{h^{s-3} y^s_{xn} k^{s-3}}{(s-3)!} \tag{2.8}$$

Arranging as

$$L[y(x),h] = C_0 y + C_1 hy + C_2 h^2 y^2 + C_3 h^3 y^3 + \dots + C_q h^q y^q \tag{2.9}$$

$$C_1 = 0 \tag{2.10}$$

$$C_2 = \frac{k^2}{2} - \frac{(k-1)^2}{2} - \sum_{j=0}^{k} \beta_j - \beta_r \tag{2.11}$$

$$C_2 = \frac{k^3}{6} - \frac{(k-1)^3}{6} - \sum_{j=0}^{k} j\beta_j - (k-r)\beta_r - \rho \tag{2.12}$$

More, generally,

$$\begin{aligned} C_q =& \frac{k^q - (k-1)^q}{q!} - \frac{\sum_{j=0}^{k} j^{q-2}\beta_j - (k-r)^{q-2}\beta_r}{(q-2)!} \\ & - \frac{k^{(q-3)}}{(q-3)}\rho \end{aligned} \tag{2.13}$$

2.1 Derivation of Method One

Using $k = 1$ and setting $C_2 - C_5$ to zero in equation (2.13) we obtain the following set of equations

$$\beta_r + \beta_0 + \beta_1 = \tfrac{1}{2} \tag{2.14}$$

$$(1-r)\beta_r + \beta_1 + \rho = \tfrac{1}{6} \tag{2.15}$$

$$(1-r)^2\beta_r + \beta_1 + 2\rho = \tfrac{1}{12} \tag{2.16}$$

$$(1-r)^3\beta_r + \beta_1 + 3\rho = \tfrac{1}{20} \tag{2.17}$$

From equations (2.14)-(2.17), the unknown parameters are obtained in terms of a free parameter β_0 as

$$r = \frac{4(1-5\beta_0)}{5(1-4\beta_0)} \tag{2.18}$$

$$\rho = \frac{1-8\beta_0}{-48+240\beta_0} \tag{2.19}$$

$$\beta_1 = \frac{-7+84\beta_0-240\beta_0^2}{-64-1600\beta_0^2} \tag{2.20}$$

$$\beta_r = \frac{-25-20\beta_0-560\beta_0^2+1600\beta_0^3}{-64-1600\beta_0^2} \tag{2.21}$$

Setting $\beta_0 = 0$ gives the exact values below

$$r = \tfrac{4}{5} \tag{2.22}$$

$$\rho = \tfrac{-1}{48} \tag{2.23}$$

$$\beta_1 = \tfrac{7}{64} \tag{2.24}$$

$$\beta_r = \tfrac{25}{64} \tag{2.25}$$

Then method one can be established successfully

$$y_{n+1} = y_n + hy'_n + \frac{7}{64}h^2 f_{n+1} + \frac{25}{64}h^2 f_{n+\frac{1}{5}} - \frac{1}{48}h^3 f'_{n+1} \tag{2.26}$$

2.2 Derivation of Method Two

Using $k = 2$ and setting $C_2 - C_6$ to zero in equation (2.22) we obtain the following set of equations

$$\beta_r + \beta_1 + \beta_2 + \beta_0 = \tfrac{3}{2} \tag{2.27}$$

$$(2-r)\beta_r + \beta_1 + 2\beta_2 + \rho = \tfrac{7}{6} \tag{2.28}$$

$$(2-r)^2\beta_r + \beta_1 + 4\beta_2 + 4\rho = \tfrac{15}{12} \tag{2.29}$$

$$(2-r)^3\beta_r + \beta_1 + 8\beta_2 + 12\rho = \tfrac{31}{20} \tag{2.30}$$

$$(2-r)^4\beta_r + \beta_1 + 16\beta_2 + 32\rho = \tfrac{63}{30} \tag{2.31}$$

$$r = \frac{5(-29+96\beta_0)}{2(-41+120\beta_0)}$$

$$\rho = \frac{199-360\beta_0}{300(-29+96\beta_0)}$$

$$\beta_2 = \frac{53241-314500\beta_0+460800\beta_0^2}{500(-29+96\beta_0)^2}$$

$$\beta_1 = \frac{-3041+12960\beta_0}{-3780+14400\beta_0}$$

$$\beta_r = -\frac{4(-41+120\beta_0^4}{1125(-21+80\beta_0)(-29+96\beta_0)^2}$$

Setting $\beta_0 = 0$ gives the exact values bellow

$$r = \frac{145}{82} \quad (2.32)$$

$$\rho = \frac{-199}{8700} \quad (2.33)$$

$$\beta_1 = \frac{3041}{3780} \quad (2.34)$$

$$\beta_2 = \frac{53241}{420500} \quad (2.35)$$

$$\beta_r = \frac{11303044}{1986825} \quad (2.36)$$

Substituting the above equations directly in equation 3.1, then the scheme for method two is given below

$$y_{n+2} = y_{n+1} + hy'_n + \frac{3041}{3780}h^2 f_{n+1} + \frac{53241}{420500}h^2 f_{n+2} + \frac{11303044}{19868625}h^2 f_{n+\frac{19}{82}} - \frac{199}{8700}h^3 f'_{n+2} \quad (2.37)$$

3. Order and error constant

Adopting Ademiluyi and Kayode 2001, and by Taylor series expansion of 2.22 and 2.33 about the point x as given in the expression below

$$L(y,h) = C_0 y_n + C_1 hy'_n + C_2 h^2 y''_n + C_3 h^3 y'''_n + C_4 h^4 y^{iv}_n + C_5 h^5 y^{v}_n + \ldots \quad (3.1)$$

and $C_0 = C_1 = \ldots = C_p = 0$ but $C_p + 1 \neq 0$ then the scheme is of order $p+1$

The order and error constant for each method is displayed on the table below

Table 1. Order and Error constant

	Order(p)	Error constant (p+1)
Method One	5	$2.7778 * 10^{-4}$
Method Two	6	$-2.3471 *^{-5}$

3.1 Consistency

A numerical method is said to be consistent if and only if;

- The order P is greater or equal to one i.e. $P \geq 1$
- The coefficient of the first characteristic polynomial must be zero $\sum_{j=0}^{k} \alpha_j = 0$

From Table1 above, it is confirmed that the methods are consistent.

3.2 Zero Stability

A numerical method is said to be zero stable if the roots of the first characteristic polynomial $\alpha \leq 1$

For method one i.e.

$$k = 1$$
$$p(r) = r - 1$$
$$r - 1 = 0$$
$$r = 1$$

so method one is zero stable.

For method two i.e. $k = 2$

$$p(r) = r^2 - r$$
$$r^2 - r = 0$$
$$r(r-1) = 0$$
$$r = 0 \text{ or } r = 1$$
$$p(r) \leq 1$$

So method two is zero stable.

4. Convergence analysis

This section discussed the convergence analysis of the schemes. The schemes converges if

Definition 4.1. *A linear multistep method for problem (1) is convergent if*

$$\lim_{h \to 0} |\Phi(h)| \leq 1 \quad (4.1)$$

Where,

$$|\Phi(h)| = \left| \frac{y_{n+k}}{y_{n+k-1}} \right|, \quad \forall k \geq 1 \quad (4.2)$$

Test of Convergence of Method 1 $(K = 1)$

When $K = 1$ equation 4.2 becomes

$$|\Phi(h)| = \left| \frac{y_{n+1}}{y_n} \right| \quad (4.3)$$

However, the following equation is obtained from equation 2.22

$$y_{n+1} = y_n + hy_n + \frac{7}{64}h^2 y_{n+1} + \frac{25}{64}h^2 y_{n+\frac{1}{5}} - \frac{1}{48}h^3 y_{n+1} \quad (4.4)$$

Table 2. Results for problem one generated by $k = 1$ and $h = 0.01$

x	y(exact)	y (computed)	error
0.01	4.058405704705347	4.058405704704999	0.000000000
0.02	4.113645937249032	4.113645788372905	1.4887×10^{-9}
0.03	4.165756033571567	4.165738219679358	1.7813×10^{-5}
0.04	4.214772179371645	4.214718765194660	5.3414×10^{-5}
0.05	4.260731375367430	4.260624032457649	1.0734×10^{-4}
0.06	4.303671402842273	4.303491435410199	1.7996×10^{-4}
0.07	4.343630789486266	4.343359160108101	2.7162×10^{-4}
0.08	4.380648775544720	4.380266130720270	3.8264×10^{-4}
0.09	4.446020868786508	4.414251975827473	5.1330×10^{-4}
0.1	4.446020868786508	4.445356995031403	6.6387×10^{-4}

Applying (4.3) in (4.4) to have

$$y_{n+1}\left(1-\frac{7}{64}h^2+\frac{1}{48}h^3\right)=y_n\left(1+h\frac{25}{64}h^2\Gamma\right) \tag{4.5}$$

Where Γ (by Tailor series) gives

$$\Gamma = y_{n+\frac{1}{5}} = y_n+\frac{1}{5}hy_n+\frac{1}{50}h^2y_n+\frac{1}{750}h^3y_n + \frac{1}{15000}h^4y_n+(0h^5) \tag{4.6}$$

Substitute equation 4.6 in equation 4.5 and simplify to equation 4.1

$$\lim|\Phi(h)| = \lim_{h\to 0}\left(\frac{1-\frac{7}{64}h^2+\frac{1}{48}h^3}{1+h+\frac{25}{64}h^2+\frac{5}{64}h^3+\frac{1}{128}h^4+\frac{1}{1920}h^5}\right) \le 1 \tag{4.7}$$

Method 1 satisfies the condition in equation 4.1; hence it is convergent.

Test of Convergence of Method 2 $(K=2)$
When $K=2$ equation 4.2 becomes

$$|\Phi(h)| = \frac{y_{n+2}}{y_{n+1}} \tag{4.8}$$

From equation 2.33, the following is obtained

$$y_{n+2} = y_{n+1}+hy_n+\frac{3041}{3780}h^2y_{n+1}+\frac{53241}{420500}h^2y_{n+2} + \frac{11303044}{19868625}h^2y_{n+\frac{19}{82}}-\frac{199}{8700}h^3y_{n+2} \tag{4.9}$$

Applying (4.8) in (4.9) to have

$$y_{n+2}\left(1-\frac{53241}{420500}h^2+\frac{199}{8700}h^3\right) = y_{n+1}\left(1+h^2+\Lambda+\frac{113030344}{19868625}h^2\Gamma\right) \tag{4.10}$$

Where Γ (by Tailor series) gives

$$\Gamma = y_{n+\frac{19}{82}} = y_n+\frac{19}{82}hy_n+\frac{361}{13448}h^2y_n+\frac{6859}{3308208}h^3y_n + \frac{130321}{1085092224}h^4y_n+(0h^5) \tag{4.11}$$

And Λ (by Tailor series) gives

$$\Lambda = y_n = y_{n+1}+hy_{n+1}+\frac{1}{2}h^2y_{n+1}+\frac{1}{6}h^3y_{n+1} + \frac{1}{24}h^4y_{n+1}+(0h^5) \tag{4.12}$$

Substitute equation 4.11 and equation 4.12 into equation 4.10 and simplify to have

$$\lim|\Phi(h)| = \lim_{h\to 0}\left(\frac{1-\frac{53241}{420500}h^2+\frac{199}{8700}h^3}{1+h+h^2+\frac{1304}{3780}h^2+\frac{11303044}{19868625}h^2\frac{1}{2}h^3+\frac{214757836}{1629227250}h^3+\frac{1}{6}h^4+\frac{4080398884}{26703432000}h^4}\right) \le 1 \tag{4.13}$$

Hence, method 2 satisfies the condition in equation 4.1; therefore it is convergent.

5. Implementation

Problem one;

- $y''+2y'+5y=0$; y(0)=4 $y'(0)=6$

Exact solution:

$$y = 4e^{-x}\cos 2x+5e^{-x}\sin 2x$$

Problem two, extracted from Kayode etal(2018). problem two ; It is a nonlinear differential equation

- $y''-x(y')^2=0$; y(0)=1 $y'(0)=0.5$

Exact solution:

$$y = 1+\frac{1}{2}ln(\frac{2+x}{2-x})$$

6. Results

The proposed scheme was tested and the result was compared with the error in Kayode *et al*(2018) the choice of this comparison was because method two have the same order(p=6) with Kayode *et al*(2018). The absolute errors were compared and the results of the absolute errors were shown on table 5.

Table 3. Results problem one generated by $k = 2$ and $h = 0.01$

x	y(exact)	y (computed)	error
0.01	4.113645937249032	4.113645937248711	0.00000000
0.02	4.165756033571567	4.165739194524353	1.6839×10^{-5}
0.03	4.214772179371645	4.214720970247788	5.1209×10^{-5}
0.04	4.260731375367430	4.260627910476646	1.0346×10^{-4}
0.05	4.303671402842273	4.303497438743769	1.7396×10^{-4}
0.06	4.343630789486266	4.343367749662243	2.6303×10^{-4}
0.07	4.380648775544720	4.380277774943675	3.7100×10^{-4}
0.08	4.414765280284209	4.414267149716310	4.9813×10^{-4}
0.09	4.446020868786508	4.445376179153847	6.4468×10^{-4}
0.10	4.474456719080371	4.473645805425398	8.1091×10^{-4}

Table 4. Results problem two generated by $k = 2$ and $h = 0.01$

x	y(exact)	y (computed)	error
0.01	1.010000333353335	1.010000853462584	5.201×10^{-7}
0.02	1.015001125151899	1.015002169047778	1.044×10^{-6}
0.03	1.020002667306850	1.020004375191243	1.707×10^{-6}
0.04	1.025005210287331	1.025007727344283	2.517×10^{-6}
0.05	1.030009004863127	1.030012481375338	3.476×10^{-6}
0.06	1.035014302180242	1.035014302180242	4.591×10^{-6}
0.07	1.040021353836768	1.040027220622601	5.866×10^{-6}
0.08	1.045030411959091	1.045037719887862	7.307×10^{-6}
0.09	1.050041729278491	1.050050649286657	8.920×10^{-6}
0.10	1.055055559208212	1.055066267471978	1.070×10^{-5}

Table 5. Results problem two, $k = 2$ and $h = 0.003125$ being compared with Kayode etal(2018)

x	Error in Kayode etal(18)	Error in new method
0.0063	9.325×10^{-15}	5.2×10^{-15}
0.0094	1.865×10^{-14}	5.0×10^{-15}
0.0125	2.798×10^{-14}	9.9×10^{-15}
0.0156	3.730×10^{-14}	1.64×10^{-14}
0.0188	4.663×10^{-14}	2.44×10^{-14}

7. Conclusion

In this paper, a multiderivative hybrid method for the integration of second order ordinary differential equation with initial conditions was developed. The two schemes developed have orders 5(for $k = 1$) and 6($k = 2$).Two test examples have been solved to demonstrate the accuracy of the methods, where we noticed little difference in the results of the comparison equation(problem 2), which shows there is an improvement in the general multistep method developed in this paper. Consequently, the method is highly recommended for use for both linear and non linear problems.

The schemes were also tested for consistency, zero stability and the absolute error was compared with Kayode *et al*(2018) and there is an obvious improvement.

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