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Chromatic completion number of corona of path and cycle graphs

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Abstract

Following the introduction of the notion of chromatic completion of a graph, this paper presents results for the *chromatic completion number* for the corona operations, $P_n \circ P_m$ and $P_n \circ C_m$, $n \geq 1$ and $m \geq 1$. From the aforesaid a general result for the chromatic completion number of $P_n \circ K_m$ came to the fore. The paper serves as a basis for further research with regards to the chromatic completion number of corona, join and other graph products.

Keywords

Chromatic completion number, chromatic completion graph, chromatic completion edge.

AMS Subject Classification

05C15, 05C38, 05C75, 05C85.

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1. Introduction

For general notation and concepts in graphs see [1, 2, 5]. It is assumed that the reader is familiar with the concept of graph coloring. Recall that in a proper coloring of G all edges are good i.e. $uv \Leftrightarrow c(u) \neq c(v)$. For any proper coloring $\varphi(G)$ of a graph G the addition of all good edges, if any, is called the chromatic completion of G in respect of $\varphi(G)$. The additional edges are called *chromatic completion edges*. The set of such chromatic completion edges is denoted by, $E_\varphi(G)$. The resultant graph G_φ is called a *chromatic completion graph* of G. See [3] for an introduction to chromatic completion of a graph.

The *chromatic completion number* of a graph G denoted by, $\zeta(G)$ is the maximum number of good edges that can be added to G over all chromatic colorings (χ-colorings). Hence, $\zeta(G) = max\{|E_\chi(G)| : over\ all\ \varphi_\chi(G)\}$.

A χ-coloring which yields $\zeta(G)$ is called a *Lucky χ-coloring* or simply, a Lucky coloring and is denoted by, $\varphi_{\mathscr{L}}(G)$. The resultant graph G_ζ is called a *minimal chromatic completion graph* of G. It is trivially true that $G \subseteq G_\zeta$. Furthermore, the graph induced by the set of completion edges, $\langle E_\chi \rangle$ is a subgraph of the complement graph, $\overline{G}$. See [4] for the notion of stability in respect of chromatic completion.

Recall that perfect Lucky χ-coloring[1] of a graph G is a

[1] Note that for many graphs a perfect Lucky coloring is equivalent to an

graph for which the vertex $V(G)$ can be partitioned in accordance to Lucky's theorem i.e. in the Lucky partition form,

$$\underbrace{\{\{\lfloor\frac{n}{\chi(G)}\rfloor\text{-element}\},\{\lfloor\frac{n}{\chi(G)}\rfloor\text{-element}\},\dots,\{\lfloor\frac{n}{\chi(G)}\rfloor\text{-element}\},}_{(\chi(G)-r)-subsets}$$
$$\underbrace{\{\lceil\frac{n}{\chi(G)}\rceil\text{-element}\},\{\lceil\frac{n}{\chi(G)}\rceil\text{-element}\},\dots,\{\lceil\frac{n}{\chi(G)}\rceil\text{-element}\}\}.}_{(r\geq 0)-subsets}$$

Else, any graph is always *near* Lucky χ-colorable (similar to near equitable colorable). The vertex partition which approximates a Lucky partition closest is called an *optimal near-completion χ-partition*. See [3, 4]. The number of times a color c_i is assigned in a graph coloring is denoted by, $\theta_G(c_i)$. If the graph context is clear we abbreviate as, $\theta(c_i)$.

Various graph parameters have been studied in respect of *sensitivity* (how critical) the parameters are in respect of edge deletion, edge addition, vertex deletion or vertex insertion and alike. Note that after chromatic completion of a graph G which has been assigned a chromatic coloring ($\chi(G)$ colors), the chromatic completion graph itself has chromatic number, $\chi(G)$. However, the addition of one or more further edges will result in an increase in chromatic number. If vertices and edges in a graph G represent modules (or entities) and initial linkages in machine learning or artificial intelligence configurations, then:
(a) Different color classes could signal destructive linkages which may not occur and,
(b) Maximum permissible linkages may be needed to enhance machine learning or artificial intelligence interactive learning.
(c) In such application the chromatic completion number of G signals the critical threshold.
Similar applications can be visioned. This justifies further research into this parameter.

2. Chromatic Completion Number of $P_n \circ P_m$

Recall that the corona between graph G or order n and graph H of order m is obtained by taking n copies of H say, H_1, H_2, H_3,..., H_n and adding the edges, $v_iu_{i,j}$, $i = 1,2,3,\dots,n$ and $j = 1,2,3,\dots,m$. Put differently, $\forall$ v_i construct $v_i + H_i$ to G. We say, *H has been corona'd with G*.

A path graph (or simply, a path) of order n denoted by, P_n, is a graph on $n \geq 1$ vertices say, $V(P_n) = \{v_1, v_2, v_3,\dots,v_n\}$ and $n-1$ edges namely, $E(P_n) = \{v_iv_{i+1} : i = 1,2,3,\dots,n-1\}$. In this section we denote, $V(P_n) = \{v_1, v_2, v_3,\dots,v_n\}$ and $V(P_m) = \{u_{i,1}, u_{i,2}, u_{i,3},\dots,u_{i,m}\}$, $i = 1,2,3,\dots,n$ and the edges accordingly. The corona operator (not necessarily commutative) will be, $P_n \circ P_m$.

equitable χ-coloring. Since it is not generally the case the alias is meant to associate the paper with Lucky's Theorem and the notion of chromatic completion in [3, 4].

For $P_1 \circ P_1$ it follows that $\zeta(P_1 \circ P_1) = 0$ because, $P_1 \circ P_1 \cong K_2$. Similarly, $\zeta(P_1 \circ P_2) = 0$ because, $P_1 \circ P_2 \cong K_3$. Also, $\zeta(P_1 \circ P_3) = 0$ because, $c(u_{1,1}) = c(u_{1,3})$ in any perfect Lucky χ-coloring. We recall two important results from [3].

Lemma 2.1. [3] For a chromatic coloring $\varphi : V(G) \mapsto \mathscr{C}$ a pseudo completion graph, $H(\varphi) = K_{n_1,n_2,n_3,\dots,n_\chi}$ exists such that,

$$\varepsilon(H(\varphi)) - \varepsilon(G) = \sum_{i=1}^{\chi-1} \theta_G(c_i)\theta_G(c_j)_{(j=i+1,i+2,i+3,\dots,\chi)} - \varepsilon(G)$$

$$\leq \zeta(G).$$

Corollary 2.1. [3] Let G be a graph. Then

$$\zeta(G) \quad = \quad max\{\varepsilon(H(\varphi)) - \varepsilon(G) : \text{ over all } \varphi : V(G) \mapsto \mathscr{C}\}.$$

Corollary 2.2. (a) For $P_1 \circ P_m$, $m \geq 4$ it follows that, $\zeta(P_1 \circ P_m) = \lceil\frac{m}{2}\rceil\lfloor\frac{m}{2}\rfloor - (m-1)$.
(b) For $P_n \circ P_1$, $n \geq 2$ it follows that, $\zeta(P_n \circ P_1) = n^2 - n + 1$.

Proof. (a) Since $\chi(P_1 \circ P_m) = 3$ let $c(v_1) = c_1$. Clearly P_m can be assigned a perfect Lucky 2-coloring with the set of colors, $\{c_2, c_3\}$. Because $P_1 \circ P_m \cong P_1 + P_m$, no chromatic completion edges, $v_1u_{1,i}$, $1 \leq i \leq m$ can be added. Hence, $\zeta(P_1 \circ P_m) = \zeta(P_m) = \lceil\frac{m}{2}\rceil\lfloor\frac{m}{2}\rfloor - (m-1)$. See [3].
(b) Because $P_n \circ P_1$ is a tree on $2n$ vertices the result is immediate from Lemma 2.1 and Corollary 2.1. □

Note the subtlety in the proof above i.e. $P_1 \circ P_m$ is not perfect Lucky 3-colorable. However, the induced subgraph P_m of $P_1 \circ P_m$, is perfect 2-colorable. Such vertex partition is called an optimal near-completion χ-partition.

Proposition 2.1. For $P_2 \circ P_m$, $m \geq 2$ it follows that,

$$\zeta(P_2 \circ P_m) = \begin{cases} \frac{5m^2}{4} - m + 2, & \text{if } m \text{ is even,} \\ \frac{5m^2+2m+1}{4} - m, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Part 1. Consider m is even. Without loss of generality, let $c(v_1) = c_1$, $c(v_2) = c_2$. Also without loss of generality let $c(u_{1,j}) = c_2$, $j = 1,3,5,\dots,(m-1)$ and $c(u_{1,j}) = c_3$, $j = 2,4,6,\dots,m$. Also let, $c(u_{2,j}) = c_1$, $j = 1,3,5,\dots,(m-1)$ and $c(u_{2,j}) = c_3$, $j = 2,4,6,\dots,m$. Therefore, $\theta(c_1) = \frac{m}{2} + 1$, $\theta(c_2) = \frac{m}{2} + 1$ and $\theta(c_3) = m$. Clearly for $m \geq 6$, the vertex partition is an optimal near-completion χ-partition.
From Lemma 2.1 and Corollary 2.1 it follow that,

$$\zeta(P_2 \circ P_m) = (\tfrac{m}{2}+1)^2 + 2m(\tfrac{m}{2}+1) - (4m-1) = \tfrac{5m^2}{4} - m + 2.$$

Part 2. Consider m is odd. Without loss of generality, let $c(v_1) = c_1$, $c(v_2) = c_2$. Also without loss of generality let $c(u_{1,j}) = c_2$, $j = 1,3,5,\dots,m$ and $c(u_{1,j}) = c_3$, $j = 2,4,6,\dots,(m-1)$. Also let, $c(u_{2,j}) = c_1$, $j = 1,3,5,\dots,m$ and $c(u_{2,j}) = c_3$, $j = 2,4,6,\dots,(m-1)$. Therefore, $\theta(c_1) = \frac{m+1}{2} + 1$, $\theta(c_2) = \frac{m+1}{2} + 1$ and $\theta(c_3) = m-1$. Clearly for $m \geq 7$, the vertex partition is an optimal near-completion χ-partition.
From Lemma 2.1 and Corollary 2.1 it follow that,

$$\zeta(P_2 \circ P_m) = (\frac{m+1}{2}+1)^2 + 2(m-1)(\frac{m+1}{2}+1) - (4m-1) = \frac{5m^2+2m+1}{4} - m.$$

□

2.1 Corona of paths, $P_n \circ P_m$, $n = 3t$, $t = 1,2,3,\ldots,$ and m is even

The families of paths considered first, i.e. $n = 0(mod\ 3)$ is meant to provide the foundation for $n = 1(mod\ 3)$ or $n = 2(mod\ 3)$ as well as for the corona, $P_n \circ C_m$, m is even.

Proposition 2.2. For $P_n \circ P_m$, $n = 3t$, $t = 1,2,3,\ldots,$ and m is even it follows that, $\zeta(P_n \circ P_m) = \frac{n^2(m-1)^2}{3} - 2nm + 1$.

Proof. Since $\chi(P_n \circ P_m) = 3$, color the vertices v_i, $i = 1,2,3,\ldots,n$ as follows, $c(v_{i+3j}) = c_i$, $i = 1,2,3$ and $j = 0,1,2,\ldots,(t-1)$. Furthermore, color the vertices of the n copies of P_m, as follows. For $j = 0,1,2,\ldots,(t-1)$ and $k_1 = 1,3,5,\ldots,(m-1)$, $k_2 = 2,4,6,\ldots,m$, let:

$$c(u_{1+3j,k_1}) = c_2,\ c(u_{1+3j,k_2}) = c_3,$$
$$c(u_{2+3j,k_1}) = c_1,\ c(u_{2+3j,k_2}) = c_3,$$
$$c(u_{3+3j,k_1}) = c_1,\ c(u_{3+3j,k_2}) = c_2.$$

It follows easily that, $\theta(c_1) = \theta(c_2) = \theta(c_3)$ which is a perfect Lucky 3-coloring of $P_n \circ P_m$.
Furthermore, $\theta(c_i) = 2t\frac{m}{2} + t = \frac{n(m-1)}{3}$, $i = 1,2,3$. Also, $\varepsilon(P_n \circ P_m) = (n-1) + nm + n(m-1) = 2nm - 1$. Therefore, from Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_n \circ P_m) = \frac{n^2(m-1)^2}{3} - 2nm + 1$. □

2.2 Corona of paths, $P_{n'} \circ P_m$, $n' = 3t+1$, $t = 1,2,3,\ldots,$ and m is even

Through immediate induction it follows we just need to extend path P_n in Proposition 2.2 to path, P_{n+1} and derive the result through similar reasoning. The result is presented as a corollary of Proposition 2.2.

Corollary 2.3. For $P_{n'} \circ P_m$, $n' = 3t+1$, $t = 1,2,3,\ldots,$ and m is even it follows that, $\zeta(P_{n'} \circ P_m) = 2(\frac{n(m-1)}{3}+1)(\frac{n(m-1)}{3}+\frac{m}{2}) + (\frac{n(m-1)}{3}+\frac{m}{2})^2 - 2m(n+1) + 1$.

Proof. Following the coloring protocol in Proposition 2.2 and without loss of generality let, $c(v_{n+1}) = c_1$. It implies that, $c(u_{(n+1),i}) = c_2$, $i = 1,3,5,\ldots,(m-1)$ and $c(u_{(n+1),i}) = c_3$, $i = 2,4,6,\ldots,m$. Hence, $\theta(c_1) = \frac{n(m-1)}{3}+1$, $\theta(c_2) = \frac{n(m-1)}{3}+\frac{m}{2}$ and $\theta(c_3) = \frac{n(m-1)}{3}+\frac{m}{2}$. Clearly, for sufficiently large m the coloring is not a perfect Lucky 3-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_{n+1} \circ P_m) = 2nm - 1 + (1 + m + (m-1)) = 2m(n+1) - 1$. Finally,

$$\zeta(P_{n'} \circ P_m) = \zeta(P_{n+1} \circ P_m) = 2(\frac{n(m-1)}{3}+1)(\frac{n(m-1)}{3}+\frac{m}{2}) + (\frac{n(m-1)}{3}+\frac{m}{2})^2 - 2m(n+1) + 1.$$

□

2.3 Corona of paths, $P_{n'} \circ P_m$, $n' = 3t+2$, $t = 1,2,3,\ldots$ and m is even

Through immediate induction it follows we just need to extend path P_{n+1} in Corollary 2.3 to path, P_{n+2} and derive the result through similar reasoning. The result is presented as a corollary of Proposition 2.2.

Corollary 2.4. For $P_{n'} \circ P_m$, $n' = 3t+2$, $t = 1,2,3,\ldots,$ and m is even it follows that, $\zeta(P_{n'} \circ P_m) = (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2}) + (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+m) + (\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2})(\frac{n(m-1)}{3}+\frac{m}{2}+m)$
$-2nm - 4m + 1$.

Proof. Following the coloring protocol in Corollary 2.3 (extended from Proposition 2.2) and without loss of generality let, $c(v_{n+2}) = c_2$. It implies that, $c(u_{(n+2),i}) = c_1$, $i = 1,3,5,\ldots,(m-1)$ and $c(u_{(n+2),i}) = c_3$, $i = 2,4,6,\ldots,m$. Hence, $\theta(c_1) = \frac{n(m-1)}{3}+\frac{m}{2}+1$, $\theta(c_2) = \frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2}$ and $\theta(c_3) = \frac{n(m-1)}{3}+\frac{m}{2}+m$. Clearly, for sufficiently large m the coloring is not a perfect Lucky 3-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_{n+2} \circ P_m) = 2nm - 1 + 2(1 + m + (m-1)) = 2nm + 4m - 1$. Finally,

$$\zeta(P_{n'} \circ P_m) = \zeta(P_{n+2} \circ P_m) = (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2}) + (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+m) + (\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2})(\frac{n(m-1)}{3}+\frac{m}{2}+m) - \varepsilon(P_{n+2} \circ P_m) = (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2}) + (\frac{n(m-1)}{3}+\frac{m}{2}+1)(\frac{n(m-1)}{3}+\frac{m}{2}+m) + (\frac{n(m-1)}{3}+\frac{m}{2}+\frac{m}{2})(\frac{n(m-1)}{3}+\frac{m}{2}+m) - 2nm - 4m + 1.$$

□

The results can be summarized as a main result for $n = i(mod\ 3)$.

Theorem 2.1. *For $P_n \circ P_m$, $n \geq 3$ and m is even it follows that,*
(a) If $m = 0(mod\ 3)$ then, $\zeta(P_n \circ P_m) = \frac{n^2(m-1)^2}{3} - 2nm + 1$.
(b) If $m = 1(mod\ 3)$ then,

$$\zeta(P_n \circ P_m) = 2(\frac{(n-1)(m-1)}{3}+1)(\frac{9n-1)(m-1)}{3}+\frac{m}{2}) + (\frac{(n-1)(m-1)}{3}+\frac{m}{2})^2 - 2mn + 1.$$

(c) If $m = 2(mod\ 3)$ then,

$$\zeta(P_n \circ P_m) = (\frac{(n-2)(m-1)}{3}+\frac{m}{2}+1)(\frac{(n-2)(m-1)}{3}+\frac{m}{2}+\frac{m}{2}) + (\frac{(n-2)(m-1)}{3}+\frac{m}{2}+1)(\frac{(n-2)(m-1)}{3}+\frac{m}{2}+m) + (\frac{(n-2)(m-1)}{3}+\frac{m}{2}+\frac{m}{2})(\frac{(n-2)(m-1)}{3}+\frac{m}{2}+m) - 2m(n-2) - 4m + 1.$$

2.4 Corona of paths, $P_n \circ P_m$, $n = 3t$, $t = 1,2,3,\dots,$ *and m is odd*

The families of paths considered first, i.e. $n = 0(mod\ 3)$ is meant to provide the foundation for $n = 1(mod\ 3)$ or $n = 2(mod\ 3)$.

Proposition 2.3. For $P_n \circ P_m$, $n = 3t$, $t = 1,2,3,\dots,$ and $m \geq 3$, m is odd it follows that, $\zeta(P_n \circ P_m) = n^2(m+1)^2 - 2nm + 1$.

Proof. Since $\chi(P_n \circ P_m) = 3$, color the vertices v_i, $i = 1,2,3,\dots,n$ as follows, $c(v_{i+3j}) = c_i$, $i = 1,2,3$ and $j = 0,1,2,\dots,(t-1)$. Furthermore, color the vertices of the n copies of P_m, as follows. For $j = 0,1,2,\dots,(t-1)$ and $k_1 = 1,3,5,\dots,m$, $k_2 = 2,4,6,\dots,(m-1)$, let:

$$c(u_{1+3j,k_1}) = c_2,\ c(u_{1+3j,k_2}) = c_3,$$
$$c(u_{2+3j,k_1}) = c_3,\ c(u_{2+3j,k_2}) = c_1,$$
$$c(u_{3+3j,k_1}) = c_1,\ c(u_{3+3j,k_2}) = c_2.$$

It follows easily that, $\theta(c_1) = \theta(c_2) = \theta(c_3)$ which is a perfect Lucky 3-coloring of $P_n \circ P_m$.
Furthermore, $\theta(c_i) = t(\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor) + t = \frac{n(m+1)}{3}$, $i = 1,2,3$. Also, $\varepsilon(P_n \circ P_m) = (n-1) + nm + n(m-1) = 2nm - 1$. Therefore, from Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_n \circ P_m) = n^2(m+1)^2 - 2nm + 1$. □

For $P_{n'} \circ P_m$, $n' = 3t+1$ or $n' = 3t+2$, $t = 1,2,3,\dots,$ and $m \geq 3$, m is odd, a corollary follows since the methodology of proof is similar to that in Subsections 2.2 and 2.3.

Corollary 2.5. For $P_{n'} \circ P_m$, $t = 1,2,3,\dots,$ and $m \geq 3$, m is odd we have that:
(a) If $n' = 3t+1$ then,

$$\zeta(P_{n'} \circ P_m) = 2(\tfrac{n(m+1)}{3} + 1)(\tfrac{n(m+1)}{3} + \tfrac{m-1}{2}) + (\tfrac{n(m+1)}{3} + \tfrac{m+1}{2})(\tfrac{n(m+1)}{3} + \tfrac{m-1}{2}) - 2m(n+1) + 1.$$

(b) If $n' = 3t+2$ then,

$$\zeta(P_{n'} \circ P_m) = (\tfrac{n(m+1)}{3} + \tfrac{m-1}{2} + 1)(\tfrac{n(m+1)}{3} + \tfrac{m+1}{2} + 1) + (\tfrac{n(m+1)}{3} + \tfrac{m-1}{2} + 1)(\tfrac{n(m+1)}{3} + m) + (\tfrac{n(m+1)}{3} + \tfrac{m+1}{2} + 1)(\tfrac{n(m+1)}{3} + m) - 2m(n+2) + 1.$$

3. Chromatic Completion Number of $P_n \circ C_m$

A cycle graph (or simply, a cycle) of order n denoted by, C_n, is a graph on $n \geq 1$ vertices say, $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and n edges namely, $E(C_n) = \{v_i v_{i+1} : i = 1,2,3,\dots,n-1\} \cup \{v_n v_1\}$.
In this section we denote, $V(C_m) = \{u_{i,1}, u_{i,2}, u_{i,3}, \dots, u_{i,m}\}$, $i = 1,2,3,\dots,n$ and the edges accordingly.

Theorem 3.1. *For $P_n \circ C_m$, $n \geq 3$ and $m \geq 4$ is even it follows that, $\zeta(P_n \circ C_m) = \zeta(P_n \circ P_m) - n$.*

Proof. Since m is even, $c(u_{i,1}) \neq c(u_{i,m})$, $\forall\ i$ in any proper coloring. Since edge $u_{i,1}u_{i,m} \in E(C_m)$ it cannot be a chromatic completion edge as yielded in the chromatic completion of $P_n \circ P_m$. Therefore, the result is immediate. □

The next general result provides for the result, $\zeta(P_n \circ C_3) = (n-1)(6n-1)$.

Theorem 3.2. *For $P_n \circ K_m$, $n \geq 1$, $m \geq 1$, it follows that, $\zeta(P_n \circ K_m) = (n-1)(\frac{nm(m+1)}{2} - 1)$.*

Proof. Consider $P_n \circ K_m$, $n \geq 1$, $m \geq 1$. Since, $P_n \circ K_m \cong H$ where H is the graph obtained from n copies of $\langle \{v_i\} \cup \{u_{i,j}\}_{\in V(K_{m,i})} \rangle \cong K_{(m+1),i}$, $i = 1,2,3,\dots,n$, $j = 1,2,3,\dots,m$ linked as a string by the edges $v_i v_{i+1}$, $i = 1,2,3,\dots,(n-1)$, it follows that $\chi(P_n \circ K_m) = m+1$. Hence, $\theta(c_i) = n$, $1 \leq i \leq (m+1)$ which is a perfect Lucky $(m+1)$-coloring. Also, $\varepsilon(P_n \circ K_m) = \frac{nm(m+1)}{2} + (n-1)$. Thus, from Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_n \circ K_m) = (\frac{n^2 m(m+1)}{2}) - (\frac{nm(m+1)}{2} + (n-1)) = (n-1)(\frac{nm(m+1)}{2} - 1)$. Immediate induction ensures that the result holds for, $n, m \in \mathbb{N}$. □

It follows easily that for, $P_1 \circ G$, G any graph, $\zeta(P_1 \circ G) = \zeta(G)$. The set of odd integers, $\{m \in \mathbb{N} : m \geq 3$ and m is odd$\}$ will be partitioned into three sets. The sets are, $O_1 = \{3 + 6t : t = 0,1,2,3,\dots\}$, $O_2 = \{5 + 6t : t = 0,1,2,\dots\}$ and $O_3 = \{7 + 6t : t = 0,1,2,\dots\}$.

3.1 Corona $P_n \circ C_m$, $m \in O_1$ and $n = 4k$, $k = 1,2,3,\dots$

First the cases $n = 2,3$ will be presented.

Proposition 3.1. For $P_2 \circ C_m$, $m \in O_1$ and $m \geq 9$ it follows that, $\zeta(P_2 \circ C_m) = \frac{m(13m-6)}{9}$.

Proof. Without loss of generality let $c(v_1) = c_1$, $c(v_2) = c_2$. Because $m \in O_1$ we have, $\theta(c_1) = \frac{m}{3} + 1$, $\theta(c_1) = \frac{m}{3} + 1$, $\theta(c_3) = \frac{2m}{3}$ and $\theta(c_4) = \frac{2m}{3}$which is not a perfect Lucky 3-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_2 \circ C_m) = 4m+1$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_2 \circ C_m) = \frac{13m^2}{9} + \frac{10m}{3} + 1 - (4m+1) = \frac{m(13m-6)}{9}$. □

Proposition 3.2. For $P_3 \circ C_m$, $m \in O_1$ and $m \geq 9$ it follows that, $\zeta(P_3 \circ C_m) = 3((\frac{2m}{3} + 1)^2 + m(\frac{2m}{3} + 1)) - 2(3m+1)$.

Proof. Without loss of generality let $c(v_1) = c_1$, $c(v_2) = c_2$, $c(v_3) = c_3$. Because $m \in O_1$ we have, $\theta(c_1) = \frac{2m}{3} + 1$, $\theta(c_2) = \frac{2m}{3} + 1$, $\theta(c_3) = \frac{2m}{3} + 1$ and $\theta(c_4) = \frac{3m}{3} = m$ which is not a perfect Lucky 3-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_3 \circ C_m) = 2(3m+1)$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_3 \circ C_m) = 3((\frac{2m}{3} + 1)^2 + m(\frac{2m}{3} + 1)) - 2(3m+1)$. □

Proposition 3.3. For $P_n \circ C_m$, $m \in O_1$ and $n = 4k$, $k = 1,2,3,\ldots$ it follows that,
$\zeta(P_n \circ C_m) = \frac{3n^2}{8}(m+1)^2 - n(m+1) - (m-1)$.

Proof. Since $\chi(P_n \circ C_m) = 4$, color the vertices v_i as follows, $c(v_{i+4j}) = c_i$, $i = 1,2,3,4$ and $j = 0,1,2,\ldots,(k-1)$. Furthermore, color the vertices of the n copies of C_m, as follows. For $j = 0,1,2,\ldots,(k-1)$ and $s_1 = 1,4,7,\ldots,(m-2)$, $s_2 = 2,5,8,\ldots,(m-1)$, $s_3 = 3,6,9,\ldots,m$ let:

$$c(u_{1+4j,s_1}) = c_2,\ c(u_{1+4j,s_2}) = c_3,\ c(u_{1+4j,s_3}) = c_4,$$
$$c(u_{2+4j,s_1}) = c_1,\ c(u_{2+4j,s_2}) = c_3,\ c(u_{2+4j,s_3}) = c_4,$$
$$c(u_{3+4j,s_1}) = c_1,\ c(u_{3+4j,s_2}) = c_2,\ c(u_{3+4j,s_3}) = c_4,$$
$$c(u_{4+4j,s_1}) = c_1,\ c(u_{4+4j,s_2}) = c_2,\ c(u_{4+4j,s_3}) = c_3.$$

Thus, the vertex partition is a perfect Lucky 4-partition yielding a perfect Lucky 4-coloring. Also, $\theta(c_i) = k(m+1)$, $1 \leq i \leq 4$ and $\varepsilon(P_n \circ C_m) = n(m+1) + (m-1)$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_n \circ C_m) = 6k^2(m+1)^2 - n(m+1) - (m-1) = \frac{3n^2}{8}(m+1)^2 - n(m+1) - (m-1)$. □

3.2 Corona $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+1$, $k = 1,2,3,\ldots$

Through immediate induction it follows that we just need to extend path P_n in Proposition 3.3 to path P_{n+1} and derive the result through similar reasoning.

Corollary 3.1. For $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+1$, $k = 1,2,3,\ldots$ it follows that,
$\zeta(P_{n'} \circ C_m) = 3[(\frac{n}{4}(m+1)+1)(\frac{n}{4}(m+1)+\frac{m}{3}) + (\frac{n}{4}(m+1)+\frac{m}{3})^2] - n(m+1) - 3m$.

Proof. Following the coloring protocol in Proposition 3.3 and without loss of generality let, $c(v_{n+1}) = c_1$. It implies that $c(u_{n+1,i}) = c_2$, $i = 1,4,7,\ldots,(m-2)$, $c(u_{n+1,i}) = c_3$, $i = 2,5,8,\ldots,(m-1)$, $c(u_{n+1,i}) = c_4$, $i = 3,6,8,\ldots,m$. Hence, $\theta(c_1) = k(m+1)+1$, $\theta(c_2) = \theta(c_3) = \theta(c_4) = k(m+1) + \frac{m}{3}$ which is not a perfect Lucky 4-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_{n'} \circ P_m) = n(m+1) + (m-1) + (m+1) + m = n(m+1) + 3m$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_{n'} \circ C_m) = 3[(\frac{n}{4}(m+1)+1)(\frac{n}{4}(m+1)+\frac{m}{3}) + (\frac{n}{4}(m+1)+\frac{m}{3})^2] - n(m+1) - 3m$. □

3.3 Corona $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+2$, $k = 1,2,3,\ldots$

Through immediate induction it follows that we just need to extend path P_n in Proposition 3.3 to path P_{n+2} and derive the result through similar reasoning.

Corollary 3.2. For $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+2$, $k = 1,2,3,\ldots$ it follows that,
$\zeta(P_{n'} \circ C_m) = ((\frac{n}{4}(m+1) + \frac{m}{3} + 1)(\frac{5n}{4}(m+1) + 3m + 1) + (\frac{n}{4}(m+1) + \frac{2m}{3})^2 - n(m+1) - 5m - 1$.

Proof. Consider the coloring of P_{n+1} in Corollary 3.2. Follow the coloring protocol in Proposition 3.3 and without loss of generality let, $c(v_{n+2}) = c_2$. It implies that $c(u_{n+2,i}) = c_1$, $i = 1,4,7,\ldots,(m-2)$, $c(u_{n+1,i}) = c_3$, $i = 2,5,8,\ldots,(m-1)$, $c(u_{n+1,i}) = c_4$, $i = 3,6,8,\ldots,m$. Hence, $\theta(c_1) = k(m+1) + \frac{m}{3} + 1$, $\theta(c_2) = k(m+1) + \frac{m}{3} + 1$, $\theta(c_3) = \theta(c_4) = k(m+1) + \frac{2m}{3}$ which is not a perfect Lucky 4-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_{n'} \circ P_m) = n(m+1) + 5m + 1$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_{n'} \circ C_m) = ((\frac{n}{4}(m+1) + \frac{m}{3} + 1)(\frac{5n}{4}(m+1) + 3m + 1) + (\frac{n}{4}(m+1) + \frac{2m}{3})^2 - n(m+1) - 5m - 1$. □

3.4 Corona $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+3$, $k = 1,2,3,\ldots$

Through immediate induction it follows that we just need to extend path P_n in Proposition 3.3 to path P_{n+3} and derive the result through similar reasoning.

Corollary 3.3. For $P_{n'} \circ C_m$, $m \in O_1$ and $n' = 4k+3$, $k = 1,2,3,\ldots$ it follows that,
$\zeta(P_{n'} \circ C_m) = 3[(k(m+1) + \frac{2m}{3} + 1)^2 + (k(m+1) + \frac{2m}{3} + 1)(k(m+1) + m)] - n(m+1) - 7m - 2$.

Proof. Consider the coloring of P_{n+2} in Corollary 3.2. Follow the coloring protocol in Proposition 3.3 and without loss of generality let, $c(v_{n+3}) = c_3$. It implies that $c(u_{n+3,i}) = c_1$, $i = 1,4,7,\ldots,(m-2)$, $c(u_{n+1,i}) = c_2$, $i = 2,5,8,\ldots,(m-1)$, $c(u_{n+1,i}) = c_4$, $i = 3,6,8,\ldots,m$. Hence, $\theta(c_1) = \theta(c_2) = \theta(c_3) = k(m+1)\frac{2m}{3} + 1$, $\theta(c_4) = k(m+1) + m$ which is not a perfect Lucky 4-coloring. However, the vertex partition is an optimal near-completion χ-partition. Therefore, the chromatic completion will yield the chromatic completion number.
Also, $\varepsilon(P_{n'} \circ P_m) = n(m+1) + 7m + 2$. From Lemma 2.1 and Corollary 2.1 it follow that, $\zeta(P_{n'} \circ C_m) = 3[(k(m+1) + \frac{2m}{3} + 1)^2 + (k(m+1) + \frac{2m}{3} + 1)(k(m+1) + m)] - n(m+1) - 7m - 2$. □

4. Conclusion

In Section 3 the family of paths were considered by a partition of order i.e. $n = 1,2,3$ and $n = 4k, (4k+1), (4k+2), (4k+3)$, $k = 1,2,3,\ldots$ Corona'd to these paths P_n, only the cycles C_m of order $m \in O_1$ were considered. It is the author's considered view that the methodology has been well established in this paper. Therefore, deriving the results for the cycles of order $m \in O_2$ and $m \in O_3$ Corona'd with the path partitions respectively, remain an exercise for the reader.

Conjecture: $\zeta(C_n \circ P_m) = \zeta(P_n \circ P_m) - 1$ and $\zeta(C_n \circ C_m) = \zeta(P_n \circ C_m) - 1$. Prove or disprove the conjecture.

It is deemed worthy research to find results for other known graph operations.

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References

[1] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications. Macmillan Press, London, 2017.

[2] F. Harary. Graph Theory. Addison-Wesley, Reading MA, 1969.

[3] J. Kok. Chromatic completion number. *Communicated.*

[4] J. Kok. Stability in respect of chromatic completion of graphs. *Communicated.*

[5] B. West. Introduction to Graph Theory. Prentice-Hall, Upper Saddle River, 1996.