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Distance strings of the vertices of certain graphs

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Abstract. The notion of the distance string of a vertex $v_i \in V(G)$ which is denoted by, $\tau(v_i)$ is introduced. Distance strings permit a new approach to determining the induced vertex stress, the total induced vertex stress and total vertex stress (sum of vertex stress over all vertices) of a graph. A seemingly under-studied topic i.e. the eccentricity of a vertex of a bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$ has been furthered. A surprisingly simple result was established, namely for $k \geq 2$, $diam(BK(n,k)) = 5$ if $n = 2k+1$ and $diam(BK(n,k)) = 3$ if $n \neq 2k+1$.

AMS Subject Classifications: 05C12, 05C30, 05C69.

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Contents

1. Introduction

It is assumed that the reader has good knowledge of graph theory. For general notation and concepts in graphs, see [2, 4, 13]. Only finite, undirected and connected simple graphs of order $n \geq 2$ will be considered. For a graph G of order n all vertices will be labeled as v_i, $1 = 1, 2, 3, \ldots, n$. Recall that the distance between vertices v_i and v_j is the length of a shortest path between v_i and v_j. The distance is denoted by $d_G(v_i, v_j)$ (or when the context is clear, simply by $d(v_i, v_j)$). A shortest v_iv_j-path is also called a v_iv_j-distance path. Since G is undirected we have that, $d_G(v_i, v_j) = d_G(v_j, v_i)$. However, for purposes of reasoning of proof or motivation of concepts a v_iv_j-distance path and a v_jv_i-distance path will distinguish between the *departure vertex* and the *destination vertex* and possibly, between two distinct shortest paths. This means that a v_iv_j-distance path has the departure vertex v_i and the destination vertex v_j whilst a v_jv_i-distance path has the departure vertex v_j and the destination vertex v_i.

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The vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash\{v\}$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u \neq w \neq v \neq u} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. Such a shortest uw-path is also called a uw-distance path. See [9, 10]. The *total vertex stress* of G is given by $\mathcal{S}(G) = \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$, [5]. From [11] we recall the definition of total induced vertex stress of a vertex v_i denoted by, $\mathfrak{s}_G(v_i)$, $v_i \in V(G)$.

Definition 1.1. *[11] Let $V(G) = \{v_i : 1 \leq i \leq n\}$. For the ordered vertex pair (v_i, v_j) let there be $k_G(i,j)$ distinct shortest paths of length $l_G(i,j)$ from v_i to v_j. Then, $\mathfrak{s}_G(v_i) = \sum\limits_{j=1, j \neq i}^{n} k_G(i,j)(\ell_G(i,j) - 1)$.*

Put differently, imagine a particle ρ moves along all possible shortest v_iv_j-paths, $j = 1, 2, 3, \ldots, i-1, i+1, \ldots, n$. Definition 1.1 provides the total number of times the particle if departing from vertex v_i will transit through internal vertices.

Let $diam(G) = k \geq 1$. Clearly a vertex v_i has a total number say, $a_{i,1} = deg(v_i)$ of paths of length 1. Similarly the vertex v_i has a total number say, $a_{i,j} \geq 0$ of paths of length $2 \leq j \leq k$. Let the *inductor vector* be $t = (0\ 1\ 2\ \cdots\ k-1)$. Define the $n \times k$ matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k} \end{pmatrix}.$$

For each vertex v_i there is a corresponding (ordered) row i.e. $\tau(v_i) = (a_{i,1} \triangleright a_{i,s} \triangleright \cdots \triangleright a_{i,k})$. Note that $\triangleright$ serves as a spacer between entries of an ordered row. The ordered row $\tau(v_i)$ is called the *distance string* of v_i. Recall that the transpose t^T is a column vector. It follows from Definition 1.1 that,

$$A \cdot t^T = \begin{pmatrix} \mathfrak{s}_G(v_1) \\ \mathfrak{s}_G(v_2) \\ \vdots \\ \mathfrak{s}_G(v_n) \end{pmatrix}.$$

The *induced-stress string* of graph G is defined by,

$$\tau(G) = (\mathfrak{s}_G(v_1) \triangleright \mathfrak{s}_G(v_2) \triangleright \cdots \triangleright \mathfrak{s}_G(v_n)).$$

A new approach to determine the total induced vertex stress of a graph G denoted and defined by, $\mathfrak{s}(G) = \sum\limits_{i=1}^{n} \mathfrak{s}_G(v_i)$ will be explored. Clearly,

$$\mathfrak{s}(G) = \sum_{\substack{j=1 \\ b_{j,1} \in A \cdot t^T}}^{n} b_{j,1} \text{ and } \mathcal{S}(G) = \tfrac{1}{2}\mathfrak{s}(G).$$

The objective of this paper is limited to determining the distance string of each vertex $v_i \in V(G)$.

2. Distance strings of certain graphs

To ensure clarity of the new concepts we begin with well-known graphs.

Proposition 2.1. *For a path P_n, $n \geq 2$ it follows that:*

$$\textit{(i)}\ \tau(v_1) = \tau(v_n) = (\underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-1\ entries)}).$$

$$\textit{(ii)}\ \tau(v_i) = \tau(v_{n-(i-1)}) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \underbrace{0 \triangleright 0 \triangleright 0 \triangleright \cdots \triangleright 0}_{(i-1\ entries)})$$
$$\textit{for, } 2 \leq i \leq \lceil \tfrac{n}{2} \rceil.$$

Proof. For convenience of reasoning assume without loss of generality that a path is depicted horizontally with the vertices consecutively labeled from left to right as, v_1, v_2,..., v_n. Note that a result for v_j, $1 \leq j \leq \lceil \frac{n}{2} \rceil$ also yields the corresponding result for $v_{n-(i-1)}$.

(i) The vertex v_1 has a unique v_1v_j-distance path for $2 \leq j \leq n$. Hence, the result as well as for the *mirror image* vertex v_n.

(ii) The upper bound $\lceil \frac{n}{2} \rceil$ with regards to i is required to settle the results for both, n is odd or even. A vertex v_i, $2 \leq i \leq \lceil \frac{n}{2} \rceil$ has a unique v_1v_j-distance path for $2 \leq j \leq i-1$. Similarly, for the mirror image vertices to the right of v_1. The aforesaid observations settle the partial entries $(\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \cdots)$. The fact that the vertex v_i has a unique v_1v_j-distance path for $2i \leq j \leq n$ settles the additional partial entries $(\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \cdots)$. Since $diam(P_n) = n-1$ and $\tau(v_i)$ is a string with $n-1$ entries, the result is finally obtained. Hence,

$$\tau(v_i) = \tau(v_{n-(i-1)}) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(i-1\ entries)} \triangleright \underbrace{1 \triangleright 1 \triangleright 1 \triangleright \cdots \triangleright 1}_{(n-(2i-1)\ entries)} \triangleright \underbrace{0 \triangleright 0 \triangleright 0 \triangleright \cdots \triangleright 0}_{(i-1\ entries)})$$
$$\text{for, } 2 \leq i \leq \lceil \tfrac{n}{2} \rceil.$$

■

Proposition 2.2. *For a cycle C_n, $n \geq 3$ it follows that:*

$$\tau(v_i) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(\lfloor \frac{n}{2} \rfloor\ entries)})$$
$$\textit{for, } 1 \leq i \leq n.$$

Proof. For convenience of reasoning assume without loss of generality that a cycle has the vertices labeled clockwise and consecutively as, v_1, v_2,..., v_n. By the symmetric property of a cycle the result for an arbitrary v_i is identical to the result of an arbitrary v_j. Without loss of generality the vertex v_1 is selected to settle the proof. It is known that $diam(C_n) = \lfloor \frac{n}{2} \rfloor$. Hence, a distance string has $\lfloor \frac{n}{2} \rfloor$ entries.

Case 1: Clockwise paths. It is trivial to see that v_1 has a unique v_1v_i-distance path in a clockwise direction to the vertices v_i, $2 \leq i \leq \lceil \frac{n}{2} \rceil$.

Case 2: Anti-clockwise paths. It is trivial to see that v_1 has a unique v_1v_j-distance path in an anti-clockwise direction, to the vertices v_i, $n - \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n$.
Clearly, all distance path from vertex v_1 have been accounted. Therefore,

$$\tau(v_i) = (\underbrace{2 \triangleright 2 \triangleright 2 \triangleright \cdots \triangleright 2}_{(\lfloor \frac{n}{2} \rfloor\ entries)})$$
$$\text{for, } 1 \leq i \leq n.$$

■

Proposition 2.3. *For a complete graph K_n, $n \geq 2$ it follows that:*

$$\tau(v_i) = (n-1)$$
$$\textit{for, } 1 \leq i \leq n.$$

Proof. Since $diam(K_n) = 1$ and $deg(v_i) = n - 1$ the result is settled. ■

Corollary 2.4. *A graph G of order $n \geq 2$ has a singleton distance string if and only if $G \cong K_n$.*

Proof. It is known that a graph G of order $n \geq 2$ has $diam(G) = 1$ if and only if $G \cong K_n$. Hence, the result follows by implication. ■

The proof of the next proposition is omitted as an exercise for the reader.

Proposition 2.5. *For a complete bipartite graph $K_{n,m}$, $n, m \geq 1$ with partition sets X, $|X| = n$ and Y, $|Y| = m$ it follows that:*

$$\textit{(i) } \tau(v_i) = (m \triangleright m(n-1)),\ v_i \in X.$$
$$\textit{(ii) } \tau(u_i) = (n \triangleright n(m-1)),\ u_i \in Y.$$

Proposition 2.6. *The Petersen graph G has $\tau(v_i) = (3 \triangleright 6)$, $\forall\ v_i \in V(G)$.*

Proof. It is known that for $G \cong$ *Petersen graph*, $diam(G) = 2$, $deg_G(v_i) = 3$, $\forall\ v_i$ and $|V(G)| = 10$. Hence, each vertex v_i has 3 incident edges or 1-distance paths and 6 distance paths of length 2. Therefore, $\tau(v_i) = (3 \triangleright 6)$, $\forall\ v_i \in V(G)$. ■

3. On bipartite Kneser graphs

It is assumed that the reader has good working knowledge of set theory. For the general notation, notions and important introductory results in set theory, see [3].

Without loss of generality let $n \geq 3$ and let $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. Let X_i, $i = 1, 2, 3, \dots, \binom{n}{k}$ be the k-element subsets of the set, $\{1, 2, 3, \dots, n\}$. Let Y_i, $i = 1, 2, 3, \dots, \binom{n}{k}$ be the $(n-k)$-element subsets of the set, $\{1, 2, 3, \dots, n\}$. Let $V_1 = \{v_i : v_i \mapsto X_i\}$ and $V_2 = \{u_i : u_i \mapsto Y_i\}$. A connected bipartite Kneser graph denoted by $BK(n, k)$ is a graph with vertex set,

$$V(BG(n, k)) = V_1 \cup V_2$$

and the edge set,

$$E(BG(n, k)) = \{v_i u_j : X_i \subset Y_j\}.$$

Claim 3.1. *Since vertex adjacency is defined identically for all vertices the, without loss of generality principle (for brevity, the wlg-principle) applies to our method of proof. All results in respect of an arbitrary vertex v_i are (immediately) valid for all $v_j \in V(BG(n, k))$. Such generalization is axiomatically valid and requires no further proof.*

Claim 3.2. *Let G be a graph in a family of graphs of well-defined (or categorized) order $n = f(i)$, $i = 1, 2, 3, \dots$ which has a well-defined adjacency definition (or regime) such that for any arbitrary vertex v_i (selected by the wlg-principle) of G, a graph theoretical parameter (or property) such as valency, eccentricity, centrality and alike is immediately valid for all vertices of G. Then formal mathematical induction on each category of n can be replaced by the principle of immediate induction.*

Theorem 3.3. [6] *A bipartite Kneser graph, $BK(n, 1)$, $n \geq 3$ has:*

$$diam(BK(n, 1)) = 3.$$

It is known that $BK(n,1)$, $n \geq 3$ is degree regular with $deg(v_i) = \binom{n-1}{1} = n-1$.

Theorem 3.4. *A bipartite Kneser graph, $BK(n,1)$, $n \geq 3$ has:*

$$\tau(v_i) = ((n-1) \triangleright (n-1)(n-2) \triangleright (n-1)(n-2)^2).$$

Proof. Because $diam(BK(n,1)) = 3$ and $BK(n,1)$ is regular it follows immediately the each vertex has exactly $(n-1)$ shortest 1-paths (or edges), exactly $(n-1)(n-2)$ shortest 2-paths and exactly $(n-1)(n-2)^2$ shortest 3-paths. Therefore the result is settled. ■

Theorem 3.5. *A bipartite Kneser graph $BK(n,2)$, $n \geq 5$ has:*

$$diam(BK(n,2)) = \begin{cases} 5, & \text{if } n = 5; \\ \\ 3, & \text{if } n \geq 6. \end{cases}$$

Proof. The diameter of a connected graph G is equal to the maximum eccentricity $\epsilon(v)$ of some vertex $v \in V(G)$. Since all bipartite Kneser graphs are vertex transitive (see Lemma 3.1 in [7]) and it is degree regular it follows that the diameter is equal to the eccentricity of any vertex. For convenience the eccentricity of $v_1 \in V_1$ will be considered. It is known that for $n > 2k$ (or $n \geq 2k+1$) the bipartite Kneser graph $BK(n,k)$ is connected which implies that a finite diameter exists. Hence, $BK(n,2)$, $n \geq 5$ is connected and has a finite diameter. For $k = 2$ the first bipartite Kneser graph to consider is $BK(5,2)$.

Case 1. Let $n = 2k+1 = 5$. Let V_1 defined as:
$v_1 \mapsto \{1,2\}$, $v_2 \mapsto \{1,3\}$, $v_3 \mapsto \{1,4\}$, $v_4 \mapsto \{1,5\}$, $v_5 \mapsto \{2,3\}$, $v_6 \mapsto \{2,4\}$, $v_7 \mapsto \{2,5\}$, $v_8 \mapsto \{3,4\}$, $v_9 \mapsto \{3,5\}$, $v_{10} \mapsto \{4,5\}$.
Let V_2 be defined as:
$u_1 \mapsto \{1,2,3\}$, $u_2 \mapsto \{1,2,4\}$, $u_3 \mapsto \{1,2,5\}$, $u_4 \mapsto \{1,3,4\}$, $u_5 \mapsto \{1,3,5\}$, $u_6 \mapsto \{1,4,5\}$, $u_7 \mapsto \{2,3,4\}$, $u_8 \mapsto \{2,3,5\}$, $u_9 \mapsto \{2,4,5\}$, $u_{10} \mapsto \{3,4,5\}$.
By utilizing an appropriate shortest path algorithm such as in [8] or [15] or in this case an exhaustive method, it is established that $\epsilon(v_1) = 5$. Without loss of generality one such *diam*-path is $v_1u_3v_7u_9v_{10}u_{10}$. Therefore, $diam(BK(5,2)) = 5$. From the standard heuristic methods used to systematically find the 2-element and the 3-element subsets[‡] it follow that the v_1u_{10}-distance path is indeed the eccentricity of v_1.

Case 2. Let $n \geq 2(k+1)$. Construct a bipartite Kneser graph $BK(6,2)$ from the $BK(5,2)$. It requires extension of the set V_1 to obtain say,

$$V_1^\star = V_1 \cup \{\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}\}.$$

Let

$$v_{11}^\star \mapsto \{1,6\}, v_{12}^\star \mapsto \{2,6\}, v_{13}^\star \mapsto \{3,6\}, v_{14}^\star \mapsto \{4,6\}, v_{15}^\star \mapsto \{5,6\}.$$

The set V_2 requires extensions as well.

$$V_2^\star = \{u_i^\star \mapsto u_i \cup \{6\} : u_i \in V_2\} \cup$$

$$\{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}.$$

Let

$$u_{11}^\star \mapsto \{1,2,3,4\}, u_{12}^\star \mapsto \{1,2,3,5\}, u_{13}^\star \mapsto \{1,2,4,5\}, u_{14}^\star \mapsto \{1,3,4,5\}, u_{15}^\star \mapsto \{2,3,4,5\}.$$

‡ See https://www.omnicalculator.com,math,subset or [4]

After applying the adjacency regime the first observation is that the $v_1u^\star_{10}$-distance path shortened to 3 i.e. $v_1u^\star_2v^\star_{14}u^\star_{10}$. In $BK(6,2)$ the eccentricity of v_1 is found to be $\epsilon(v_1)=3$. Without loss of generality a *diam*-path which is obtained similar in method to that used for $BK(5,2)$ is given by, $v_1u^\star_3v^\star_{15}u^\star_{10}$. In fact, by first generating the subsets in the conventional algorithmic fashion we have $V_1(BK(6,2))$ defined as:
$v_1\mapsto\{1,2\}$, $v_2\mapsto\{1,3\}$, $v_3\mapsto\{1,4\}$, $v_4\mapsto\{1,5\}$, $v_5\mapsto\{1,6\}$, $v_6\mapsto\{2,3\}$, $v_7\mapsto\{2,4\}$, $v_8\mapsto\{2,5\}$, $v_9\mapsto\{2,6\}$, $v_{10}\mapsto\{3,4\}$, $v_{11}\mapsto\{3,5\}$, $v_{12}\mapsto\{3,6\}$, $v_{13}\mapsto\{4,5\}$, $v_{14}\mapsto\{4,6\}$, $v_{15}\mapsto\{5,6\}$.
Let $V_2(BK(6,2))$ be defined as:
$u_1\mapsto\{1,2,3,4\}$, $u_2\mapsto\{1,2,3,5\}$, $u_3\mapsto\{1,2,3,6\}$, $u_4\mapsto\{1,2,4,5\}$, $u_5\mapsto\{1,2,4,6\}$, $u_6\mapsto\{1,2,5,6\}$, $u_7\mapsto\{1,3,4,5\}$, $u_8\mapsto\{1,3,4,6\}$, $u_9\mapsto\{1,3,5,6\}$, $u_{10}\mapsto\{1,4,5,6\}$, $u_{11}\mapsto\{2,3,4,5\}$, $u_{12}\mapsto\{2,3,4,6\}$, $u_{13}\mapsto\{2,3,5,6\}$, $u_{14}\mapsto\{2,4,5,6\}$, $u_{15}\mapsto\{3,4,5,6\}$.
A *diam*-path which is obtained by similar method to that used for $BK(5,2)$ is given by, $v_1u_6v_{15}u_{15}$.

Assume the result $diam(BK(n,2))=3$ holds for $7\leq n\leq\ell$. Obviously the vertex changes and the addition of exactly 2ℓ new vertices as n progresses consecutively through ℓ to $\ell+1$ (in fact as n progresses consecutively through $\ell,\ell+1,\ell+2\dots$) remain consistent. By similar reasoning to show the result for the progression from $n=5$ to $n=6$, it follows by immediate induction that the results holds for the progression from $n=\ell$ to $n=\ell+1$. Finally by applying the wlg-principle for $n\geq 6$ read with the well-defined 2-element subsets and $(n-2)$-element subsets, the principle of immediately induction is valid. Furthermore, a heuristic method to be used in general for $BK(n,2)$, $n\geq 6$ is:
(a) Select $v_1\mapsto\{1,2\}$ and link to $u_{\binom{n-2}{2}}\mapsto\{1,2,5,6,\dots,n\}$.
(b) From $u_{\binom{n-2}{2}}\mapsto\{1,2,5,6,\dots,n\}$ link to $v_{\binom{n}{2}}\mapsto\{n-1,n\}$ then,
(c) Link $v_{\binom{n}{2}}\mapsto\{n-1,n\}$ to $u_{\binom{n}{2}}\mapsto\{3,4,\dots,n-1,n\}$.
Therefore,

$$diam(BK(n,2))=\begin{cases}5, & \text{if } n=5;\\ \\ 3, & \text{if } n\geq 6.\end{cases}$$

■

Lemma 3.6. *A bipartite Kneser graph $BK(n,k)$, $n=2k+1$, $k\geq 2$ has $diam(BK(2k+1,k))=5$.*

Proof. The result for $BK(5,2)$ follows from Theorem 3.5. For $BK(7,3)$ we utilize

https://www.omnicalculator.com>math>subset

to systematically (conventionally) generate the 3-element and 4-element subsets respectively. Label the respective subsets as generated consecutively as:
$v_1\mapsto\{1,2,3\},\dots,v_{35}\mapsto\{5,6,7\}$ and $u_1\mapsto\{1,2,3,4\},\dots,u_{35}\mapsto\{4,5,6,7\}$.
By utilizing an appropriate shortest path algorithm such as in [8] or [15] or in this case an exhaustive method, it is established that $\epsilon(v_1)=5$. Without loss of generality one such *diam*-path for $BK(7,3)$ is $v_1u_4v_{31}u_{34}v_{35}u_{35}$. Therefore, $diam(BK(7,3))=5$.
By induction reasoning similar to that stated in the proof of Theorem 3.5 and utilizing Claim 3.2 we may utilize immediate induction for the result as stated. Hence, $BK(n,k)$, $n=2k+1$, $k\geq 2$ has:

$$diam(BK(2k+1,k))=5.$$

■

From the proof of Theorem 3.5, Lemma 3.6 read together with Claim 3.3 an immediate generalized result is permitted.

Theorem 3.7. *A bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$, $k \geq 3$ has:*

$$diam(BK(n,k)) = \begin{cases} 5, & \text{if } n = 2k+1; \\ \\ 3, & \text{if } n \geq 2(k+1)(\text{or } n > 2k+1). \end{cases}$$

The next result is a direct consequence of Theorem 3.7 and the fact that $deg_{BK(n,k)}(v_i) = \binom{n-k}{k}$.

Theorem 3.8. *A bipartite Kneser graph $BK(n,k)$, $n \geq 2k+1$, $k \geq 3$ and $t = \binom{n-k}{k}$ has:*

$$\tau(v_i) = \begin{cases} (t \triangleright t(t-1) \triangleright t(t-1)^2 \triangleright t(t-1)^3 \triangleright t(t-1)^4), & \text{if } n = 2k+1; \\ \\ (t \triangleright t(t-1) \triangleright t(t-1)^2), & \text{if } n \geq 2(k+1). \end{cases}$$

Some authors define the adjacency of bipartite Kneser graphs as:

$$E(BG(n,k)) = \{v_i u_j : X_i \subseteq Y_j\}.$$

It implies that for $n = 2k$ the graph $BK(2k,k)$ is a matching graph. Furthermore, since the empty set is a proper subset of a set, the graph $BK(n,0) \cong K_2$ with $v_1 \mapsto \emptyset$ and $u_1 \mapsto \{1,2,3,\ldots,n\}$. Since $BK(n,k) \cong BK(n,n-k)$, Theorem 3.7 can be stated as:

Theorem 3.9. Alternative *A bipartite Kneser graph $BK(n,k)$, $n \geq 5$, $k \geq 3$ has:*

$$diam(BK(n,k)) = \begin{cases} 5, & \text{if } n = 2k+1; \\ \\ 3, & \text{if } n \neq 2k+1. \end{cases}$$

4. A research avenue

For $n = 0,1,2,3,\ldots$ the integer sequence $A001349$ found in the on-line encyclopedia of integer sequences (OEIS) presents the number of distinct (non-isomorphic) connected simple graphs on n vertices. Let a_n consecutively map onto

$$1,1,1,2,6,21,112,853,11117,261080,11716571,\ldots$$

This study only considers connected simple graphs of order $n \geq 2$. Consider a graph theoretical property $\sqcup$. Assume that for $n \geq 2$ at least $\ell_n \geq 0$ graphs of order n have property $\sqcup$. Hence, the portion (or ratio) of graphs of order n which have the property is $\frac{\ell_n}{a_n}$. If $\lim\limits_{n\to\infty} \frac{\sum\limits_{m=2}^{n} \ell_m}{\sum\limits_{m=2}^{n} a_m} = 1$, it is said that *almost all* graphs have property $\sqcup$.

Conjecture 4.1. *For almost all graphs G and its distinct vertex pairs $v_i, v_j \in V(G)$ it follow that:*

$$\sum_{a_{i,k} \in \tau(v_i)} a_{i,k} = \sum_{a_{j,k} \in \tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

An example to show that Conjecture 4.1 is not valid for all graphs is $K_n - e$, $n \geq 4$. If in K_n, $n \geq 4$ and without loss of generality, the edge $e = v_1v_2$ is deleted then, $\tau(v_1) = (n-2 \triangleright n-2)$ and $\tau(v_n) = (n-1 \triangleright 0)$. Clearly, $2(n-2) \neq n-1$ for $n \geq 4$.

Conjecture 4.2. *For almost all graphs G it follows that:*

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} \leq \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j) = \varepsilon(G).$$

For certain families of graphs we can show that equality holds in Conjectures 4.1 and 4.2.

Theorem 4.3. *For all paths P_n, $n \geq 2$ and all its distinct vertex pairs $v_i, v_j \in V(P_n)$ it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(P_n) = n-1.$$

(ii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j).$$

Proof. (i) From Proposition 2.1(i) it follows that:

$$\sum_{a_{1,k}\in\tau(v_1)} a_{1,k} = \sum_{a_{n,k}\in\tau(v_n)} a_{n,k} = n-1,\ 1 \leq k \leq diam(P_n) = n-1.$$

From Proposition 2.1(ii) it follows that:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = 2(i-1) + \Big[n-(2i-1)\Big] = n-1,\ 1 \leq k \leq diam(P_n) = n-1.$$

The result is settled.
(ii) The sum of vertex degrees in a path is given by, $2+2(n-2) = 2n-2 = 2(n-1)$. Therefore,

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j).$$

■

The proof of the next result follows similar reasoning to that found in the proof Theorem 4.3. The proof is omitted as an exercise to the reader.

Theorem 4.4. *For all cycles C_n, $n \geq 4$ and all its distinct vertex pairs $v_i, v_j \in V(C_n)$ it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(C_n) = \lfloor\tfrac{n}{2}\rfloor.$$

(ii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j)$$

and n is even.

(iii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} < \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j)$$

and n is odd.

Theorem 4.5. *For all complete bipartite graphs $K_{n,m}$, $n, m \geq 1$ and all its distinct vertex pairs $v_i, v_j \in V(K_{n,m})$ or $v_i, u_j \in V(K_{n,m})$ or $u_i, u_j \in V(K_{n,m})$ or simply referred to as vertices v_i, v_j, it follows that:*
(i)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(K_{n,m}) = 2.$$

(ii)

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \frac{1}{2}\sum_{v_j\in V(G)} deg(v_j).$$

Proof. Both results is a direct result from the fact that, $m + m(n-1) = n + n(m-1) = 2nm$. ■

The notion of *stress regular* graphs was introduced in [10]. A graph G for which $\mathcal{S}_G(v_i) = \mathcal{S}_G(v_j)$ for all distinct pairs $v_i, v_j \in V(G)$ is said to be stress regular.

Theorem 4.6. *For a stress regular graph G it follows that:*

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1\leq k\leq diam(G).$$

Proof. Since G is stress regular it implies that $\mathcal{S}_G(v_i) = \mathcal{S}_G(v_j) \Leftrightarrow \mathfrak{s}_G(v_i) = \mathfrak{s}_G(v_j),\ \forall\ v_i, v_j \in V(G)$. The column vector t^T is a "constant" vector hence, $\tau(v_i) = \tau(v_j),\ \forall\ v_i, v_j \in V(G)$. If the latter is not true then G is not stress regular which is a contradiction. Thus,

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1\leq k\leq diam(G).$$

■

We recall further results from [10]

Theorem 4.7. [10] *Every distance regular graph is stress regular.*

Corollary 4.8. [10] *Every strongly regular graph is stress regular.*

Corollary 4.9. [10] *Every distance transitive graph is stress regular.*

Let X_i, $i = 1, 2, 3, \ldots, \binom{n}{k}$ be the k-element subsets of the set, $\{1, 2, 3, \ldots, n\}$. A Kneser graph denoted by $KG(n, k)$, $n, k \in \mathbb{N}$ is the graph with vertex set,

$$V(KG(n,k)) = \{v_i : v_i \mapsto X_i\}$$

and the edge set,

$$E(KG(n,k)) = \{v_iv_j : X_i \cap X_j = \emptyset\}.$$

It is known that the family of Kneser graphs $KG(n, 2)$ are distance regular graphs. Therefore, from Theorem 4.7 it follows that the Kneser graphs $KG(n, 2)$ are stress regular. Furthermore, it is known from [1] that every distance regular graph G with $diam(G) = 2$, is strongly regular. We recall some results from [6].

Corollary 4.10. [6] *Kneser graphs $KG(n, k_1)$, $k_1 \in \mathbb{N}\backslash\{1, 2\}$, $n \geq 3k_1 - 1$ are stress regular.*

In fact, a general result (without further proof) is permitted from the knowledge that all Kneser graphs $KG(n, k)$, $n \geq k$ are vertex transitive.

Theorem 4.11. [6] *All Kneser graphs $KG(n, k)$, $n \geq k$ are stress regular.*

The immediate above read together with Theorem 4.6 permits the next corollary without further proof.

Corollary 4.12. *(i) Every distance regular graph has:*

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1\leq k\leq diam(G).$$

(ii) Every strongly regular graph has:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

(iii) Every distance transitive graph has:

$$\sum_{a_{i,k}\in\tau(v_i)} a_{i,k} = \sum_{a_{j,k}\in\tau(v_j)} a_{j,k},\ 1 \leq k \leq diam(G).$$

5. Conclusion

Sections 1, 2 and 3 laid the foundation for further research related to the notion of distance strings of vertices and the induced-stress string of a graph. A wide scope for further research remains open. Prove or disprove the next conjecture.

Conjecture 5.1. *For the induced-stress string $\tau(G)$ of a graph G it follows that:*

$$\mathfrak{s}_G(v_i) = min\{\mathfrak{s}_G(v_j) : \mathfrak{s}_G(v_j) \in \tau(G)\}, \text{ is even.}$$

The author holds the view that the eccentricity of a vertex in bipartite Kneser graphs in general did not receive adequate attention in the literature. As stated before, Theorem 21 in [6] established that $diam(BK(n,1)) = 3$, $n \geq 3$. In this paper a result for $diam(BK(n,k))$, $k \geq 2$, $n \geq 2k+1$ was established. With the results of Section 3 it is now possible to determine all the vertex stress related parameters for bipartite Kneser graphs.

Section 4 presents two interesting conjectures. Statements of the form "... for almost all graphs (disconnected and connected) ..." is an interesting field for further research. More so with the vigorous research in experimental mathematics and the introduction of AI-mathematics. For example from [14]:

Theorem 5.2. [14] *Almost all graphs have diameter* 2.

Corollary 5.3. [14] *Almost all graphs have every edge in a triangle.*

Corollary 5.4. [14] *Almost all graphs are connected.*

From Corollary 5.1 we state the following.

Corollary 5.5. *Almost all ICT-networks, AI-networks, data structure networks, social media networks and alike will remain incomplete networks.*

From Corollary 5.2 we state the following.

Corollary 5.6. *Almost all graphs G have chromatic number, $\chi(G) \geq 3$.*

From Corollary 5.3 we state that:

Corollary 5.7. *All results which are common for connected graphs are valid for almost all graphs.*

Author is of the view that statements of the form "... for almost all graphs (disconnected and connected) ..." have important implications in the field of experimental mathematics.

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