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On various quasi ideals in b-semirings

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Abstract

We introduce six types of quasi ideals in b-semirings. Each quasi ideals generated by single element(set) is established. We characterize various 1-regular(2-regular, regular) by using generalized 1-quasi (generalized 2-quasi, generalized quasi) ideal, 1-quasi (2-quasi, quasi) ideal, weak1(2)-right(left) ideal and weak1(2)-ideal. Examples are provided to strengthen our results.

Keywords

1-quasi ideal, 2-quasi ideal, generalized 1-quasi ideal, generalized 2-quasi ideal.

AMS Subject Classification

16Y60

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1. Introduction

The concept of b-semirings [4] was introduced by Ronnason in 2009. The concept of weak 1(2)-right ideal, weak 1(2)- left ideal, weak 1(2)-ideal in b-semirings are introduced by Mohanraj et al [3]. By introducing the 1-regular(2-regular, regular) b-semirings. The 1-regular(2-regular, regular) b-semirings are characterized by using various weak-ideals by Mohanraj et al [1].We initiated the notions of k-regular b-semirings using they are various weak k-ideals [2].

2. Preliminaries

The algebraic structure $(S,+,\cdot)$ is called a b-semiring if $(S,+)$ and $(S,\cdot)$ are semigroups, connected by four distributive laws that "$\cdot$" distributes over "$+$" from left and right and "$+$" distributes over "$\cdot$" from left and right[4]. The subset A of S is called a sub b-semiring in S if A is itself a b-semiring. The subset A of S is called a weak-1 right ideal (weak-1 left ideal) in S if $a_1+a_2 \in A$ and $a_1 \cdot s \in A$ $(s \cdot a_1 \in A)$ for all $a_1, a_2 \in A$ and $s \in S$ [1]. The subset A of S is called a weak-2 right ideal (weak-2 left ideal) in S if $a_1 \cdot a_2 \in A$ and $a_1 + s \in A$ $(s + a_1 \in A)$ for all $a_1, a_2 \in A$ and $s \in S$ [1]. The subset A of S is called a weak-1 ideal (weak-2 ideal) in S if it is both weak-1 right ideal (weak-2 right ideal) and weak-1 left ideal(weak-2 left ideal) in S [1]. The subset A of S is called a right ideal (left ideal) in S if it is both weak-1 right ideal (weak-1 left ideal) and weak-2 right ideal(weak-2 left ideal) in S [1]. The b-semiring S [1] is called 1-regular [2-regular] if for each $a \in S$ there exists $x \in S$ such that $a \cdot (x \cdot a) = a$ $[a + (x + a) = a]$. The b-semiring S [1] is called regular if it is both 1-regular and 2-regular in S.

3. 1-quasi ideals in b-semirings

Throughout this paper, S denotes b-semirings unless otherwise noted. The intersection of a weak-1 right ideal and weak-1 left ideal in S is neither weak-1 right ideal nor weak-1 left ideal in S by the following Example 3.1. Naturally one question arises;

What is the intersection of weak 1(2)-right ideal with weak 1(2)-left ideal? We answer the questions by introducing 1-quasi(2-quasi)ideal.

Example 3.1. *Consider the b-semiring* $S = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ *by the following table.*

$+$	g_1	g_2	g_3	g_4	g_5	g_6
g_1	g_1	g_2	g_3	g_4	g_5	g_6
g_2	g_1	g_2	g_3	g_4	g_5	g_6
g_3	g_1	g_2	g_3	g_4	g_5	g_6
g_4	g_1	g_2	g_3	g_4	g_5	g_6
g_5	g_1	g_2	g_3	g_4	g_5	g_6
g_6	g_1	g_2	g_3	g_4	g_5	g_6

$\cdot$	g_1	g_2	g_3	g_4	g_5	g_6
g_1	g_1	g_1	g_1	g_1	g_5	g_5
g_2	g_1	g_2	g_1	g_4	g_5	g_6
g_3	g_1	g_1	g_3	g_1	g_5	g_5
g_4	g_4	g_4	g_4	g_4	g_6	g_6
g_5	g_1	g_1	g_5	g_1	g_5	g_5
g_6	g_4	g_4	g_6	g_4	g_6	g_6

Now, $A = \{g_1, g_5\}$ and $B = \{g_5, g_6\}$ are weak-1 right ideal and weak-1 left ideal respectively, but $A \cap B$ is neither weak-1 right ideal nor weak-1 left ideal in S.

Definition 3.2. *(i) The subset Q of S is called a generalized 1-quasi ideal in S if $(Q \cdot S) \cap (S \cdot Q) \subseteq Q$.*
(ii) The generalized 1-quasi ideal Q is called a 1-quasi ideal in S if Q is a sub b-semiring.

Lemma 3.3. *The generalized 1-quasi ideal Q is a 1-quasi ideal in S if Q is closed under "$+$".*

Proof. Suppose that Q is a generalized 1-quasi ideal which is closed under "$+$". Now, $Q \cdot Q \subseteq Q \cdot S$ and $Q \cdot Q \subseteq S \cdot Q$ imply $Q \cdot Q \subseteq (Q \cdot S) \cap (S \cdot Q) \subseteq Q$. Thus,$Q$ is a 1-quasi ideal in S. □

Remark 3.4. *The generalized 1-quasi ideal fails to be a 1-quasi ideal in S by the Example 3.5.*

Example 3.5. *Consider the b-semiring $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ by the following table.*

$+$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_5	a_5
a_2	a_1	a_2	a_1	a_4	a_5	a_6
a_3	a_1	a_1	a_3	a_1	a_5	a_5
a_4	a_4	a_4	a_4	a_4	a_6	a_6
a_5	a_1	a_1	a_5	a_1	a_5	a_5
a_6	a_4	a_4	a_6	a_4	a_6	a_6

$\cdot$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_2	a_2	a_2	a_2	a_2	a_2
a_3	a_3	a_3	a_3	a_3	a_3	a_3
a_4	a_4	a_4	a_4	a_4	a_4	a_4
a_5	a_5	a_5	a_5	a_5	a_5	a_5
a_6	a_6	a_6	a_6	a_6	a_6	a_6

Clearly, $\{a_2, a_5\}$ is a generalized 1-quasi ideal, but $a_5 + a_2 \notin \{a_2, a_5\}$ implies $\{a_2, a_5\}$ is not 1-quasi ideal in S.

Lemma 3.6. *Every weak-1 right (left) ideal is a 1-quasi ideal in S.*

Remark 3.7. *Converse of the Lemma 3.6 fails by the Example 3.8.*

Example 3.8. *Consider the b-semiring $(S, +, \cdot)$ by the following table.*

$+$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_2	x_3	x_4	x_5	x_6
x_2	x_1	x_2	x_3	x_4	x_5	x_6
x_3	x_1	x_2	x_3	x_4	x_5	x_6
x_4	x_1	x_2	x_3	x_4	x_5	x_6
x_5	x_1	x_2	x_3	x_4	x_5	x_6
x_6	x_1	x_2	x_3	x_4	x_5	x_6

$\cdot$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_2	x_4	x_4	x_4
x_3	x_1	x_2	x_3	x_4	x_5	x_6
x_4	x_1	x_2	x_2	x_4	x_4	x_4
x_5	x_1	x_2	x_3	x_4	x_5	x_6
x_6	x_1	x_2	x_3	x_4	x_5	x_6

Clearly, $\{x_1, x_2\}$ is a 1-quasi ideal, but not weak-1 right ideal in S.

Theorem 3.9. *The intersection of weak-1 right ideal with weak-1 left ideal in S is a 1-quasi ideal.*

Proof. For the weak-1 right ideal A and weak-1 left ideal B in S,$A \cap B$ is a sub b-semiring. Now,$[(A \cap B) \cdot S] \cap [S \cdot (A \cap B)] \subseteq (A \cdot S) \cap (S \cdot B) \subseteq A \cap B$ implies $A \cap B$ is a 1- quasi ideal in S. □

Theorem 3.10. *For any $a \in S$, the generalized 1-quasi ideal generated by "a", denoted by $< a >_{g1q}$ is given by $\{a\} \cup [(a \cdot S) \cap (S \cdot a)]$.*

Proof. Now, $x \in (a \cdot S) \cap (S \cdot a)$, then $(x \cdot S) \cap (S \cdot x) \subseteq (a \cdot S) \cap (S \cdot a)$. Thus, $\{a\} \cup [(a \cdot S) \cap (S \cdot a)]$ is a generalized 1-quasi ideal in S. If A is a generalized 1-quasi ideal in S such that $a \in A$, then $\{a\} \cup [(a \cdot S) \cap (S \cdot a)] \subseteq A$. Thus $< a >_{g1q}$ is the generalized 1-quasi ideal generated by "a". □

Corollary 3.11. *For a subset A of S, $A \cup [(A \cdot S) \cap (S \cdot A)]$ is the generalized 1-quasi ideal generated by a set A in S.*

Lemma 3.12. *[1] For $n \in \mathbb{Z}^+$ and $a \in S$,*
(i) $(na \cdot s) = n(a \cdot s) = (a \cdot ns)$.
(ii) $(s \cdot na) = n(s \cdot a) = (ns \cdot a)$, where $na = a + a + \ldots n$ times.
(iii) $(a^n + s) = (a + s)^n = (a + s^n)$.
(iv) $(s + a^n) = (s + a)^n = (s^n + a)$, where $a^n = a \cdot a \cdot \ldots n$ times.

Theorem 3.13. *For any $a \in S$, the 1-quasi ideal generated by "a", denoted by $< a >_{1q}$ is given by $\{na | n \in \mathbb{Z}^+\} \cup [(a \cdot S) \cap (S \cdot a)]$.*

Proof. Clearly, $\{na | n \in \mathbb{Z}^+\} \cup [(a \cdot S) \cap (S \cdot a)]$ is generalized 1-quasi ideal. For $x, y \in [(a \cdot S) \cap (S \cdot a)]$, $x + y = a \cdot (s_1 + s_3) \in a \cdot S$. Similarly, $x + y = (s_2 + s_4) \cdot a \in S \cdot a$ imply $x + y \in (a \cdot S) \cap (S \cdot a)$. For $x \in \{na | n \in \mathbb{Z}^+\}$ and $y \in [(a \cdot S) \cap (S \cdot a)]$ and by Lemma 3.12, $x + y = na + (a \cdot s_3) = a \cdot [(n+1)(na + s_3)] \in a \cdot S$ and $x + y = na + (s_4 \cdot a) = [(n+1)(na + s_4)] \cdot a \in S \cdot a$. Thus $x + y \in [(a \cdot S) \cap (S \cdot a)]$. Similarly $y + x \in [(a \cdot S) \cap (S \cdot a)]$. By Lemma 3.3, $\{na | n \in \mathbb{Z}^+\} \cup [(a \cdot S) \cap (S \cdot a)]$ is a 1-quasi ideal in S. If A is a 1-quasi ideal in S such that $a \in A$, then $\{na | n \in \mathbb{Z}^+\} \cup [(a \cdot S) \cap (S \cdot a)] \subseteq A$. Hence $< a >_{1q}$ is the 1-quasi ideal generated by "a". □

Notation 3.14. *For a subset A of S and* $i = 1,2,3,...,n$
(i) $\sum A = \{(a_1 + a_2 + ... + a_n) | a_i \in A\}$.
(ii) $\prod A = \{(a_1 \cdot a_2 \cdot ... \cdot a_n) | a_i \in A\}$.
(iii) $\sum(A \cdot S) = \{(a_1 \cdot s_1) + (a_2 \cdot s_2) + ... + (a_n \cdot s_n) | a_i \in A,\ s_i \in S\}$.
(iv) $\prod(A + S) = \{(a_1 + s_1) \cdot (a_2 + s_2) \cdot ... \cdot (a_n + s_n) | a_i \in A,\ s_i \in S\}$.

Corollary 3.15. *For the subset A of S,* $\sum A \cup [\sum(A \cdot S) \cap \sum(S \cdot A)]$ *is the 1-quasi ideal generated by a set A in S.*

Theorem 3.16. *[1] (i) The b-semiring S is 1-regular if and only if*
$R \cap L = R \cdot L$, for every weak-1 right ideals R and every weak-1 left ideals L in S.
(ii) The b-semiring S is 2-regular if and only if $R \cap L = R + L$, for every weak-2 right ideals R and every weak-2 left ideals L in S .

Theorem 3.17. *For a b-semiring S, the following conditions are equivalent.*
(1) S is 1-regular.
(2) $R \cap Q_1 \subseteq R \cdot Q_1$, for the weak-1 right ideals R and generalized 1-quasi ideals Q_1.
(3) $R \cap Q \subseteq R \cdot Q$, for the weak-1 right ideals R and 1-quasi ideals Q.
(4) $Q_1 \cap L \subseteq Q_1 \cdot L$, the for generalized 1-quasi ideals Q_1 and weak-1 left ideals L.
(5) $Q \cap L \subseteq Q \cdot L$, for the 1-quasi ideals Q and weak-1 left ideals L.
(6) $R \cap L = R \cdot L$, for the weak-1 right ideals R and weak-1 left ideals L.

Proof. First, we prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (6) \Rightarrow (1)$ and $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$.
$(1) \Rightarrow (2)$ For $a \in R \cap Q_1$, then there exampleist $s \in S$ such that $a = (a \cdot s) \cdot a$. Thus, $R \cap Q_1 \subseteq R \cdot Q_1$.
$(2) \Rightarrow (3)$ Straightforward.
$(3) \Rightarrow (6)$ By Lemma 3.6, $R \cap L \subseteq R \cdot L$. Now, $R \cdot L \subseteq R \cdot S \subseteq R$ and $R \cdot L \subseteq S \cdot L \subseteq L$. Then (6) follows.
$(6) \Rightarrow (1)$ The proof follows from Theorem 3.16.
$(1) \Rightarrow (4)$ For $a \in Q_1 \cap L$, then there exampleist $s \in S$ such that $a = a \cdot (s \cdot a)$. Thus, $Q_1 \cap L \subseteq Q_1 \cdot L$.
$(4) \Rightarrow (5)$ Straightforward.
$(5) \Rightarrow (6)$ The proof follows from Lemma 3.6. □

Theorem 3.18. *For a b-semiring S, the following conditions are equivalent.*
(1) S is 1-regular.
(2) $Q_1 \cap I \cap Q_2 \subseteq Q_1 \cdot I \cdot Q_2$, for the generalized 1-quasi ideals Q_1 and Q_2 and weak-1 ideals I.
(3) $Q_1 \cap I \cap Q \subseteq Q_1 \cdot I \cdot Q$, for the generalized 1-quasi ideals Q_1, weak-1 ideals I and 1-quasi ideals Q.
(4) $Q \cap I \cap Q_2 \subseteq Q \cdot I \cdot Q_2$, for the 1-quasi ideals Q, weak-1 ideals I and generalized 1-quasi ideals Q_2.
(5) $Q \cap I \cap Q \subseteq Q \cdot I \cdot Q$, for the 1-quasi ideals Q and weak-1 ideals I.
(6) $Q_1 \cap I \cap L \subseteq Q_1 \cdot I \cdot L$, for the generalized 1-quasi ideals Q_1, weak-1 ideals I and weak-1 left ideals L.
(7) $Q \cap I \cap L \subseteq Q \cdot I \cdot L$, for the 1-quasi ideals Q, weak-1 ideals I and weak-1 left ideals L.
(8) $R \cap I \cap Q_2 \subseteq R \cdot I \cdot Q_2$, for the weak-1 right ideals R, weak-1 ideals I and generalized 1-quasi ideals Q_2.
(9) $R \cap I \cap Q \subseteq R \cdot I \cdot Q$, for the weak-1 right ideals R, weak-1 ideals I and 1-quasi ideals Q .
(10) $R \cap I \cap L \subseteq R \cdot I \cdot L$, for the weak-1 right ideals R, weak-1 ideals I and weak-1 left ideals L.
(11) $R \cap L = R \cdot L$, for the weak-1 right ideals R and weak-1 left ideals L.
(12) $Q_1 \cap I \subseteq Q_1 \cdot I \cdot Q_1$, for the generalized 1-quasi ideals Q_1 and weak-1 ideals I.
(13) $Q \cap I \subseteq Q \cdot I \cdot Q$, for the 1-quasi ideals Q and weak-1 ideals I.
(14) $Q_1 = Q_1 \cdot S \cdot Q_1$, for the generalized 1-quasi ideals Q_1.
(15) $Q = Q \cdot S \cdot Q$, for the 1-quasi ideals Q.

Proof. First, we prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (11) \Rightarrow (1)$, $(3) \Rightarrow (6) \Rightarrow (7) \Rightarrow (11)$, $(2) \Rightarrow (4) \Rightarrow (8) \Rightarrow (9) \Rightarrow (10) \Rightarrow (11)$, $(12) \Rightarrow (14) \Rightarrow (1)$ and $(2) \Rightarrow (12) \Rightarrow (13) \Rightarrow (15) \Rightarrow (1)$.
$(1) \Rightarrow (2)$ For $a \in Q_1 \cap I \cap Q_2$, then there exists $s \in S$ such that $a = a \cdot s \cdot a$. Thus, $a = a \cdot (s \cdot a \cdot s) \cdot a \in Q_1 \cdot I \cdot Q_2$. Thus (2) holds.
$(2) \Rightarrow (3)$ Straightforward.
$(3) \Rightarrow (5)$ Straightforward.
$(5) \Rightarrow (11)$ Taking $I = S$ in (5), $R \cap L \subseteq R \cdot L$. Thus, $R \cap L = R \cdot L$.
$(11) \Rightarrow (1)$ The proof follows from Theorem 3.16.
$(3) \Rightarrow (6)$ By Lemma 3.6, (6) holds.
$(6) \Rightarrow (7)$ Straightforward.
$(7) \Rightarrow (11)$ By taking $I = S$, $R \cap L \subseteq R \cdot L$.Thus, $R \cap L = R \cdot L$.
$(2) \Rightarrow (4)$ Straightforward.
$(4) \Rightarrow (8)$ By Lemma 3.6, we get the result.
$(8) \Rightarrow (9)$ Straightforward.
$(9) \Rightarrow (10)$ The proof follows from Lemma 3.6.
$(10) \Rightarrow (11)$ Taking $I = S$ in (10), $R \cap L \subseteq R \cdot L$. Thus, $R \cap L = R \cdot L$.
$(12) \Rightarrow (14)$ By (12), $Q_1 \subseteq Q_1 \cdot S \cdot Q_1 \subseteq [(Q_1 \cdot S) \cap (S \cdot Q_1)] \subseteq Q_1$ implies $Q_1 = Q_1 \cdot S \cdot Q_1$.
$(14) \Rightarrow (1)$ For any $a \in S$, $a \in < a >_{g1q} \cdot S \cdot < a >_{g1q}$ and by Theorem 3.10, $a \in \Big[a \cdot S \cdot a\Big] \cup \Big[a \cdot S \cdot [(a \cdot S) \cap (S \cdot a)]\Big] \cup \Big[[(a \cdot S) \cap (S \cdot a)] \cdot S \cdot a\Big] \cup \Big[[(a \cdot S) \cap (S \cdot a)] \cdot S \cdot [(a \cdot S) \cap (S \cdot a)]\Big]$. Thus, $a \in a \cdot S \cdot a$. Therefore S is 1-regular.
$(2) \Rightarrow (12)$ Taking $Q_2 = Q_1$ in (2), we get the result.
$(12) \Rightarrow (13)$ Straightforward.
$(13) \Rightarrow (15)$ By (13), $Q \subseteq Q \cdot S \cdot Q \subseteq [(Q \cdot S) \cap (S \cdot Q)] \subseteq Q$ implies $Q = Q \cdot S \cdot Q$.
$(15) \Rightarrow (1)$ For any $a \in S$ by (15), $a \in < a >_{1q} \cdot S \cdot < a >_{1q}$ and by Theorem 3.13 and Lemma 3.12, $a \in \Big[na \cdot S \cdot ma\Big] \cup \Big[na \cdot S \cdot [(a \cdot S) \cap (S \cdot a)]\Big] \cup \Big[[(a \cdot S) \cap (S \cdot a)] \cdot S \cdot ma\Big] \cup \Big[[(a \cdot S) \cap (S \cdot a)] \cdot S \cdot [(a \cdot S) \cap (S \cdot a)]\Big]$. Thus, $a \in a \cdot S \cdot a$. Hence S is 1-regular. □

4. 2-quasi ideals in b-semirings

Definition 4.1. *(i) The subset Q of S is called a generalized 2-quasi ideal in S if $(Q+S)\cap(S+Q)\subseteq Q$.*
(ii) The generalized 2-quasi ideal Q is called a 2-quasi ideal in S if Q is a sub b-semiring.

Definition 4.2. *The generalized 1-quasi ideal Q is called a generalized quasi ideal if it is generalized 2-quasi ideal.*

Definition 4.3. *The sub b-semiring Q of S is called a quasi ideal if it is Q is generalized quasi ideal.*

Lemma 4.4. *The generalized 2-quasi ideal Q is a 2-quasi ideal in S if Q is closed under "$\cdot$".*

Proof. Suppose that Q is generalized 2-quasi ideal in S which is closed under "$\cdot$". Now, $Q+Q\subseteq Q+S$ and $Q+Q\subseteq S+Q$ implies $Q+Q\subseteq(Q+S)\cap(S+Q)\subseteq Q$. Thus, Q is a 2-quasi ideal in S. □

Remark 4.5. *The generalized 2-quasi ideal fails to be a 2-quasi ideal in S by the Example 4.6.*

Example 4.6. *Consider the b-semiring $(S,+,\cdot)$ by the following table.*

$+$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_1	s_1	s_1	s_1	s_1
s_2	s_2	s_2	s_2	s_2	s_2	s_2
s_3	s_3	s_3	s_3	s_3	s_3	s_3
s_4	s_4	s_4	s_4	s_4	s_4	s_4
s_5	s_5	s_5	s_5	s_5	s_5	s_5
s_6	s_6	s_6	s_6	s_6	s_6	s_6

$\cdot$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_2	s_2	s_2	s_2
s_3	s_1	s_3	s_3	s_3	s_3	s_3
s_4	s_1	s_2	s_2	s_4	s_4	s_4
s_5	s_1	s_3	s_3	s_5	s_5	s_5
s_6	s_1	s_3	s_3	s_6	s_6	s_6

Clearly, $\{s_1,s_2,s_5\}$ is a generalized 2-quasi ideal, but $s_5\cdot s_2\notin\{s_1,s_2,s_5\}$ implies $\{s_1,s_2,s_5\}$ is not 2-quasi ideal in S.

Lemma 4.7. *Every weak-2 right (left) ideal is a 2-quasi ideal in S.*

Remark 4.8. *Converse of the Lemma 4.7 fails by the Example 4.9.*

Example 4.9. *In Example 4.6, $\{s_3,s_5\}$ is a 2-quasi ideal, but $s_1+s_5\notin\{s_3,s_5\}$ implies $\{s_3,s_5\}$ not weak-2-left ideal in S.*

Theorem 4.10. *The intersection of weak-2 right ideal with weak-2 left ideal in S is a 2-quasi ideal.*

Proof. For the weak-2 right ideal A and weak-2 left ideal B in S, $A\cap B$ is a sub b-semiring. Now, $[(A\cap B)+S]\cap[S+(A\cap B)]\subseteq(A+S)\cap(S+B)\subseteq A\cap B$ implies $A\cap B$ is a 2-quasi ideal in S. □

Theorem 4.11. *For any $a\in S$, the generalized 2-quasi ideal generated by "a", denoted by $<a>_{g2q}$ is given by $\{a\}\cup[(a+S)\cap(S+a)]$.*

Proof. Now, $x\in(a+S)\cap(S+a)$, then $(x+S)\cap(S+x)\subseteq(a+S)\cap(S+a)$. Thus $\{a\}\cup[(a+S)\cap(S+a)]$ is a generalized 2-quasi ideal in S. If A is a generalized 2-quasi ideal in S such that $a\in A$, then $\{a\}\cup[(a+S)\cap(S+a)]\subseteq A$. Therefore $<a>_{g2q}$ is the generalized 2-quasi ideal generated by "a". □

Corollary 4.12. *For a subset A of S, $A\cup[(A+S)\cap(S+A)]$ is the generalized 2-quasi ideal generated by a set A in S.*

Theorem 4.13. *For any $a\in S$, the 2-quasi ideal generated by "a", denoted by $<a>_{2q}$ is given by $\{a^m|m\in\mathbb{Z}^+\}\cup[(a+S)\cap(S+a)]$.*

Proof. Clearly, $\{a^m|m\in\mathbb{Z}^+\}\cup[(a+S)\cap(S+a)]$ is generalized 2-quasi ideal. For $x,y\in[(a+S)\cap(S+a)]$, $x\cdot y=a+(s_1\cdot s_3)\in a+S$ and $x\cdot y=(s_2\cdot s_4)+a\in S+a$ imply $x\cdot y\in[(a+S)\cap(S+a)]$. For $x\in\{a^m|m\in\mathbb{Z}^+\}, y\in[(a+S)\cap(S+a)]$ and by Lemma 3.12, $x\cdot y=a^m\cdot(a+s_3)=a+[(a^m\cdot s_3)^{m+1}]\in a+S$ and $x\cdot y=a^m\cdot(s_4+a)=[(a^m\cdot s_4)^{m+1}]+a\in S+a$. Thus, $x\cdot y\in[(a+S)\cap(S+a)]$. Similarly, $y\cdot x\in[(a+S)\cap(S+a)]$. By Lemma 4.4, $\{a^m\}\cup[(a+S)\cap(S+a)]$ is a 2-quasi ideal in S. If A is a 2-quasi ideal in S such that $a\in A$, then $\{a^m\}\cup[(a+S)\cap(S+a)]\subseteq A$. Hence $<a>_{2q}$ is the 2-quasi ideal generated by "a". □

Corollary 4.14. *For a subset A of S, $\prod A\cup[\prod(A+S)\cap\prod(S+A)]$ is the 2-quasi ideal generated by a set A in S.*

Theorem 4.15. *For a b-semiring S, the following conditions are equivalent.*
(1) S is 2-regular.
(2) $R\cap Q_1\subseteq R+Q_1$, for the weak-2 right ideals R and generalized 2-quasi ideals Q_1.
(3) $R\cap Q\subseteq R+Q$, for the weak-2 right ideals R and 2-quasi ideals Q.
(4) $Q_1\cap L\subseteq Q_1+L$, for the generalized 2-quasi ideals Q_1 and weak-2 left ideals L.
(5) $Q\cap L\subseteq Q+L$, for the 2-quasi ideals Q and weak-2 left ideals L.
(6) $R\cap L=R+L$, for the weak-2 right ideals R and weak-2 left ideals L.

Proof. First, we prove that $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(6)\Rightarrow(1)$ and $(1)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)$.
$(1)\Rightarrow(2)$ For $a\in R\cap Q_1$, then there exist $s\in S$ such that $a=(a+s)+a$. Thus, $R\cap Q_1\subseteq R+Q_1$.
$(2)\Rightarrow(3)$ Straightforward.
$(3)\Rightarrow(6)$ By Lemma 4.7, $R\cap L\subseteq R+L$. Now, $R+L\subseteq R+S\subseteq R$ and $R+L\subseteq S+L\subseteq L$. Then (6) follows.
$(6)\Rightarrow(1)$ The proof follows from Theorem 3.16.
$(1)\Rightarrow(4)$ For $a\in Q_1\cap L$, then there exist $s\in S$ such that $a=a+(s+a)$. Thus, $Q_1\cap L\subseteq Q_1+L$.
$(4)\Rightarrow(5)$ By Lemma 4.5, (5) holds.
$(5)\Rightarrow(6)$ The proof follows from Lemma 4.7. □

Theorem 4.16. *For a b-semiring S, the following conditions are equivalent.*
(1) S is 2-regular.
(2) $Q_1 \cap I \cap Q_2 \subseteq Q_1 + I + Q_2$, for the generalized 2-quasi ideals Q_1 and Q_2 and weak-2 ideals I.
(3) $Q_1 \cap I \cap Q \subseteq Q_1 + I + Q$, for the generalized 2-quasi ideals Q_1, weak-2 ideals I and 2-quasi ideals Q.
(4) $Q \cap I \cap Q_2 \subseteq Q + I + Q_2$, for the 2-quasi ideals Q, weak-2 ideals I and generalized 2-quasi ideals Q_2.
(5) $Q \cap I \cap Q \subseteq Q + I + Q$, for the 2-quasi ideals Q and weak-2 ideals I.
(6) $Q_1 \cap I \cap L \subseteq Q_1 + I + L$, for the generalized 2-quasi ideals Q_1, weak-2 ideals I and weak-2 left ideals L.
(7) $Q \cap I \cap L \subseteq Q + I + L$, for the 2-quasi ideals Q, weak-2 ideals I and weak-2 left ideals L.
(8) $R \cap I \cap Q_2 \subseteq R + I + Q_2$, for the weak-2 right ideals R, weak-2 ideals I and generalized 2-quasi ideals Q_2.
(9) $R \cap I \cap Q \subseteq R + I + Q$, for the weak-2 right ideals R, weak-2 ideals I and 2-quasi ideals Q.
(10) $R \cap I \cap L \subseteq R + I + L$, for the weak-2 right ideals R, weak-2 ideals I and weak-2 left ideals L.
(11) $R \cap L = R + L$, for the weak-2 right ideals R and weak-2 left ideals L.
(12) $Q_1 \cap I \subseteq Q_1 + I + Q_1$, for the generalized 2-quasi ideals Q_1 and weak-2 ideals I.
(13) $Q \cap I \subseteq Q + I + Q$, for the 2-quasi ideals Q and weak-2 ideals I.
(14) $Q_1 = Q_1 + S + Q_1$, for the generalized 2-quasi ideals Q_1.
(15) $Q = Q + S + Q$, for the 2-quasi ideals Q.

Proof. First, we prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (11) \Rightarrow (1)$, $(3) \Rightarrow (6) \Rightarrow (7) \Rightarrow (11)$, $(2) \Rightarrow (4) \Rightarrow (8) \Rightarrow (9) \Rightarrow (10) \Rightarrow (11)$, $(12) \Rightarrow (14) \Rightarrow (1)$ and $(2) \Rightarrow (12) \Rightarrow (13) \Rightarrow (15) \Rightarrow (1)$.
$(1) \Rightarrow (2)$ For $a \in Q_1 \cap I \cap Q_2$, then there exists $s \in S$ such that $a = a + s + a$. Thus, $a = a + (s + a + s) + a \in Q_1 + I + Q_2$. Then (2) follows.
$(2) \Rightarrow (3)$ Straightforward.
$(3) \Rightarrow (5)$ Straightforward.
$(5) \Rightarrow (11)$ By taking $I = S$ in (5), $R \cap L \subseteq R + L$. Thus, $R \cap L = R + L$.
$(11) \Rightarrow (1)$ The proof follows from Theorem 3.16.
$(3) \Rightarrow (6)$ The proof follows from Lemma 4.7.
$(6) \Rightarrow (7)$ Straightforward.
$(7) \Rightarrow (11)$ By taking $I = S$, $R \cap L \subseteq R + L$. Thus, $R \cap L = R + L$.
$(2) \Rightarrow (4)$ Straightforward.
$(4) \Rightarrow (8)$ By Lemma 4.7, the result holds.
$(8) \Rightarrow (9)$ Straightforward.
$(9) \Rightarrow (10)$ By Lemma 4.7, (10) holds.
$(10) \Rightarrow (11)$ Taking $I = S$ in (10), $R \cap L \subseteq R + L$. Thus, $R \cap L = R + L$.
$(12) \Rightarrow (14)$ By (12), $Q_1 \subseteq Q_1 + S + Q_1 \subseteq [(Q_1 + S) \cap (S + Q_1)] \subseteq Q_1$. Thus, (14) holds .
$(14) \Rightarrow (1)$ For any $a \in S$, $a \in < a >_{g2q} + S + < a >_{g2q}$ and by Theorem 4.11 , $a \in \left[a + S + a\right] \cup \left[a + S + [(a+S) \cap (S + a)]\right] \cup \left[[(a+S) \cap (S+a)] + S + a\right] \cup \left[[(a+S) \cap (S+a)] + S + [(a+S) \cap (S+a)]\right]$. Thus, $a \in a + S + a$. Therefore S is 2-regular.
$(2) \Rightarrow (12)$ Taking $Q_2 = Q_1$ in (2), we get the result.
$(12) \Rightarrow (13)$ Straightforward.
$(13) \Rightarrow (15)$ By (13), $Q \subseteq Q + S + Q \subseteq [(Q + S) \cap (S + Q)] \subseteq Q$ implies $Q = Q + S + Q$.
$(15) \Rightarrow (1)$ For any $a \in S$ by (15), $a \in < a >_{2q} + S + < a >_{2q}$ and by Theorem 4.13 and Lemma 3.12, $a \in \left[a^n + S + a^m\right] \cup \left[a^n + S + [(a+S) \cap (S+a)]\right] \cup \left[[(a+S) \cap (S+a)] + S + a^m\right] \cup \left[[(a+S) \cap (S+a)] + S + [(a+S) \cap (S+a)]\right]$. Thus, $a \in a + S + a$. Hence S is 2-regular. □

Theorem 4.17. *For a b-semiring S, the following conditions are equivalent.*
(1) S is regular.
(2) $R \cap Q_1 \subseteq (R \cdot Q_1) \cap (R + Q_1)$, for the right ideals R and generalized quasi ideals Q_1.
(3) $R \cap Q \subseteq (R \cdot Q) \cap (R + Q)$, for the right ideals R and quasi ideals Q.
(4) $Q_1 \cap L \subseteq (Q_1 \cdot L) \cap (Q_1 + L)$, for the generalized quasi ideals Q_1 and left ideals L.
(5) $Q \cap L \subseteq (Q \cdot L) \cap (Q + L)$, for the quasi ideals Q and left ideals L.
(6) $R \cap L = (R \cdot L) \cap (R + L)$, for the right ideals R and left ideals L.

Proof. First, we prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (6) \Rightarrow (1)$ and $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$.
$(1) \Rightarrow (2)$ By Theorem 3.17, $R \cap Q_1 \subseteq R \cdot Q_1$ and by Theorem 4.15, $R \cap Q_1 \subseteq R + Q_1$. Thus, $R \cap Q_1 \subseteq (R \cdot Q_1) \cap (R + Q_1)$
$(2) \Rightarrow (3)$ The proof follows from Theorem 3.17 and 4.15.
$(3) \Rightarrow (6)$ By Theorem 3.17, $R \cap L = R \cdot L$ and by Theorem 4.15, $R \cap L = R + L$. Then (6) follows.
$(6) \Rightarrow (1)$ Now, $R \cdot L \subseteq R \cap L = (R \cdot L) \cap (R + L) \subseteq R \cdot L$, then by Theorem 3.16, S is 1-regular. Similarly, $R + L \subseteq R \cap L = (R \cdot L) \cap (R + L) \subseteq R + L$. Then by Theorem 3.16, S is 2-regular. Thus, S is regular.
$(1) \Rightarrow (4)$ By Theorem 3.17 and 4.15, $Q_1 \cap L \subseteq Q_1 \cdot L$ and $Q_1 \cap L \subseteq Q_1 + L$. Then (4) follows.
$(4) \Rightarrow (5)$ The proof follows from Theorem 3.17 and 4.15.
$(5) \Rightarrow (6)$ By Lemma 3.6 and 4.7, (6) holds. □

Theorem 4.18. *For a b-semiring S, the following conditions are equivalent.*
(1) S is regular.
(2) $Q_1 \cap I \cap Q_2 \subseteq (Q_1 \cdot I \cdot Q_2) \cap (Q_1 + I + Q_2)$, for the generalized quasi ideals Q_1 and Q_2 and ideals I.
(3) $Q_1 \cap I \cap Q \subseteq (Q_1 \cdot I \cdot Q) \cap (Q_1 + I + Q)$, for the generalized quasi ideals Q_1, ideals I and quasi ideals Q.
(4) $Q \cap I \cap Q_2 \subseteq (Q \cdot I \cdot Q_2) \cap (Q + I + Q_2)$, for the quasi ideals Q, ideals I and generalized quasi ideals Q_2.
(5) $Q \cap I \cap Q \subseteq (Q \cdot I \cdot Q) \cap (Q + I + Q)$, for the quasi ideals Q and ideals I.
(6) $Q_1 \cap I \cap L \subseteq (Q_1 \cdot I \cdot L) \cap (Q_1 + I + L)$, for the generalized quasi ideals Q_1, ideals I and left ideals L.
(7) $Q \cap I \cap L \subseteq (Q \cdot I \cdot L) \cap (Q + I + L)$, for the quasi ideals Q, ideals I and left ideals L.

(8) $R\cap I\cap Q_2\subseteq (R\cdot I\cdot Q_2)\cap(R+I+Q_2)$, for the right ideals R, ideals I and generalized quasi ideals Q_2.
(9) $R\cap I\cap Q\subseteq (R\cdot I\cdot Q)\cap(R+I+Q)$, for the right ideals R, ideals I and quasi ideals Q .
(10) $R\cap I\cap L\subseteq (R\cdot I\cdot L)\cap(R+I+L)$, for the right ideals R, ideals I and left ideals L.
(11) $R\cap L=(R\cdot L)\cap(R+L)$, for the right ideals R and left ideals L.
(12) $Q_1\cap I\subseteq (Q_1\cdot I\cdot Q_1)\cap(Q_1+I+Q_1)$, for the generalized quasi ideals Q_1 and ideals I.
(13) $Q\cap I\subseteq (Q\cdot I\cdot Q)\cap(Q+I+Q)$, for the quasi ideals Q and ideals I.
(14) $Q_1=(Q_1\cdot S\cdot Q_1)\cap(Q_1+S+Q_1)$, for the generalized quasi ideals Q_1.
(15) $Q=(Q\cdot S\cdot Q)\cap(Q+S+Q)$, for the quasi ideals Q.

Proof. First, we prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (11) $\Rightarrow$ (1), (3) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (11), (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (11), (12) $\Rightarrow$ (14) $\Rightarrow$ (1) and (2) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (15) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) By Theorem 3.18, $Q_1\cap I\cap Q_2\subseteq (Q_1\cdot I\cdot Q_2)$ and by Theorem 4.16, $Q_1\cap I\cap Q_2\subseteq (Q_1+I+Q_2)$. Thus, (2)holds.
(2) $\Rightarrow$ (3) By Theorem 3.18 and 4.16, (3) holds.
(3) $\Rightarrow$ (5) The proof follows from Lemma 3.6 and 4.7.
(5) $\Rightarrow$ (11) By Theorem 3.18 and 4.16, (11) holds.
(11) $\Rightarrow$ (1) The proof follows from Theorem 4.17.
(3) $\Rightarrow$ (6) The proof follows from Theorem 3.18 and 4.16.
(6) $\Rightarrow$ (7) By Theorem 3.18 and 4.16, (7) holds.
(7) $\Rightarrow$ (11) By Theorem 3.18, $R\cap L=R\cdot L$ and by Theorem 4.16, $R\cap L=R+L$. Thus, (11) holds.
(2) $\Rightarrow$ (4) By Theorem 3.18 and 4.16, (4) holds.
(4) $\Rightarrow$ (8) The proof follows from Lemma 3.6 and 4.7.
(8) $\Rightarrow$ (9) The proof follows from Theorem 3.18 and 4.16.
(9) $\Rightarrow$ (10) By Lemma 3.6 and Lemma 4.7, (10) holds.
(10) $\Rightarrow$ (11) The proof follows from Theorem 3.18 and 4.16.
(12) $\Rightarrow$ (14) By Theorem 3.18 and 4.16, $Q_1=Q_1\cdot S\cdot Q_1$ and $Q_1=Q_1+S+Q_1$. Thus, (14)holds.
(14) $\Rightarrow$ (1) Now, $Q_1\subseteq Q_1\cdot S\cdot Q_1\subseteq[(Q_1\cdot S)\cap(S\cdot Q_1)]\subseteq Q_1$. Then $Q_1=Q_1\cdot S\cdot Q_1$. By Theorem 3.18, S is 1-regular. Then, $Q_1\subseteq Q_1+S+Q_1\subseteq[(Q_1+S)\cap(S+Q_1)]\subseteq Q_1$. Thus $Q_1=Q_1+S+Q_1$. By Theorem 4.16, S is 2-regular. Thus, S is regular.
(2) $\Rightarrow$ (12) Taking $Q_2=Q_1$ in (2), we get the result.
(12) $\Rightarrow$ (13) By Theorem 3.18 and 4.16, (13) holds.
(13) $\Rightarrow$ (15) By Theorem 3.18 and 4.16, $Q=Q\cdot S\cdot Q$ and $Q=Q+S+Q$. Thus, (15)holds.
(15) $\Rightarrow$ (1) Now, $Q\subseteq Q\cdot S\cdot Q\subseteq[(Q\cdot S)\cap(S\cdot Q)]\subseteq Q$. Then $Q=Q\cdot S\cdot Q$. By Theorem 3.18, S is 1-regular. Then, $Q\subseteq Q+S+Q\subseteq[(Q+S)\cap(S+Q)]\subseteq Q$. Thus $Q=Q+S+Q$. By Theorem 4.16, S is 2-regular. Thus, S is regular. ☐

Theorem 4.19. *For any $a\in S$, the generalized quasi ideal generated by "a", denoted by $<a>_{gq}$ is given by $\{a\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$.*

Proof. By Theorem 3.10 and 4.11,$\{a\}\cup[(a\cdot S)\cap(S\cdot a)]$ and $\{a\}\cup[(a+S)\cap(S+a)]$ is a generalized 1-quasi ideal and generalized 2-quasi ideal of S respectively.
For $x\in[(a\cdot S)\cap(S\cdot a)]$, $x+s'=(a\cdot s_1)+s'\in[(a\cdot S)+S]$ and $s''+x=s''+(s_2\cdot a)\in[S+(S\cdot a)]$ imply $\Big[[(a\cdot S)\cap(S\cdot a)]+S\Big]\cap\Big[S+[(a\cdot S)\cap(S\cdot a)]\Big]\subseteq\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$.
For $x\in[(a+S)\cap(S+a)]$, $x\cdot s'=(a+s_1)\cdot s'\in[(a\cdot S)+S]$ and $s''\cdot x=s''\cdot(s_2+a)\in[S+(S\cdot a)]$ imply $\Big[[(a+S)\cap(S+a)]\cdot S\Big]\cap\Big[S\cdot[(a+S)\cap(S+a)]\Big]\subseteq\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$.
Now, $[(a\cdot S)+S]+S\subseteq[(a\cdot S)+S]$ and $S+[S+(S\cdot a)]\subseteq[S+(S\cdot a)]$ imply $\Big[\big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\big]+S\Big]\cap\Big[S+\big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\big]\Big]\subseteq\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$. Now, $[(a\cdot S)+S]\cdot S\subseteq[(a\cdot S)+S]$ and $S\cdot[S+(S\cdot a)]\subseteq[S+(S\cdot a)]$ imply $\Big[\big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\big]\cdot S\Big]\cap\Big[S\cdot\big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\big]\Big]\subseteq\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$.
Thus, $\{a\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$ is a generalized quasi ideal in S. If A is a generalized quasi ideal in S such that $a\in A$, then $\{a\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]\subseteq A$. Thus, $<a>_{gq}$ is a generalized quasi ideal generated by "a". ☐

Theorem 4.20. *For any $a\in S$, the quasi ideal generated by "a", denoted by $<a>_q$ is given by $\{na|n\in\mathbb{Z}^+\}\cup\{a^m|m\in\mathbb{Z}^+\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$.*

Proof. By Theorem 4.19, $\{a\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$ is a generalized quasi ideal of S. Now,
$\Big[(\{na|n\in\mathbb{Z}^+\}+S)\cap(S+\{na|n\in\mathbb{Z}^+\})\Big]\subseteq[(a+S)\cap(S+a)]$ and $\Big[(\{a^m|m\in\mathbb{Z}^+\}\cdot S)\cap(S\cdot\{a^m|m\in\mathbb{Z}^+\})\Big]\subseteq[(a\cdot S)\cap(S\cdot a)]$.
By Theorem 3.13 and 4.13, $Q=\{na|n\in\mathbb{Z}^+\}\cup\{a^m|m\in\mathbb{Z}^+\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$, $\{na|n\in\mathbb{Z}^+\}\cup[(a\cdot S)\cap(S\cdot a)]$ and $\{a^m|m\in\mathbb{Z}^+\}\cup[(a+S)\cap(S+a)]$ are generalized quasi ideal and sub b-semirings of S respectively.
Now, $\{na|n\in\mathbb{Z}^+\}+\{a^m|m\in\mathbb{Z}^+\}\subseteq[(a+S)\cap(S+a)]\subseteq Q$ and $\{a^m|m\in\mathbb{Z}^+\}+\{na|n\in\mathbb{Z}^+\}\subseteq[(a+S)\cap(S+a)]\subseteq Q$ and $\{na|n\in\mathbb{Z}^+\}\cdot\{a^m|m\in\mathbb{Z}^+\}\subseteq[(a\cdot S)\cap(S\cdot a)]\subseteq Q$ and $\{a^m|m\in\mathbb{Z}^+\}\cdot\{na|n\in\mathbb{Z}^+\}\subseteq[(a\cdot S)\cap(S\cdot a)]\subseteq Q$.
Let $x\in\{na|n\in\mathbb{Z}^+\}$ and $y\in[(a+S)\cap(S+a)]$. Then $x+y=na+(a+s_1)=(n+1)a+s_1=a+(na+s_1)\in a+S$ and $x+y=na+(s_2+a)=(na+s_2)+a\in S+a$ imply $\{na|n\in\mathbb{Z}^+\}+[(a+S)\cap(S+a)]\subseteq[(a+S)\cap(S+a)]\subseteq Q$. Similarly, $[(a+S)\cap(S+a)]+\{na|n\in\mathbb{Z}^+\}\subseteq[(a+S)\cap(S+a)]\subseteq Q$.
Now, $x\cdot y=na\cdot(a+s_1)=[a\cdot(a+s_1)]+[(n-1)a\cdot(a+s_1)]\in[(a\cdot S)+S]$ and $x\cdot y=na\cdot(s_2+a)=(na\cdot s_2)+(na\cdot a)\in[S+(S\cdot a)]$ imply $\{na|n\in\mathbb{Z}^+\}\cdot[(a+S)\cap(S+a)]\subseteq\Big[[(a\cdot$

$S)+S] \cap [S+(S\cdot a)]\Big] \subseteq Q$ and $[(a+S)\cap(S+a)]\cdot\{na|n \in \mathbb{Z}^+\} \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Let $x \in \{na|n \in \mathbb{Z}^+\}$ and $y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$. Then $x+y = na+[(a\cdot s_1)+s_2] = [(n+1)a\cdot(na+s_1)]+s_2 \in [(a\cdot S)+S]$ and $x+y = (na+s_3)+(s_4\cdot a) \in [S+(S\cdot a)]$ imply $x+y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Now, $x\cdot y = na\cdot[(a\cdot s_1)+s_2] = \Big[(a\cdot n(a\cdot s_1))+(na\cdot s_2)\Big] \in [(a\cdot S)+S]$ and $x\cdot y = na\cdot[s_3+(s_4\cdot a)] = \Big[(na\cdot s_3)+((na\cdot s_4)\cdot a)\Big] \in [S+(S\cdot a)]$ imply $x\cdot y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $y+x$ and $y\cdot x \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Let $x \in \{a^m|m \in \mathbb{Z}^+\}$ and $y \in [(a\cdot S)\cap(S\cdot a)]$. Then , $x+y = a^m+(a\cdot s_1) = (a\cdot a^{m-1})+(a\cdot s_1) = [a+(a\cdot s_1)]\cdot[a^{m-1}+(a\cdot s_1)] \in [(a\cdot S)+S]$ and $x+y = a^m+(s_3\cdot a) = (a^m+s_3)\cdot(a^m+a) = [(a^m+s_3)\cdot a^m]+[(a^m+s_3)\cdot a] \in [S+(S\cdot a)]$ imply $x+y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now ,$x\cdot y = a^m\cdot(a\cdot s_1) \in a\cdot S$ and $x\cdot y = a^m\cdot(s_2\cdot a) = (a^m\cdot s_2)\cdot a \in S\cdot a$ imply $x\cdot y \in [(a\cdot S)\cap(S\cdot a)] \subseteq Q$. Similarly, $y+x \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$ and $y\cdot x \in [(a\cdot S)\cap(S\cdot a)] \subseteq Q$.

Let $x \in \{a^m|m \in \mathbb{Z}^+\}$ and $y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]$. Now, $x+y = a^m+[(a\cdot s_1)+s_2] = (a\cdot a^{m-1})+[(a\cdot s_1)+s_2] = \Big[a+[(a\cdot s_1)+s_2]\Big]\cdot\Big[a^{m-1}+[(a\cdot s_1)+s_2]\Big] \in [(a\cdot S)+S]$ and $x+y = (a^m+s_4)+(s_5\cdot a)] \in [S+(S\cdot a)]$ imply $x+y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Now, $x\cdot y = a^m\cdot[(a\cdot s_1)+s_2] = a\cdot[a^{m-1}\cdot(a\cdot s_1)]+(a^m\cdot s_2) \in [(a\cdot S)+S]$ and $x\cdot y = a^m\cdot[s_4+(s_5\cdot a)] = (a^m\cdot s_4)+[(a^m\cdot s_5)\cdot a)] \in [S+(S\cdot a)]$ imply $x\cdot y \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $y+x$ and $y\cdot x \in \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $(a\cdot S)+(a+S) \subseteq (a\cdot S)+S$ and $(S\cdot a)+(S+a) = [S+(S+a)]\cdot[(a+S)+a] \subseteq S\cdot(S+a)$ imply $\Big[[(a\cdot S)\cap(S\cdot a)]+[(a+S)\cap(S+a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$ and $\Big[[(a+S)\cap(S+a)]+[(a\cdot S)\cap(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $(a\cdot S)\cdot(a+S) = [(a\cdot S)\cdot a]+[(a\cdot S)\cdot S] \subseteq (a\cdot S)+S$ and $(S\cdot a)\cdot(S+a) = [(S\cdot a)\cdot S]+[(S\cdot a)\cdot a] \subseteq S\cdot(S+a)$ imply $\Big[[(a\cdot S)\cap(S\cdot a)]\cdot[(a+S)\cap(S+a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$ and $\Big[[(a+S)\cap(S+a)]\cdot[(a\cdot S)\cap(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $(a\cdot S)+[(a\cdot S)+S] \subseteq [(a\cdot S)+S]$ and $(S\cdot a)+[S+(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[(a\cdot S)\cap(S\cdot a)\Big]+\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]+\Big[(a\cdot S)\cap(S\cdot a)\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $(a\cdot S)\cdot[(a\cdot S)+S] \subseteq [(a\cdot S)+S]$ and $(S\cdot a)\cdot[S+(S\cdot a)] = [(S\cdot a)\cdot S]+[(S\cdot a)\cdot(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[(a\cdot S)\cap(S\cdot a)\Big]\cdot\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]\cdot\Big[(a\cdot S)\cap(S\cdot a)\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $(a+S)+[(a\cdot S)+S] \subseteq (a+S)+[(a+S)\cdot S] = [(a+S)+(a+S)]\cdot[(a+S)+S] \subseteq (a+S)\cdot S = [(a\cdot S)+S]$ and $(S+a)+[S+(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[(a+S)\cap(S+a)\Big]+\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]+\Big[(a+S)\cap(S+a)\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Now, $(a+S)\cdot[(a\cdot S)+S] \subseteq [(a+S)\cdot S] = [(a\cdot S)+S]$ and $(S+a)\cdot[S+(S\cdot a)] \subseteq [(S+a)\cdot S]+[(S+a)\cdot(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[(a+S)\cap(S+a)\Big]\cdot\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Similarly, $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]\cdot\Big[(a+S)\cap(S+a)\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$.

Now, $[(a\cdot S)+S]+[(a\cdot S)+S] \subseteq [(a\cdot S)+S]$ and $[S+(S\cdot a)]+[S+(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]+\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Now, $[(a\cdot S)+S]\cdot[(a\cdot S)+S] \subseteq [(a\cdot S)+S]\cdot S \subseteq [(a\cdot S)+S]$ and $[S+(S\cdot a)]\cdot[S+(S\cdot a)] \subseteq S\cdot[S+(S\cdot a)] \subseteq [S+(S\cdot a)]$ imply $\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big]\cdot\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq \Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq Q$. Thus, Q is a quasi ideal in S. If A is a quasi ideal in S such that $a \in A$, then $\{na|n \in \mathbb{Z}^+\}\cup\{a^m|m \in \mathbb{Z}^+\}\cup[(a\cdot S)\cap(S\cdot a)]\cup[(a+S)\cap(S+a)]\cup\Big[[(a\cdot S)+S]\cap[S+(S\cdot a)]\Big] \subseteq A$. Thus, $<a>_q$ is the quasi ideal generated by "a".

□

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