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Mod difference labeling of some classes of digraphs

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Abstract

A graph is a *difference graph* if there is a bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x)-f(y)| \in S$. A digraph $D=(V,E)$ is a *mod difference digraph* if there exist a positive integer m and labeling $L: V \to \{1,2,...,m-1\}$ such that $(x,y) \in E$ if and only if $L(y)-L(x) \equiv L(w)(mod\ m)$ for some $w \in V$. In this paper, we prove that the complete bipartite digraphs, oriented binary trees, ladder graphs and fan graphs are mod difference digraphs.

Keywords

Difference labeling, mod difference labeling, digraphs.

AMS Subject Classification

05C20, 05C78.

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Contents

1. Introduction

In this paper, we consider only finite simple graphs. Let S be a finite multiset of real numbers, i.e. a finite collection of real numbers in which repetitions is permitted but order is irrelevant. The difference graph with signature S is a finite digraph G with vertices labelled bijectively by f from a vertex labelled x to a vertex labelled y exactly when $x-y \in S$. In [4] Harary introduced the concept of difference graphs similar to sum graphs and similar works we refer [8, 9]. Some classes of difference graphs (paths, trees, cycles, special wheels, complete graphs, complete bipartite graphs etc.) were investigated by Bloom, Burr, Eggleton, Gervacio, Hell and Taylor in the undirected [2, 3, 12] as well as in the directed case [1].

The concept of mod difference digraph was introduced by S.M.Hegde and Vasudeva [6]. For the recent works we refer [7, 10, 11].

Definition 1.1. *A graph is a difference graph if there is a bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x)-f(y)| \in S$. A digraph $D=(V,E)$ is a mod difference digraph if there exist a positive integer m and labeling $L: V \to \{1,2,...,m-1\}$ such that $(x,y) \in E$ if and only if $L(y)-L(x) \equiv L(w)(mod\ m)$ for some $w \in V$.*

They have shown some of the structural properties of mod difference digraphs in [5]. It is also proved that complete symmetric digraphs, unipaths and unicycles are mod difference digraphs [6].

2. Mod difference labeling of some classes of digraphs

In this section we present some results on mod difference labeling of some classes of digraphs.

Lemma 2.1. *If S is a proper signature of $\overleftrightarrow{K}_{n,n}$ then S is partitioned into two disjoint sets V_1 and V_2 as follows:*

$$V_1 = \{v_i = 3i-2, 1 \le i \le n\}$$
$$V_2 = \{v_i = 3i-1, 1 \le i \le n\}$$

Proof. Consider an edge $x = uv$ in $\overleftrightarrow{K}_{n,n}$. Assume that both $u,v \in V_1$ (or V_2.) Without the loss of generality, let $u,v \in V_1$ then $u-v \in S$ for some $1 \le i_1,\ i_2 \le n$ i.e $(3i_1-2)-(3i_2-2) \in S$, for some $1 \le i_1, i_2 \le n$ i.e $3(i_1-i_2) \in S$, but $3(i_1-i_2) \equiv 0(mod 3)$, a contradiction. Hence$V_1$ and V_2 form the partition of $\overleftrightarrow{K}_{n,n}$. □

Lemma 2.2. *Let V_1 and V_2 be the bipartite vertex sets of $\overleftrightarrow{K}_{n,n}$ as defined in the above lemma. If $v_i \in V_1$ and $v_j \in V_2$ with $v_i < v_j, 1 \leq i, j \leq n$ then $v_i - v_j \in V_1$.*

Proof. If $x = v_i v_j$ with $v_i < v_j$ is an edge in $\overleftrightarrow{K}_{n,n}$, then by Lemma 2.1, $v_i = 3k_1 - 2$ and $v_j = 3k_2 - 1$, for some $1 \leq k_1, k_2 \leq n$. Therefore, $v_j - v_i = (3k_2 - 1) - (3k_1 - 2) = 3(k_2 - k_1) + 1$. Hence $0 \leq k_1, k_2 \leq (n-1)$.

Taking,

$$\begin{aligned} k_2 - k_1 &= 0, \text{we get, } v_j - v_i = 1 = 3(1) - 2 \in V_1 \\ k_2 - k_1 &= 1, \text{we get, } v_j - v_i = 4 = 3(2) - 2 \in V_1 \\ &\vdots \\ k_2 - k_1 &= n-1, \text{we get, } v_j - v_i = 1 = 3(n-1) + 1 \in V_1 \end{aligned}$$

Hence the proof. □

Theorem 2.3. *If the labeling L given by $L(v_i) = a_i, \forall i, i = 1, 2, ..., 2n$ is a mod difference labeling of $\overleftrightarrow{K}_{n,n}$, then there exits an integer a such that $a_i = (3n_i - k)a$ for $k = 2$, $n_i = \frac{i+1}{2}$, when i is odd and $k = 1$, $n_i = \frac{i}{2}$, when i is even and $m = 3na$.*

Proof. Without the loss of generality we take $a_1 < a_2 < \cdots < a_{2n}$. Set $a_1 = a = (3(1) - 2)a$. Since L is a mod difference labeling, consider $a_2 - a_1 = a_l$ for some l. Since $a_1 < a_2$,we must have $a_l = a_1$.Then $a_2 = a_1 + a_1 = 2a = (3(1) - 1)a$. Consider $a_3 - a_2 = a_m$ for some m. Again $a_2 < a_3$, we must have $a_m = a_1$ or a_2. If $a_m = a_1$, then $a_3 = a_1 + a_2 \Rightarrow a_3 - a_1 = a_2 \Rightarrow$ there is an edge $a_1 a_3$ in $\overleftrightarrow{K}_{n,n}$, a contradiction to the Lemma 2.1. Therefore, $a_m = a_2$ and hence $a_3 = a_2 + a_2 = 4a = [3(2) - 2]a$.

We have proved the theorem is true for $n = 1, 2, 3$. We assume the theorem is true for all $n < k$. Without the loss of generality consider k to be even. Consider $a_k - a_l = a_j$ for some j, with $a_l < a_k$. Also $a_j < a_k$ We have observed that $a_j = a_{k-l} \in V_1$.

If k is even ,then l is odd. By Lemma 2.2, j is also odd. By induction assumption, we have $a_l = [3l_i - 2]a$ and $a_j = [3j_i - 2]a$, for some $1 \leq l_i, j_i \leq k$. Therefore,

$$\begin{aligned} a_k &= [3j_i - 2]a + [3l_i - 2]a \\ &= \left[3\left[\frac{j+1}{2}\right] - 2\right]a + \left[3\left[\frac{l+1}{2}\right] - 2\right]a \\ &= \left\{3\left[\frac{j+l+2}{2}\right] - 4\right\}a \\ &= 3\left\{\left[\frac{k-l+l+2}{2}\right] - 4\right\}a \\ &= 3\left\{\left[\frac{k+2}{2}\right] - 4\right\}a \\ &= 3\left\{\left[\frac{k}{2}\right] - 1\right\}a \\ &= [3k_i - 1]a \end{aligned}$$

Therefore, the result is true for k. Hence by induction it is true for all positive n.

Also consider $a_1 - a_{2n} \equiv a_j (mod \text{m})$ for some j, since $a_1 \in V_1$ and $a_{2n} \in V_2$. Hence

$$m = \left[(3n-1) - 1 + 3\left(\frac{j}{2}\right) - 1\right]a = \left[3n - 3 + 3\left(\frac{j}{2}\right)\right]a.$$

In the above $j \neq 2n - 1$, and $1 \leq j \leq n$. If $j > 2$, then $m = 3(n+1)a$, which contradicts labeling condition. Therefore, $j = 2$ which implies $m = 3na$. Hence the proof. □

Corollary 2.4. *Any complete bipartite digraph $\overleftrightarrow{K}_{n,n}$ is a mod difference digraph.*

Proof. Label the vertices $v_1, v_2, \ldots, v_{2n}$ of $\overleftrightarrow{k}_{n,n}$, using the labeling $f(v_i) = ia$, for $1 \leq i \leq 3n - 1$, with $i \not\equiv 0(\text{mod}3)$ and a is a positive integer. We prove that f is mod difference labeling of $\overleftrightarrow{K}_{n,n}$ with $m = 3na$. Now, for all $i, j, i \neq j, ia - ja \equiv ka$ (mod m), where

$$k = \begin{cases} i - j, & \text{if} \quad i > j \\ 3n + (i - j), & \text{if} \quad i < j. \end{cases}$$

In both the cases $k \leq 2n$. Therefore, there exits $k \leq 2n$ such that $f(v_i) - f(v_j) \equiv f(v_k)(\text{mod } m)$, for all i, j with $i \neq j$. Hence f is a mod difference digraph. □

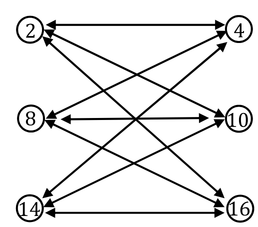

Figure 1. A mod difference labeling of the bi-directional digraph of $K_{3,3}$ with $m = 18$ and $a = 2$.

Definition 2.5. *An m-ary tree $(m \geq 2)$ is a rooted tree in which each vertex has less than or equal to m children. When $m = 2$, the rooted tree is called the binary tree.*

Definition 2.6. *An oriented binary tree $\overrightarrow{T_n}$ with n vertices is said to be inspoken if all the parents in each level has indegree two.*

Theorem 2.7. *A oriented binary tree $\overrightarrow{T_n}$ whose internal vertices have indegree 2 and outdegree 1 is a mod difference digraph.*

Proof. Let $\overrightarrow{T_n} = (V, E)$ be the oriented binary tree and v be the root vertex. Let $v_1, v_2, \ldots, v_p$ be the end vertices in $\overrightarrow{T_n}$. First label all the p leaves h as $l(v_1) = a$ and $l(v_i) = 2l(v_{i-1}) + 1$, $2 \leq i \leq p$. Then label all the internal vertices in the $h - 1$ level by adding the labels of its children. Continuing the same procedure of labeling to all the non-pendent vertices in the previous level and so on., we finally reach the root v.

Now, for the vertices u, v in $\overrightarrow{T}$ with $l(u > l(v)$, it is easy to observe that if $uv \in E$ with $l(u) > l(v)$, then u is a parent of v, so $l(u) = l(v) + l(w)$ for some child w of u. Hence $l(u) - l(v) \equiv l(w)$ whenever $uv \in E$.

Further, the label of i^{th} pendent vertex v_i is $l(v_i) = 2^i(a+1) - 1$ and the level of the internal vertices is $l(u_i) = (2^{i_1} + 2^{i_2} + \cdots + 2^{i_{2^k}})(a+1) - 2^k$ for some $k \in Z^+$.

if u_i and u_j be any two non-adjacent vertices of $\overrightarrow{T}$, then we have the following cases

CASE 1: u_i and u_j are pendent vertices

In this case $l(u_i) - l(u_j) = l(v_i) - l(v_j) = [2^i(a+1) - 1] - [2^j(a+1) - 1] = (2^i - 2^j)(a+1) \notin l(S) \Rightarrow u_iu_j \notin E$.

CASE 2: u_i is a pendent and u_j is an internal vertex.

In this case $l(u_j) - l(u_i) = (2^{i_1} + 2^{i_2} + \cdots + 2^{i_{2^k}})(a+1) - 2^k - (2^i)(a+1) + 1 = (2^{i_1} + 2^{i_2} + \cdots + 2^{i_{2^k}} + 2^i)(a+1) - (2^k - 1) \notin l(S) \Rightarrow u_iu_j \notin E$

CASE 3: u_i and u_j are internal vertices

In this case $l(u_j) - l(u_i) = (2^{j_1} + 2^{j_2} + \cdots + 2^{j_{2^k}})(a+1) - 2^k - (2^{i_1} + 2^{i_2} + \cdots + 2^{i_{2^l}})(a+1) - 2^l = (2^{j_1} + 2^{j_2} + \cdots + 2^{j_{2^k}} - 2^{i_1} - 2^{i_2} - \cdots - 2^{i_{2^l}})(a+1) - (2^k + 2^l) \notin l(S) \Rightarrow u_iu_j \notin E$ whenever $k \neq l$ (i.e $u_i \neq u_j$).

Hence $\overrightarrow{T}$ is a mod difference digraph. □

A mod difference digraph of complete oriented binary tree $\overrightarrow{T_7}$

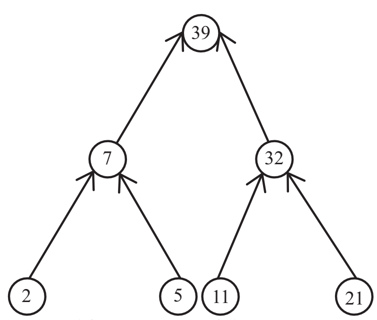

Figure 2. A binary tree with $a = 2$ and $m = 40$.

Definition 2.8. *A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K}$ is the empty graph on m vertices and P_n is the path graph of n vertices. The case $m = 1$,corresponds to the usual fan graphs, while $m = 2$ corresponds to the double fan graphs etc.*

Definition 2.9. *An oriented fan graph $\overrightarrow{F_{1,n}}$ is said to be an unipath fan, if the path of the fan is unidirectional.*

Definition 2.10. *An oriented fan graph $\overrightarrow{F_{1,n}}$ with $n+1$ vertices is called outspoken(inspoken), if indegree (outdegree) of the apex vertex is 0.*

Theorem 2.11. *An unipath fan $\overrightarrow{F_{1,n}}$ with n+1 vertices is mod difference digraph, if indegree of the apex vertex is one.*

Proof. Let $\overrightarrow{F_{1,n}} = (V,E)$ be an unipath fan. Let $V = \{v_0, v_1, v_2, \ldots, v_n\}$ and $E = \{v_0v_i : 1 \leq i \leq n\} \bigcup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$, where v_0 is the apex vertex.

Label the apex vertex v_0 by 2 and label each vertex v_i of the path by $2i - 1, i = 1, 2, \ldots, n$. With modular value $m = 2(n+1)$, this labeling scheme generates the signature for $\overrightarrow{F_{1,n}}$.

For $v_i - v_0 = (2i-1) - 2 = 2(i-1) - 1 = v_{i-1}$, for all $i = 2, 3, 4, \ldots, n$.

For $i = 1$, we have $v_0 - v_1 = 2 - 1 = 1 \in V$.

Also for $1 \leq i, j \leq n, i = j+1$,

$$v_i - v_j = (2i-1) - (2j-1) = 2(i-j) = 2 = v_0.$$

Since for every $i, j \in \{1, 2, \ldots, n\}$, we have $v_i - v_j$ is an even number greater than 2 and is not in V under modulo m. Hence the labeling does not induces any additional edges. Hence $\overrightarrow{F_{1,n}}$ is mod difference digraph. □

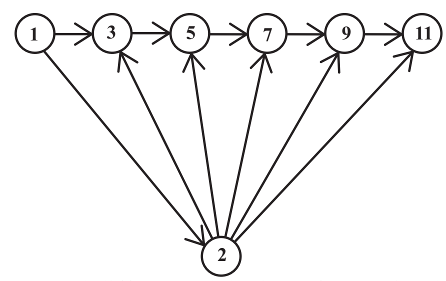

Figure 3. A mod difference labeling of an Unipath Fan with $m = 14$.

If the signature of the mod difference digraph contains 0 integer, then such a digraph is named as *mod** difference digraph. S.M Hegde and Vasudeva [5] introduced *mod** difference digraph and defined as follows:

Definition 2.12. *A simple digraph D is called a mod* difference digraph if there exists a positive integer m and a labeling f of the vertices of D with distinct elements of $f = \{0, 1, 2, \ldots, m-1\}$ such that for the vertices u and v there exists an arc from u to v (denoted as $u \to v$) if and only if there is a vertex w such that $(f(v) - f(u)) \equiv f(w)(modm)$. The function f is called a mod* difference labeling of digraph D.*

Theorem 2.13. *An unipath outspoken fan $\overrightarrow{F_{1,n}}$ with n+1 vertices is a mod* difference digraph.*

Proof. Consider the signature $S = \{0, 1, 2, \ldots, 2^{n-1}\}$ and modular value $m = 2^n$. We prove that if S is the signature of an unipath outspoken fan. Let $\{v_0, v_1, \ldots, v_n\}$ be the vertices of unipath outspoken fan where v_0 is the apex vertex.

Let $v_0 = 0, v_i = 2^{i-1}$ for $i = 1, 2, 3, \ldots, n$. Now, $v_i - v_0 = v_i$ for $i = 1, 2, \ldots, n$. For the vertices on the path and $i \neq j$, we have $v_i - v_j = 2^{i-1} - 2^{j-1} \in S$ if and only if $i = j+1$. Hence the proof. □

Definition 2.14. *An oriented Ladder graph $\overrightarrow{L_n} = \overrightarrow{P_n} \times \overrightarrow{P_2}$ is said to be oriented unipath ladder if the path P_n is unidirectional.*

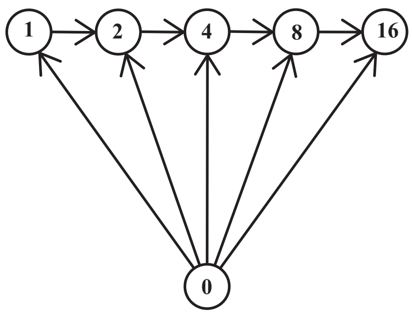

Figure 4. A mod difference labeling of an unipath outspoken Fan with $m = 32$.

Theorem 2.15. *An oriented unipath Ladder $\overrightarrow{L_n}$ is a mod difference digraph.*

Proof. Let $\overrightarrow{L_n} = \overrightarrow{P_n} \times \overrightarrow{P_2}$ be an oriented unipath ladder. We label the vertices of $\overrightarrow{L_n}$ as $v_1, v_2, \ldots, v_{2n}$. A ladder contains two paths, one path contains the vertices of the form v_{2i-1}, $1 \leq i \leq n-1$ and the other path contains the vertices of the form $v_{2i}, 1 \leq i \leq n$. The edge set contains the edges of the form :
$\{v_{2i}v_{2i+2}, v_{2i-1}v_{2i+1}, v_{2i}v_{2i-1}, 1 \leq i \leq n\}$.

We define the labeling function f as:

$$\begin{aligned} f(v_{2i}) &= 2^i, 1 \leq i \leq n \\ f(v_{2i-1}) &= 2^{i+1} - 3, 1 \leq i \leq (n-1) \end{aligned}$$

We prove that f is a mod difference labeling of $\overrightarrow{L_n}$ with $m = 2(2^{n+1} - 3)$. That is, we show that $f(v_i) - f(v_j) \equiv f(v_k)$ (mod m), for some $k, 1 \leq k \leq 2n$ if and only if $v_j v_i$ is an edge in $\overrightarrow{L_n}$.

For an edge $v_{2i}v_{2i+2}, 1 \leq i \leq (n-1)$, we have,

$$\begin{aligned} f(v_{2i+2}) - f(v_{2i}) &= 2^{i+1} - 2^i \\ &= 2^i(2-1) \\ &= 2^i \\ &= f(v_{2i})(mod\ m) \end{aligned}$$

For an edge $v_{2i-1}v_{2i+1}, 1 \leq i \leq (n-1)$ we have,

$$\begin{aligned} f(v_{2i+1}) - f(v_{2i-1}) &= [2^{(i+1)+1} - 3] - [2^{i+1} - 3] \\ &= 2.2^{i+1} - 2^{i+1} \\ &= 2^{i+1} \\ &= f(v_{2(i+1)})(mod\ m) \end{aligned}$$

For an edge $v_{2i}v_{2i-1}$, for $i > 1$, we have,

$$\begin{aligned} f(v_{2i}) - f(v_{2i-1}) &= 2^i - [2^{i+1} - 3] \\ &= 2.2^i - 2^i - 3 \\ &= 2^i - 3 \\ &= f(v_{2(i-1)-1}) \\ &= f(v_{2i-3})(mod\ m) \end{aligned}$$

Finally, for i=1, $f(v_2) - f(v_1) = 2 - 1 = 1$ (mod m). Hence $\overrightarrow{L_n}$ is a mod difference graph. □

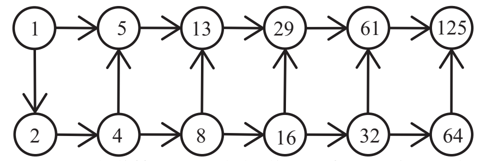

Figure 5. A mod difference labeling of an oriented unipath Ladder with $m = 251$.

3. Conclusion

Cayley graphs are known to be excellent model for interconnection network due to their various properties like vertex transitivity, regularity, connectivity etc., Cayley graph $Cay_g(A,S)$ is connected if and only if S generates the group A. In particular, if $S = A - \{e\}$, where e is the identity of A, then $Cay(A,S)$ turns out to be a complete graph, which is a mod difference graph whenever the group operation is the usual addition. The Subgraph of a Cayley graph induced by S is a problem of our interest. A graph G that is a mod difference graph is a subgraph of certain Cayley graphs of a group of superset of $V(G)$.

The investigations made in this paper may enlighten a new direction for further development of a good interconnection networks which are subgraphs of Cayley Networks. The links are specific and can be identified as difference of the addresses. So, one can easily develop a routing algorithm to communicate between two nodes with the list of addresses of the nodes. .

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