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Properties of disjunctive domination in product graphs

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Abstract

In this paper properties of disjunctive domination in some graph products are studied. We examine whether disjunctive domination number is multiplicative with respect to different graph products, that is, $\gamma_2^d(G_1 * G_2) \geq \gamma_2^d(G_1)\gamma_2^d(G_2)$ for all graphs G_1 and G_2 or $\gamma_2^d(G_1 * G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$ for all graphs G_1 and G_2 where $*$ denotes lexicographic, tensor, strong or Cartesian product of graphs. Some other inequalities involving disjunctive domination number of product graphs and the graphs attaining these inequalities are also given.

Keywords

Domination, disjunctive domination, disjunctive domination number, graph product. .

AMS Subject Classification

05C69.

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Contents

1. Introduction

Various graph products clearly model processor connections in multiprocessor systems. The fast transmission of information between the processors is very important in communication systems. Hence the study of graph theoretic properties of product graphs is important. Domination number in product graphs has been studied for a long time. Among various products, the Cartesian product is the centre of study in almost all works in literature. These studies are focused largely on Vizing's conjecture. Here an attempt to determine the disjunctive domination number of different types of graph products is made.

2. Preliminaries

Domination in graphs is an important parameter in graph theory because of its wide applications. Tremendous research has been made by many researchers on this topic. A brilliant survey of studies related to domination is given in [2] by Haynes et al. A variation of classical domination defined as secondary dominations is studied in [3]. Another variation of domination, defined as disjunctive domination, was introduced and studied by Goddard et al. in [4]. For more details on graph products and its applications, we suggest the reader to refer [7].

Definition 2.1. *A subset S of the vertex set V is a disjunctive dominating set or DD-set, if for any vertex $u \notin S$ one of the following two conditions are true.*

1. *there is a vertex $v \in S$ which is adjacent to u or*

2. *there are two vertices $v_1, v_2 \in S$ such that $d(u,v_1) = d(u,v_2) = 2$.*

The disjunctive domination number or DD-number, $\gamma_2^d(G)$ of a graph G is $min\{|S| : S \text{ is a } DD-set \text{ in } G\}$ [4, 5]. If the above condition is true for every vertex in $u \in S$, then S is called a total disjunctive dominating set or TDD-set of G . Total disjunctive domination number or TDD-number, $\gamma_t^d(G)$ of G is $min\{|S| : S \text{ is a } TDD-set \text{ in } G\}$ [6].

Definition 2.2. *A vertex v in a graph is called a universal vertex or full degree vertex if $N[v] = V(G)$.*

Definition 2.3. *A graph parameter ϕ is multiplicative with respect to a graph product $*$ if $\phi(G_1 * G_2) \geq \phi(G_1)\phi(G_2)$ for all graphs G_1 and G_2 or $\phi(G_1 * G_2) \leq \phi(G_1)\phi(G_2)$ for all graphs G_1 and G_2.*

For all standard terminology and notation we follow [1]. The terms related to domonation in graphs are used as in [2].

3. Main Results

Disjunctive domination in lexicographic products

The Lexicographic product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1[G_2]$ whose vertex set is $V_1 \times V_2$ in which $((u_1, v_1), (u_2, v_2))$ is an edge if

- $u_1u_2 \in E_1$ or
- u_1, u_2 are equal and $v_1v_2 \in E_2$.

Theorem 3.1. *Disjunctive domination number is multiplicative with respect to Lexicographic product.*

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times S_2$ is a DD-set of $G_1[G_2]$.

claim
Let (u, v) be a vertex in $G_1[G_2]$ which is not in $S_1 \times S_2$.

case (i)
Let $u \in V_1 \setminus S_1$ and $v \in S_2$. If u is adjacent to $u_1 \in S_1$, then (u, v) is adjacent to $(u_1, v) \in S_1 \times S_2$. If u is disjunctively dominated by $u_1, u_2 \in S_1$, then $(u_1, v), (u_2, v) \in S_1 \times S_2$ and $d((u, v), (u_1, v)) = d((u, v), (u_2, v)) = 2$. So (u, v) is disjunctively dominated by $S_1 \times S_2$.

case (ii)
Let $u \in S_1$ and $v \in V_2 \setminus S_2$. If v is adjacent to $v_1 \in S_2$, then (u, v) is adjacent to $(u, v_1) \in S_1 \times S_2$. If v is disjunctively dominated by $v_1, v_2 \in S_2$, then $(u, v_1), (u, v_2) \in S_1 \times S_2$ and $d((u, v), (u, v_1)) = d((u, v), (u, v_2)) = 2$ so that (u, v) is disjunctively dominated by $S_1 \times S_2$.

case (iii)
Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$.
If u is adjacent to $u_1 \in S_1$ and v_1 is any vertex in S_2, then (u, v) is adjacent to $(u_1, v_1) \in S_1 \times S_2$. If u is disjunctively dominated by $u_1, u_2 \in S_1$, then $(u_1, v_1), (u_2, v_1) \in S_1 \times S_2$ and $d((u, v), (u_1, v_1)) = d((u, v), (u_2, v_1)) = 2$ so that (u, v) is disjunctively dominated by $S_1 \times S_2$.

From the above cases it follows that in each case (u, v) is either dominated or disjunctively dominated by elements of $S_1 \times S_2$. Thus $S_1 \times S_2$ is a DD-set in $G_1[G_2]$. Hence $\gamma_2^d(G_1[G_2]) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$ for all graphs G_1 and G_2. □

Remark 3.2.
1. *The above bound is sharp. If $G_1 = P_2$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1[G_2]) = 2$, and so, $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)\gamma_2^d(G_2)$.*
2. *Strict inequality may occur in the above result. For example consider the graphs $G_1 = P_2$ and $G_2 = S_4 \circ K_1$. Then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 4$, $\gamma_2^d(G_1[G_2]) = 2$. Here $\gamma_2^d(G_1[G_2]) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.*

Theorem 3.3.
1. *$\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)$ if G_2 has a universal vertex. In particular for a positive integer n, $\gamma_2^d(G[K_n]) = \gamma_2^d(G)$.*
2. *$\gamma_2^d(G_1[G_2]) = 2$, if G_1 has a universal vertex, but G_2 has no such vertex. In particular, if $G_1 = K_n$ and G_2 has no universal vertex, then $\gamma_2^d(G_1[G_2]) = 2$.*
3. *If both G_1 and G_2 have a universal vertex, then $\gamma_2^d(G_1[G_2]) = 1$. In particular if $G_1 = K_n$ and $G_2 = K_m$, where m, n are positive integers, then $\gamma_2^d(G_1[G_2]) = 1$.*

Proof.
1. Let v be a universal vertex of G_2 and S_1 be a γ_2^d-set of G_1. Then $S_1 \times v$ disjunctively dominates $G_1[G_2]$. The minimality of $S_1 \times v$ follows from the minimality of the γ_2^d-set S_1 of G_1. Thus, $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)$.
2. Let u be a universal vertex of G_1 and v_1, v_2 are any two vertices in G_2. $\{(u, v_1), (u, v_2)\}$ forms a γ_2^d-set of $G_1[G_2]$, for if (u', v') is an arbitrary vertex in $G_1[G_2] \setminus \{(u, v_1), (u, v_2)\}$, then it is dominated by both (u, v_1) and (u, v_2) whenever $u \neq u'$ and disjunctively dominated by $\{(u, v_1), (u, v_2)\}$ whenever $u = u'$.
3. Let u and v be universal vertices in G_1 and G_2 respectively. Then (u, v) dominates all the vertices in $G_1[G_2]$. So, $\gamma_2^d(G_1[G_2]) = 1$.

□

Corollary 3.4. *$\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ if G_2 has a universal vertex.*

Theorem 3.5. *Let G_1 be a graph without isolated vertices and G_2 be a non-trivial graph. Then,*

$$\gamma_2^d(G_1[G_2]) \leq 2\gamma_2^d(G_1).$$

Proof. Let S be a DD-set of G_1 and x, y are any two distinct vertices in G_2. We can show that $(S \times x) \cup (S \times y)$ is a DD-set of $G_1[G_2]$. Clearly, $S \times x$ dominates or disjunctively dominates all the vertices in $(G_1 \setminus S) \times G_2$. Now, let (u, v) be a vertex in $S \times G_2$. Let u' be a vertex in G_1 which is adjacent to u in G_1. Then (u, v) is adjacent to (u', x) which is adjacent to $(u, x) \in S \times x$ and $(u, y) \in S \times y$ in $G_1[G_2]$. It shows that every vertex in $S \times G_2$ has at least two vertices in $(S \times x) \cup (S \times y)$ at a distance 2 from it in $G_1[G_2]$. Thus $(S \times x) \cup (S \times y)$ is a DD-set in $G_1[G_2]$, proving that $\gamma_2^d(G_1[G_2]) \leq 2\gamma_2^d(G_1)$. □

Remark 3.6.
1. *If G_1 has a universal vertex, but G_2 has no such vertex, then equality occurs in the above relation.*

2. *If both G_1 and G_2 have a universal vertex then, strict inequality occurs in the above result.*

3. *If G_1 has a γ_2^d -set in which a pair of vertices are adjacent or if some vertex in G_1 is dominated by two different vertices in S, then strict inequality occurs in 3.5.*

Theorem 3.7. *If G_1 has no isolated vertex, then for all graphs G_2, $\gamma_2^d(G_1[G_2]) \leq \gamma_t^d(G_1)$, where $\gamma_t^d(G_1)$ is the total disjunctive domination number of G_1.*

Proof. Let S be a TDD-set of G_1. For any vertex $x \in G_2$, we can show that $S \times x$ is a DD-set in $G_1[G_2]$. It is clear that $S \times x$ dominates or disjunctively dominates $(G_1 \setminus S) \times G_2$. Now let (u,v) be any vertex in $S \times x$. u is either adjacent to $u' \in S$ or has two vertices u_1 and u_2 in S at a distance 2 from it. Then (u,v) is either dominated by $(u',x) \in S \times x$ or disjunctively dominated by $(u_1,x),(u_2,x) \in S \times x$, showing that $S \times x$ is a disjunctive dominating set in $G_1[G_2]$. This proves that, $\gamma_2^d(G_1[G_2]) \leq \gamma_t^d(G_1)$. □

Remark 3.8. *The bound given in the above theorem is sharp. If G_1 has a universal vertex and G_2 has no such vertex, then $\gamma_2^d(G_1[G_2]) = \gamma_t^d(G_1) = 2$. We may also note that strict inequality in the bound can be achieved. Consider the graphs $G_1 = P_5$, $G_2 = P_2$. Then $\gamma_t^d(G_1) = 3$, $\gamma_2^d(G_1[G_2] = 2$ and hence $\gamma_2^d(G_1[G_2]) < \gamma_t^d(G_1)$.*

Disjunctive domination in tensor products

Tensor product or Cross Product of graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is the graph $G_1 \times G_2$ whose vertex set is $V_1 \times V_2$ and edge set is $\{((u_1,v_1),(u_2,v_2)) : u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2\}$. There is no consistent relation between the disjunctive domination number of the tensor product of two graphs and the product of their disjunctive domination numbers. There are graphs in which $\gamma_2^d(G_1 \times G_2) > \gamma_2^d(G_1)\gamma_2^d(G_2)$, $\gamma_2^d(G_1 \times G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ and $\gamma_2^d(G_1 \times G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

Example 3.9. 1. $\gamma_2^d(P_5 \times P_3) = 4 > \gamma_2^d(P_5)\gamma_2^d(P_3)$.

2. $\gamma_2^d(C_3 \times C_4) = 2 = \gamma_2^d(C_3)\gamma_2^d(C_4)$.

3. *If G_1 is the graph given in fig.1, then $\gamma_2^d(G_1 \times G_1) = 2 < \gamma_2^d(G_1)\gamma_2^d(G_1)$.*

Theorem 3.10. *For any two graphs G_1 and G_2 with at least two vertices and G_2 having no isolated vertices,*

$$\gamma_2^d(G_1 \times G_2) \leq min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$$

Proof. Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times V_2$ and $V_1 \times S_2$ are both *DD*-sets in $G_1 \times G_2$.

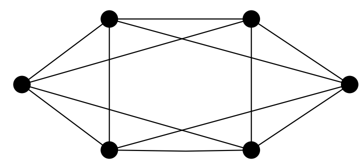

Figure 1. G_1

claim

Let (u,v) be a vertex in $G_1 \times G_2$.

If $u \in S_1$, then $(u,v) \in S_1 \times V_2$. If $u \notin S_1$, then u is either dominated by $x \in S_1$ or disjunctively dominated by two different vertices $x_1,x_2 \in S_1$. If u is dominated by $x \in S_1$, then the vertex (u,v) in $G_1 \times G_2$ is dominated by $(x,v') \in S_1 \times V_2$, where v' is some vertex adjacent to v in G_2. If u is disjunctively dominated by $x_1,x_2 \in S_1$, then the vertices $(x_1,v),(x_2,v) \in S_1 \times V_2$ are such that $d((u,v),(x_1,v)) = d((u,v),(x_2,v)) = 2$. That is, (u,v) has two vertices in $S_1 \times V_2$ at a distance two from it. So, $(u,v) \in G_1 \times G_2$ is disjunctively dominated by $S_1 \times V_2$. Thus $S_1 \times V_2$ is a *DD*-set of $G_1 \times G_2$. Similarly, $V_1 \times S_2$ is also a *DD*-set of $G_1 \times G_2$. From these it follows that, $\gamma_2^d(G_1 \times G_2) \leq min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$. □

Remark 3.11. 1. *This bound is sharp. For example, if $G_1 = P_2$, $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \times G_2) = 4$. In this case, $\gamma_2^d(G_1 \times G_2) = min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$*

2. *Strict inequality may occur in the above result. If $G_1 = P_3$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1 \times G_2) = 5$, $min\{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\} = 6$. Here, $\gamma_2^d(G_1 \times G_2) < min\{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$.*

Disjunctive domination in strong products

The strong product or normal product of graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is the graph $G_1 \boxtimes G_2$ whose vertex set is $V_1 \times V_2$ in which (u_1,v_1) is adjacent to (u_2,v_2) if and only if either

- $u_1 = u_2$ and $v_1v_2 \in E_2$ or
- $u_1u_2 \in E_1$ and $v_1 = v_2$ or
- $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Theorem 3.12. *For any two non trivial graphs G_1 and G_2,*

$$\gamma_2^d(G_1 \boxtimes G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2).$$

Proof. Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ have γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times S_2$ is a *DD*- set of $G_1 \boxtimes G_2$.

claim
Let $(u,v) \notin S_1 \times S_2$ be a vertex in $G_1 \boxtimes G_2$.

case (i)
Let $u \in V_1 \setminus S_1$ and $v \in S_2$. Then either u is dominated by $x \in S_1$ or is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$. If u is dominated by $x \in S_1$, then (u,v) is dominated by $(x,v) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$. If u is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$, then $(x_1,v),(x_2,v) \in S_1 \times S_2$ and $d((u,v),(x_1,v)) = d((u,v),(x_2,v)) = 2$ so that (u,v) is disjunctively dominated by $S_1 \times S_2$ in $G_1 \boxtimes G_2$.

case (ii)
Let $u \in V_1$ and $v \in V_2 \setminus S_2$. Then either v is dominated by $y \in S_2$ or is disjunctively dominated by two different vertices $y_1, y_2 \in S_2$. If v is dominated by $y \in S_2$, (u,v) is dominated by $(u,y) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$. If v is disjunctively dominated by two vertices $y_1, y_2 \in S_2$, then $(u,y_1),(u,y_2) \in S_1 \times S_2$ and $d((u,v),(u,y_1)) = d((u,v),(u,y_2)) = 2$ so that (u,v) is disjunctively dominated by $S_1 \times S_2$ in $G_1 \boxtimes G_2$.

case (iii)
Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$. If u is dominated by $x \in S_1$ and v is dominated by $y \in S_2$, then (u,v) is dominated by $(x,y) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$.

If u is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$ in G_1 and v is dominated by $y \in S_2$ in G_2, then (u,v) is adjacent to (u_1,y) which is again adjacent to $(x_1,y) \in S_1 \times S_2$. Similarly, (u,v) is also adjacent to (u_2,y) which is again adjacent to $(x_2,y) \in S_1 \times S_2$. Thus $d((u,v),(x_1,y)) = d((u,v),(x_2,y)) = 2$. In other words (u,v) is disjunctively dominated by two different vertices $(x_1,y),(x_2,y) \in S_1 \times S_2$. Similarly if u is dominated by $x \in S_1$ in G and v is disjunctively dominated $y_1, y_2 \in S_2$ in G_2, then (u,v) is disjunctively dominated by $(x,y_1),(x.y_2) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$.

If u and v are both disjunctively dominated by S_1 in G_1 and S_2 in G_2 respectively, then there exist $x_1, x_2 \in S_1$ and $y_1, y_2 \in S_2$ such that $d(u,x_1) = d(u,x_2) = 2$ in G_1 and $d(v,y_1) = d(v,y_2) = 2$ in G_2.Then there exist $u_1, u_2 \in V_1 \setminus S_1$ such that u is adjacent to u_1 and u_2 where u_1, u_2 are respectively adjacent to x_1 and x_2 in G. Similarly, there exist $v_1, v_2 \in V_2 \setminus S_2$ such that v is adjacent to v_1 and v_2 where v_1, v_2 are respectively adjacent to y_1 and y_2 in G_2. Thus in $G_1 \boxtimes G_2$, vertex (u,v) is adjacent to (u_1,v_1) and (u_2,v_2) which are respectively adjacent to (x_1,y_1) and (x_2,y_2) in $S_1 \times S_2$. Then, $d((u,v),(x_1,y_1)) = d((u,v),(x_2,y_2)) = 2$, proving that (u,v) is disjunctively dominated by $S_1 \times S_2$.

The above cases show that $S_1 \times S_2$ is a DD-set in $G_1 \boxtimes G_2$. Thus $\gamma_2^d(G_1 \boxtimes G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$. □

Remark 3.13.
1. *The above bound is sharp. For example if $G_1 = P_2$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1 \boxtimes G_2) = 2$. So $\gamma_2^d(G_1 \boxtimes G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$.*
2. *Strict inequality occurs if $G_1 = G_2 = P_4$. Then $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \boxtimes G_2) = 2$. Hence, $\gamma_2^d(G_1 \boxtimes G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.*

Disjunctive domination in cartesian products

The Cartesian Product $G_1 \Box G_2$ of graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is the graph with vertex set $V_1 \times V_2$ in which $(u_1,v_1),(u_2,v_2)$ is an edge if and only if either

- $u_1 = u_2$ and $v_1v_2 \in E_2$ or
- $u_1u_2 \in E_1$ and $v_1 = v_2$

Theorem 3.14. *For any two graphs G_1 and G_2,*

$$\gamma_2^d(G_1 \Box G_2) \leq min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$$

Proof. Let G_1 and G_2 are two graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times V_2$ and $V_1 \times S_2$ are both DD-sets of $G_1 \Box G_2$.

claim
Let (u,v) be a vertex in $G_1 \Box G_2$. If $u \in S_1$, then $(u,v) \in S_1 \times V_2$. If $u \notin S_1$, then u is either dominated by $x \in S_1$ or disjunctively dominated by two different vertices $x_1, x_2 \in S_1$. If u is dominated by $x \in S_1$, then (u,v) is adjacent to $(x,v) \in S_1 \times V_2$. If u is disjunctively dominated by $x_1, x_2 \in S_1$, then the vertices $(x_1,v),(x_2,v) \in S_1 \times V_2$ are such that $d((u,v),(x_1,v)) = d((u,v),(x_2,v)) = 2$. That is, (u,v) has two vertices in $S_1 \times V_2$ at a distance two from it. Thus it is disjunctively dominated by $S_1 \times V_2$. Hence $S_1 \times V_2$ is a DD-set of $G_1 \Box G_2$. Similarly, $V_1 \times S_2$ is also a DD-set of $G_1 \Box G_2$. Thus $\gamma_2^d(G_1 \Box G_2) \leq min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$. □

Remark 3.15.
1. *Equality comes in the above theorem if $G_1 = P_2$ or P_3 and $G_2 = P_2$.*
2. *Strict inequality occurs if $G_1 = P_2$ and $G_2 = P_7$.*

Remark 3.16. *The Vizing's like inequality $\gamma_2^d(G_1 \Box G_2) \geq \gamma_2^d(G_1)\gamma_2^d(G_2)$ is not true in disjunctive domination. There are graphs in which $\gamma_2^d(G_1 \Box G_2) > \gamma_2^d(G_1)\gamma_2^d(G_2)$, $\gamma_2^d(G_1 \Box G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ and $\gamma_2^d(G_1 \Box G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.*
For example,

1. *If $G_1 = P_7 and G_2 = P_2$, then $\gamma_2^d(G_1 \Box G_2) = 3 > \gamma_2^d(G_1)\gamma_2^d(G_2)$.*
2. *If $G_1 = C_4$ and $G_2 = P_2$, then $\gamma_2^d(G_1 \Box G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2) = 2$.*
3. *If $G_1 = G_2 = C_4$, then $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \Box G_2) = 2$. Hence $\gamma_2^d(G_1 \Box G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.*

Theorem 3.17. *For any two graphs G_1 and G_2, where G_1 has a γ- set which is such that the vertices not in this set are twice dominated, $\gamma_2^d(G_1 \Box G_2) \leq \gamma(G_1)\gamma(G_2)$.*

Proof. Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs with γ- sets S_1 and S_2 respectively. Let the elements of $V_1 \setminus S_1$ are dominated by two different vertices in S_1. We can show that $S_1 \times S_2$ is a disjunctive dominating set of $G_1 \Box G_2$. Let (u,v) be a vertex in $G_1 \Box G_2$.

case (i)
If $u \in S_1$ and $v \in S_2$, then $(u,v) \in S_1 \times S_2$.

case (ii)
Let $u \in S_1$ and $v \in V_2 \setminus S_2$. If v is dominated by $x \in S_2$ in G_2, then (u,v) is dominated by $(u,x) \in S_1 \times S_2$ in $G_1 \Box G_2$. Similar is the case when $u \in V_1 \setminus S_1$ and $v \in S_2$.

case (iii)
Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$. By hypothesis u is adjacent to two different vertices $x_1, x_2 \in S_1$ in G_1 and v is adjacent to $y \in S_2$ in G_2. Then in $G_1 \Box G_2$, (u,v) is adjacent to (u,y) which is adjacent to (x_1,y) and $(x_2,y) \in S_1 \times S_2$. Thus there are two different vertices (x_1,y), $(x_2,y) \in S_1 \times S_2$ such that $d((u,v),(x_1,y)) = d((u,v),(x_2,y)) = 2$. Hence (u,v) is disjunctively dominated by $S_1 \times S_2$.

The above cases show that $S_1 \times S_2$ is a disjunctive dominating set of $G_1 \Box G_2$. Hence $\gamma_2^d(G_1 \Box G_2) \leq \gamma(G_1)\gamma(G_2)$. □

Remark 3.18. *The above result is not true in general. The following examples show this.*

1. *If* $G_1 = G_2 = P_6, \gamma(G_1) = \gamma_2^d(G_1) = 2, \gamma(G_2) = \gamma_2^d(G_2) = 2, \gamma_2^d(G_1 \Box G_2) = 6 > \gamma_2^d(G_1)\gamma_2^d(G_2) = \gamma(G_1)\gamma(G_2)$.
2. *If* $G_1 = G_2 = P_7$, $\gamma(G_1) = \gamma(G_2) = 3$, $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1 \Box G_2) = 8$, $\gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma_2^d(G_1 \Box G_2) < \gamma(G_1)\gamma(G_2)$.
3. *If* $G_1 = G_2 = P_{10}, \gamma(G_1) = \gamma(G_2) = 4$, $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 3$, $\gamma_2^d(G_1 \Box G_2) = 15$. *Here,* $\gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma_2^d(G_1 \Box G_2) < \gamma(G_1)\gamma(G_2)$.
4. *If* $G_1 = G_2 = P_{11}$, $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 3$, $\gamma(G_1) = \gamma(G_2) = 4$, $\gamma_2^d(G_1 \Box G_2) = 18$. *Here* $\gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma(G_1)\gamma(G_2) < \gamma(G_1 \Box G_2)$.

Theorem 3.19. *For any two positive integers* m,n, $\gamma_2^d(K_m \Box K_n) = 2$.

Proof. Let $(u_1,v_1),(u_2,v_2)$ are two distinct vertices in $K_m \Box K_n$. A vertex $(x,y) \in K_m \Box K_n$ which not dominated by these vertices is such that $d((u_1,v_1),(x,y)) = d((u_2,v_2),(x,y)) = 2$. Hence $\{(u_1,v_1),(u_2,v_2)\}$ is a *DD*-set in $K_m \Box K_n$ which gives $\gamma_2^d(K_m \Box K_n) \leq 2$. If $u_1 \neq u_2$ and $v_1 \neq v_2$ then (u_1,v_1) and (u_2,v_2) are not adjacent in $K_m \Box K_n$. So there does not exist a universal vertex in $K_m \Box K_n$ which implies that $\gamma_2^d(K_m \Box K_n) \geq 2$. Therefore $\gamma_2^d(K_m \Box K_n) = 2$. □

4. Conclusion

In this paper we have tried to find properties of disjunctive domination in certain product of graphs. Further investigations are possible to find *DD*-number of product of important classes of graphs. The problem of determining $\gamma_2^d(G_1 * G_2)$ precisely for different classes of graphs would be interesting.

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