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Existence and regularity of solutions in α-norm for some second order partial neutral functional differential equations with finite delay in Banach spaces

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Abstract. In this paper, we investigate the regularity and existence of solutions in the α-norm for some second order partial neutral functional differential equations with finite delay in Banach spaces. To do this, we use the cosine families theory and Schauder's fixed point theorem to establish the existence of solutions and then we give some sufficient conditions that ensure the regularity of solutions. Finally, we give an example to illustrate the theoretical results.

AMS Subject Classifications: 47D09, 34K40, 45N05, 35A01.

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Contents

1. Introduction

The aim of this work, we to study the existence and regularity of solutions in α-norm for the following second order neutral partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}[u'(t) - g(t, u_t)] = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_\alpha, \\ u'_0 = \varphi' \in \mathcal{C}_\alpha, \end{cases} \tag{1.1}$$

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where A is the (possibly unbounded) infinitesimal generator of strongly continuous cosine family of linear operators in X. $\mathcal{C}_\alpha = C^1([-r,0], D((-A)^\alpha))$, $0 < \alpha < 1$, denotes the space of continuous differentiable functions from $[-r,0]$ into $D((-A)^\alpha)$, $(-A)^\alpha$ is the fractional α-power of A. This operator $((-A)^\alpha, D((-A)^\alpha))$ will be describe later. C_α is endowed with the following norm $\|h\|_{\mathcal{C}_\alpha} = \|h\|_\alpha + \|h'\|_\alpha$ for all $h \in \mathcal{C}_\alpha = C^1([-r,0], X_\alpha)$, where $\|h\|_\alpha = \sup_{-r\leq 0\leq 0} |h(\theta)|_\alpha$. The norm $|.|_\alpha$ will be specified later. For $u \in C^1([-r,b], D((-A)^\alpha))$, $t \geq 0, b > 0$, and $t \in [0,b]$ u_t denotes the history function of $\mathcal{C}_\alpha$ defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \in [-r,0],$$

$f : \mathbb{R}^+ \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ and $g : \mathbb{R}^+ \times \mathcal{C}_\alpha \to X_\alpha$ are given functions.
In [3] the authors study firstly the abstract semi-linear second order initial value problem and secondly they unify and simplify some ideas from strongly continuous cosine families of linear operators in Banach spaces.
In [7], the authors reveal three properties of cosine families, distinguishing them from semi-groups of operators.
In [1] by use of the theory of cosine families of linear operators in Banach space, the author studied the existence of solutions of following second order partial neutral functional differential equation

$$\begin{cases} \dfrac{d}{dt}[u'(t) - g(t,u_t)] = Au(t) + f(t,u_t,u'(t)),\ t \in J = [0,T] \\ \\ u_0 = \varphi \in \mathcal{B},\ u'(0) = z \in X. \end{cases} \tag{1.2}$$

To the best of the authors knowledge, the equation (1.2) and most similar other problems using cosine families theory are studied without delay arguments. However time-delay is known to have a significant impact on the asymptotic behavior and stability of these dynamic systems, it is inevitable that it be included in the mathematical description of phenomena. For this purpose, in [5], Zabsonre et al. studied the existence and regularity of solution for some nonlinear second order differential with finite delay in Banach spaces.

This present work is a generalization of [4] and a continuation of [1]. The neutral functional differential equations, on the other hand, received a lot of attention in recent years due to the fact that they are present in many areas of applied mathematics.
By use of the theory of strongly continuous cosine families of linear operator in Banach space, we will prove in this paper the existence of mild and strict solution.
The organization of this work as follows, in Section 2, we recall some preliminary results about cosine families theory and fractional α-power, in Section 3, we prove the existence and uniqueness of mild solution in the α-norm for (1.1). In Section 4, we study the regularity of solutions. Finally, we illustrate our results, in Section 5 by examining an example.

2. Preliminary Results

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and A be the infinitesimal generator of strongly continuous $(C(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(-A)$. Note that if the assumption $0 \in \rho(-A)$ is not satisfied, one can substitute the operator $-A$ by the operator $(-A - \sigma I)$ with σ large enough such that $0 \in \rho(-A - \sigma I)$. This allows us to define the fractional power $(-A)^\alpha$ for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D((-A)^\alpha)$ dense in X. The closeness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$, endowed with the graph norm of $(-A)^\alpha$, $|x| = \|x\| + \|(-A)^\alpha x\|$, is a Banach space. Since $(-A)^\alpha$ is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|(-A)^\alpha x\|$. Thus, $D((-A)^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α.

Definition 2.1. *[3] A one parameter family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if*

$i)$ $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s,t\in\mathbb{R}$
$ii)$ $C(0)=I$
$iii)$ $C(t)x$ is continuous on $\mathbb{R}$ for each fixed $x\in X$.

The strongly continuous sine family $\{S(t), t\in\mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t), t\in\mathbb{R}\}$ by

$$S(t)x=\int_0^t C(s)xds, \text{ for } x\in X, t\in\mathbb{R}. \tag{2.1}$$

Definition 2.2. *The infinitesimal generator of strongly continuous cosine family $\{C(t), t\in\mathbb{R}\}$ is the operator $A: X\longrightarrow X$ define by*

$$Ax=\frac{d^2C(t)x}{dt^2}\Big|_{t=0}.$$

$D(A)=\{x\in X:$ *C(t)x is a twice continuously differentiable function of t*$\}$.

We shall also make use of the set

$$E=\{x: C(t)x \text{ is a once continuously differentiable function of t}\}.$$

Lemma 2.3. *Let $C(t),\in\mathbb{R}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true.*
$i)$ $D(A)$ is dense in X and A is closed operator in X;

$ii)$ if $x\in X$ and $s,r\in\mathbb{R}$ then $z=\int_s^r = S(u)xdu\in D(A)$ and $Az=C(s)x-C(r)x$;
$iii)$ if $x\in X$, $s,r\in\mathbb{R}$ then $z=\int_0^s\int_0^r C(u)C(v)xdudv\in D(A)$ and
$Az=\frac{1}{2}(C(s+r)x-C(s-r)x)$;
$iv)$ if $x\in X$, $S(t)x\in E$;
$v)$ if $\in X$, the $S(t)x\in D(A)$ and $\frac{dC(t)}{dt}=AS(t)x$:

$vi)$ if $x\in D(A)$, then $C(t)x\in D(A)$ and $\frac{d^2C(t)}{dt^2}=AC(t)x=C(t)Ax$;
$vii)$ if $x\in E$, then $\lim_{t\to 0} AS(t)=0$;
$viii)$ if $x\in E$, then $S(t)x\in D(A)$ and $\frac{d^2S(t)}{dt^2}=AS(t)x$;
$ix)$ if $x\in D(A)$, then $S(t)x\in D(A)$ and $AS(t)x=S(t)Ax$;
$x)$ $C(t+s)+C(t-s)=2AS(t)S(s)$ for all $s,t\in\mathbb{R}$.

In [3], for $0<\alpha<1$ the fractional powers $(-A)^\alpha$ exist as closed linear operators in X,

$D((-A)^\alpha)\subset D((-A)^\beta)$ for $0\le\beta\le\alpha\le 1$ and $(-A)^\alpha(-A)^\beta=(-A)^{\alpha+\beta}$ for $0\le\alpha+\beta\le 1$.

For our objective we assume that
$(\mathbf{H}_0)$ A is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X.

By Lemma2.3, $(\mathbf{H}_{0)})$ implies that the operator A is densely defined in X, i.e $\overline{D(A)}=X$. We have the following result.

Lemma 2.4. *[3] Assume that (H_0) hols. Then there are constants $M \geq 1$ and $\omega \geq 0$ such that*
$\|C(t)\| \leq Me^{\omega|t|}$ *and* $\|S(t_1) - S(t_2)\| \leq M\Big|\int_{t_1}^{t_2} e^{\omega|s|}ds\Big|$, *for all* $t_1, t_2 \in \mathbb{R}$.

From previous inequality, since $S(0) = 0$ we can deduce that

$$\|S(t)\| \leq \frac{M}{\omega}e^{\omega t} \text{ for } t \in \mathbb{R}^+.$$

In the sequel, let us pose $M_1 = \max\Big(M, \dfrac{M}{\omega}\Big)$.

Theorem 2.5. *[3] If $k : \mathbb{R}^+ \to X$ is continuous, $h : \mathbb{R}^+ \to X$ is continuous and u is a solution of equation* (1.1)*, then u is a solution of integral equation*

$$u(t) = C(t)x + S(t)y + \int_0^t C(t-s)k(s)ds + \int_0^t S(t-s)h(s)ds.$$

($\mathbf{A}_1$): For $0 < \alpha < 1$, $(-A)^\alpha$ maps onto X and $1-1$, so that $D((-A)^\alpha)$ endowed with the norm $|x|_\alpha = \|(-A)^\alpha x\|$ is a Banach space. We denote by X_α this space. In addition we assume that A^{-1} is compact. To establish our results, we need the following Lemmas.

Lemma 2.6. *[4] Assume that $(\boldsymbol{H}_0)$ holds. The following are true*
(i) For $0 < \alpha < 1$, $(-A)^{-\alpha}$ is compact if and only if A^{-1} is compact.
(ii) For $0 < \alpha < 1$, and $t \in \mathbb{R}$ $(-A)^{-\alpha}C(t) = C(t)(-A)^{-\alpha}$ and $(-A)^{-\alpha}S(t) = S(t)(-A)^{-\alpha}$.

Recall from [10], $(-A)^{-\alpha}$ is given by the following formula

$$(-A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^{+\infty} t^{-\alpha}(tI - A)^{-1}dt.$$

Lemma 2.7. *[4] Assume that $(\boldsymbol{H}_0)$ holds. Let $v : \mathbb{R} \longrightarrow x$ such that v is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then*
(i) q is twice continuously differentiable and for $t \in \mathbb{R}$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t-s)v(s)ds$$

and

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

(ii) For $0 < \alpha < 1$ and $t \in \mathbb{R}$, $(-A)^{\alpha-1}q'(t) \in E$.

Theorem 2.8. *(Heine's theorem)*
Let f be a continuous function on a compact set K, then f is uniformly continuous on K.

Theorem 2.9. *(Arzela-Ascoli theorem)*
Let (X, d_X) and (Y, d_Y) be compact metric spaces, $C(X, Y)$ be the set of continuous functions from X to Y and Let $\mathcal{F}$ be q subset of $C(X, Y)$. If $\mathcal{F}$ is closed and equicontinuous then, it is compact.

Theorem 2.10. *(Schauder's fixed point theorem)*
Let X be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f : K \longrightarrow K$ there exists $x \in K$ such that $f(x) = x$.

3. Existence of mild solutions

Definition 3.1. *A continuous function $u\ :]-r,+\infty[\to X_\alpha$ is said a strict solution of equation* (1.1) *if the following conditions hold*

(i) $u \in C^1([0,+\infty[;X_\alpha)\cap C^2([0,\infty[;X_\alpha)$
(ii) u *satisfies equation* (1.1) *on* $[0,+\infty[$.
(iii) $u(\theta)=\varphi(\theta)$ *for* $-r\le\theta\le 0$.

Proposition 3.2. *Assume that* $(\boldsymbol{H}_{0)})$ *holds. If* u *is a strict solution of equation* (1.1)*, then*

$$u(t)=C(t)\phi(0)+S(t)(\phi'(0)-g(0,\varphi))+\int_0^t C(t-s)g(s,u_s)ds+\int_0^t S(t-s)f(s,u_s,u_s')ds. \tag{3.1}$$

Proof. It is just the consequence of Theorem 2.5. In fact, let us pose $k(t)=g(t,u_t)$ and $h(t)=f(t,u_t,u_t')$ for $t\ge 0$. The we get the desired results.■

Remark 3.3. *The converse is not true. In fact if* u *satisfies equation* (3.1)*,* u *may be not twice continuously differentiable, that is why we distinguish between mild and strict solutions.*

Definition 3.4. *A continuous function* $u:]-r,+\infty[\to X_\alpha$*, for* $b>0$ *is said to a mild solution of equation* (1.1) *if*

$$\begin{cases} u(t)=C(t)\varphi(0)+S(t)(\varphi'(0)-g(0,\varphi))+\displaystyle\int_0^t C(t-s)g(s,u_s)ds+\int_0^t S(t-s)f(s,u_s,u_s')ds \text{ for } t\in[0,b],\\ u_0=\varphi(0),\\ u_0'=\varphi'(0).\end{cases}$$

In the following, we give a local existence of mild solutions of equation(1.1). We will use the Schauder's fixed point theorem. For this purpose, we make this following assumptions.

$(\mathbf{H}_1)$The function $f:[0,b]\times\mathcal{C}_\alpha\to X$ satisfies the following conditions

i) $f:[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha\to X$ is continuously differentiable.

ii) There exists a continuous nondecreasing function $\beta:[0,b]\to\mathbb{R}^+$ such that

$\|f(t,\varphi,\varphi')\|\le\beta(t)\|\varphi\|_\alpha$ for $(t,\varphi)\in[0,b]\times\mathcal{C}_\alpha$.

$(\mathbf{H}_2)$ $g:[0,b]\times\mathcal{C}_\alpha\to X_\alpha$ is continuously differentiable and for each $b>0$ there exist $0<L_g<1$ such that

$$|g(t,\varphi)-g(t,\psi)|_\alpha\le L_g\|\varphi-\psi\|_\alpha \text{ for every } t\in[0,b] \text{ and } \varphi,\psi\in\mathcal{C}_\alpha.$$

$(\mathbf{H}_3)$ A^{-1} is compact on X.

Theorem 3.5. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathcal{C}_\alpha$ *such that* $\varphi(0) \in D(A)$, $\varphi'(0) - g(0,\varphi) \in E$ *and assume that*

$$L_g M_1 e^{\omega b} + \|(-A)^{\alpha-1}\| \sup_{t\in[0,b]} \Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big] < 1.$$

Then equation (1.1) *has at least one mild solution on* $[0,b]$.

Proof. Let $k > \|\varphi\|_{\mathcal{C}_\alpha}$, we define the following set

$$B_k = \{u \in C([0,b], X_\alpha) : \ u(0) = \varphi(0) \text{ and } |u|_\infty \leq k\},$$

where $|u|_\infty = \sup\limits_{t\in[0,b]} |u(t)|_\alpha$. For $u \in B_k$, define the $\widetilde{u}(t) : [-r,b] \to X_\alpha$ by

$$\widetilde{u}(t) = \begin{cases} u(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in [-r,0]. \end{cases}$$

The function $t \to \widetilde{u}_t$ is continuous from $[0,b]$ to $\mathcal{C}_\alpha$. Now, define the operator $\mathcal{K}$ on B_k by

$$\mathcal{K}(u)(t) = C(t)\varphi(0) + S(t)(\varphi'(0)g(0,\varphi)) + \int_0^t C(t-s)g(s,\widetilde{u}_s)ds + \int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t \in [0,b].$$

It is sufficient to show that $\mathcal{K}$ has a fixed point in B_k. We give the proof in several steps.

Step 1: There is a positive $k > \|\varphi\|_\alpha$ such that $\mathcal{K}(B_k) \subset B_k$.

If not, then for each $k > \|\varphi\|_{\mathcal{C}_\alpha}$, there exist $u_k \in B_k$ and $t_k \in [0,b]$ such that $|(\mathcal{K}u_k)(t_k)|_\alpha > k$.

$$\begin{aligned}
&k < |(\mathcal{K}u_k)(t_k)|_\alpha \\
&= \Big| C(t_k)\varphi(0) + S(t_k)(\varphi'(0) - g(0,\varphi)) + \int_0^{t_k} C(t_k-s)g(s,\widetilde{u}_s)ds + \int_0^{t_k} S(t_k-s)f(s,\widetilde{u}_s)ds \Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha + \Big\| -(-A)^{\alpha-1}\int_0^{t_k} AS(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \Big\| \\
&\quad + \Big| \int_0^{t_k} \frac{d}{ds}\big(S(s)g(t_k-s,\widetilde{u}_{t_k-s})\big)ds - \int_0^{t_k} S(s)\frac{d}{ds}\big(g(t_k-s,\widetilde{u}_{t_k-s})\big)ds \Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha \\
&\quad + \Big| \int_0^{t_k} \frac{d}{ds}\big(S(s)g(t_k-s,\widetilde{u}_{t_k-s})\big)ds - \int_0^{t_k} S(s)\frac{d}{ds}\big(g(t_k-s,\widetilde{u}_{t_k-s})\big)ds \Big|_\alpha \\
&\quad + \Big\| (-A)^{\alpha-1}\Big[\int_0^{t_k} \frac{d}{ds}\big(C(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)\Big] \Big\|
\end{aligned}$$

$$
\begin{aligned}
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)(\varphi'(0) - g(0,\varphi))|_\alpha + |S(t_k)g(0,\widetilde{u}_0)|_\alpha + M_1 e^{\omega b}|g(t_k,\widetilde{u}_{t_k}) - g(0,\widetilde{u}_0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\Big(\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k})\| + \|C(t_k)f(0,\widetilde{u}_0,\widetilde{u'}_0)\| + Me^{\omega b}\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k}) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big) \\
&< M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0,\varphi))|_\alpha\Big) + M_1 e^{\omega b}\sup_{s\in[0,b]}|g(s,0)|_\alpha + M_1 e^{\omega b}L_g\|\widetilde{u}_{t_k}\|_\alpha \\
&\quad + 2M_1 e^{\omega b}|g(0,\varphi)|_\alpha + \|(-A)^{\alpha-1}\|\Big[\big(\beta(t_k) + Me^{\omega b}\big)\|\widetilde{u}_{t_k}\|_\alpha + 2Me^{\omega b}\beta(0)\|\widetilde{u}_0\|_\alpha\Big].
\end{aligned}
$$

Since $\|\widetilde{u}_t\|_\alpha \leq k$ for all $t \in [0,b]$ and $u \in B_k$. Then we have

$$
\begin{aligned}
k &< M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0\varphi))|_\alpha\Big) + M_1 e^{\omega b}L_g k + M_1 e^{\omega b}\sup_{s\in[0,b]}|g(s,0)|_\alpha + 2M_1 e^{\omega b}|g(0,\widetilde{u}_0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1 + 2Me^{\omega b}) + Me^{\omega b}\Big]k.
\end{aligned}
$$

Dividing above sides of above inequality by k, it follows that

$$
\begin{aligned}
1 &< \frac{M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |(\varphi'(0) - g(0,\varphi))|_\alpha\Big)}{k} + L_g M_1 e^{\omega b} + \frac{M_1 e^{\omega b}\sup\limits_{s\in[0,b]}|g(s,0)|_\alpha}{k} + \frac{2M_1 e^{\omega b}|g(0,\varphi)|_\alpha}{k} + \\
&\quad + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1 + 2Me^{\omega b}) + Me^{\omega b}\Big].
\end{aligned}
$$

When $k \to 0$, we have

$$
1 < L_g M_1 e^{\omega b} + \|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1 + 2Me^{\omega b}) + Me^{\omega b}\Big],
$$

which gives contradiction.

Step 2: $\mathcal{K}$ is continuous.

Let $(u^n)_n \subset B_k$ with $u^n \to u$ and ${u'}^n \to u'$ in B_k. Then, the set

$$
\Delta = \{(s,\widetilde{u}_s^n,\widetilde{u'}_s^n)),\ (s,\widetilde{u}_s,\widetilde{u}_s)) :\ s \in [0,b],\ n \geq 1\}
$$

and

$$
\wedge = \{(s,\widetilde{u}_s^n),\ (s,\widetilde{u}_s) :\ s \in [0,b],\ n \geq 1\}
$$

are compact respectively in $[0,b] \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha$ and $[0,b] \times \mathcal{C}_\alpha$. Heine's theorem implies that f and g are uniformly continuous respectively in Δ and $\wedge$. Then, we have

$$\begin{aligned}
&|\mathcal{K}(u^n)(t) - \mathcal{K}(u)(t)|_\infty \\
&\le \sup_{t\in[0,b]} \Big| \int_0^t C(t-s)\Big(g(s,\widetilde{u}_s^n) - g(s,\widetilde{u}_s)\Big)ds\Big|_\alpha \\
&\quad + \sup_{t\in[0,b]} \Big\| -(-A)^{\alpha-1}\int_0^t AS(t-s)\Big(f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big)\Big\| \\
&\le \sup_{t\in[0,b]} \Big| \int_0^t \frac{d}{ds}\Big(S(s)g(t_k-s,\widetilde{u}_{t_k-s}^n) - g(t_k-s,\widetilde{u}_{t_k-s})\Big)ds \\
&\quad - \int_0^t S(s)\frac{d}{ds}\Big(g(t_k-s,\widetilde{u}_{t_k-s}^n) - g(t_k-s,\widetilde{u}_{t_k-s})\Big)ds\Big|_\alpha \\
&\quad + \sup_{t\in[0,b]} \Big\|(-A)^{\alpha-1}\Big[\int_0^t \frac{d}{ds}\big(C(t-s)f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds \\
&\quad - \int_0^t C(t-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s^n,\widetilde{u'}_s^n) - f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\big)\Big]\Big\| \\
&\le \sup_{t\in[0,b]} \Big[|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + M_1e^{\omega b}\Big(|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + |g(t,\widetilde{u}_t^n) - g(t,\widetilde{u}_t)|_\alpha\Big] \\
&\quad + \sup_{t\in[0,b]} \|(-A)^{\alpha-1}\|\Big[\big(f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\big) - C(t)\big(f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\| \\
&\quad + Me^{\omega b}\|f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\big) - \big(f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\Big] \\
&\le \sup_{t\in[0,b]} \Big[(1+Me^{\omega b})|g(0,\widetilde{u}_0^n) - g(0,\widetilde{u}_0)|_\alpha + M_1e^{\omega b}|g(t,\widetilde{u}_t^n) - g(t,\widetilde{u}_t)|_\alpha\Big] \\
&\quad + \sup_{t\in[0,b]} \|(-A)^{\alpha-1}\|\Big[(1+Me^{\omega b})\|f(t,\widetilde{u}_t^n,\widetilde{u'}_t^n) - f(t,\widetilde{u}_t,\widetilde{u'}_t)\| \\
&\quad + 2Me^{\omega b}\|f(0,\widetilde{u}_0^n,\widetilde{u'}_0^n) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big] \to 0 \text{ as } n\to\infty,
\end{aligned}$$

and this yield the continuity of $\mathcal{K}$ on B_k.

Step 3: The set $\{\mathcal{K}(u)(t):\ u\in B_k\}$ is relatively compact for each $t\in]0,b]$.

Let $t\in]0,b]$ be fixed and $\gamma>0$ be such that $\alpha<\gamma<1$. Using the same reasoning like previously, it follows that

$$\begin{aligned}
\|(-A)^\gamma\mathcal{K}(u)\| &\le \|(-A)^{\gamma-1}\|\Big[M_1e^{\omega b}\Big(\|A\varphi(0)\| + \|A(\varphi'(0) - g(0,\varphi))\| + \sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2Me^{\omega b}) + Me^{\omega b}\Big]k \\
&\quad + M_1e^{\omega b}\Big[L_gk + \sup_{s\in[0,b]}|g(s,0)|_\gamma + |g(0,\varphi)|_\gamma\Big] < \infty.
\end{aligned}$$

Consequently for $t\in]0,b]$ fixed, the set $\{(-A)^\gamma\mathcal{K}(u)(t):\ u\in B_k\}$ is bounded in X. By ($\mathbf{H}_3$), we deduce that $(-A)^{-\gamma}: X\to X_\alpha$ is compact. It follows that the set $\{\mathcal{K}(u)(t):\ u\in B_k\}$ is relatively compact for each $t\in]0,b]$ in X_α.

Step 4: The set $\{\mathcal{K}(u):\ u\in B_k\}$ is an equicontinuous family of functions.

Let $u\in B_k$ and $0\leq\tau_1<\tau_2\leq b$ then, we have

$$
\begin{aligned}
|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha &\leq |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha\\
&\quad+\Big|\int_0^{\tau_2}C(\tau_2-s)g(s,\widetilde{u}_s)ds-\int_0^{\tau_1}C(\tau_1-s)g(s,\widetilde{u}_s)ds\Big|_\alpha\\
&\quad+\Big|\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds-\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big|\\
&\leq |[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_\alpha\\
&\quad+\Big|\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]g(s,\widetilde{u}_s)ds-\int_{\tau_1}^{\tau_2}[C(\tau_2-s)g(s,\widetilde{u}_s)ds\Big|_\alpha\\
&\quad+\Big|\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\Big|\\
&\quad+\Big|\int_{\tau_2}^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big|,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha\\
&\leq |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha\\
&\quad+\Big|\int_0^{\tau_1}\frac{d}{ds}\Big([S(\tau_2-s)-S(\tau_1-s)]g(s,\widetilde{u}_s)\Big)ds-\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]\frac{d}{ds}g(s,\widetilde{u}_s)ds\Big|_\alpha\\
&\quad+\Big|\int_{\tau_1}^{\tau_2}\frac{d}{ds}\big(S(\tau_2-s)g(s,u_s)\big)ds-\int_{\tau_1}^{\tau_2}S(\tau_2-s)\frac{d}{ds}\big(g(s,u_s)\big)ds\Big|_\alpha\\
&\quad+\Big\|(-A)^{\alpha-1}\Big[\int_0^{\tau_1}\frac{d}{ds}\Big([C(\tau_2-s)-C(\tau_1-s)]f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\\
&\quad-\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]\frac{d}{ds}\big(]f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|\\
&\quad+\Big\|(-A)^{\alpha-1}\int_{\tau_1}^{\tau_2}\frac{d}{ds}\big(C(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds-\int_{\tau_1}^{\tau_2}C(\tau_2-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&|\mathcal{K}(u)(\tau_2)-\mathcal{K}(u)(\tau_1)|_\alpha\\
&\leq |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-g(0,\varphi))|_\alpha+|(S(\tau_2-\tau_1)g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha\\
&\quad+\|S(\tau_2)-S(\tau_1)\|\,|g(0,\widetilde{u}_0)|_\alpha+|S(\tau_2)-S(\tau_1)\|\,|(g(\tau_1,\widetilde{u}_{\tau_1}))-(g(0,\widetilde{u}_0))|_\alpha
\end{aligned}
$$

$$+M_1 e^{\omega b}|g(\tau_2,\widetilde{u}_{\tau_2})-g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha+\|(-A)^{\alpha-1}\|\Big[\|(C(\tau_2-\tau_1)-I)f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{u'}_{\tau_1})\|$$

$$+\|[C(\tau_2)-C(\tau_1)]f(0,\widetilde{u}_0,\widetilde{u'}_0)\|+\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2})-C(\tau_2-\tau_1)f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\|$$

$$+Me^{\omega b}\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2)}-f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\Big]\to 0 \text{ as } \tau_1\to\tau_2.$$

Since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}})$ is uniformly continuous on compact subset of X. Thus $\mathcal{K}$ maps B_k into an equicontinuous family of functions.
So from **Step 1** to **Step 4** and by Ascoli-Arzela theorem, we can conclude that $\mathcal{K}:B_k\to B_k$ is completely continuous. Hence by Schauder's fixed point theorem, we conclude that $\mathcal{K}$ has least one fixed point in B_k which is a mild solution of equation (1.1) on $[0,b]$. ■

Our next objective is to prove the uniqueness of mild solution. For this purpose formulate the followings assumptions

($\mathbf{H}_4$): $f:[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha\to X$ is continuously differentiable and locally Lipschitzian with the respect on second variable. Then there exists $c_0(r)>0$ such that for $\varphi,\ \psi\in\mathcal{C}_\alpha$ with $\|\varphi\|_{\mathcal{C}_\alpha},\ \|\psi\|_{\mathcal{C}_\alpha}\le r$, we have

$$\|f(t,\varphi_1,\varphi_1',)-f(t,\varphi_2,\varphi_2',)\le c_0(r)\|\varphi_1-\varphi_2\|_{\mathcal{C}_\alpha} \text{ for } \in[0,b],\ \varphi_1,\varphi_2\in\mathcal{C}_\alpha.$$

($\mathbf{H}_5$) The maps $t\mapsto AC(t)$ is locally bounded.

Theorem 3.6. *Assume that* ($\boldsymbol{H}_0$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_4$) *and* ($\boldsymbol{H}_5$) *hold. Let* $\varphi\in\mathcal{C}_\alpha$ *such that* $\varphi(0)\in D(A)$ *and* $\varphi'(0)-g(0,\varphi)\in E$. *Assume that*

$$\Big[L_g\big(1+(Me^{\omega b}+\mu b)b\big)+\|(-A)^{\alpha-1}\|\mu c_0(r)b(1+b)\Big]<1.$$

Then Equation (1.1) *has unique mild solution.*

Proof. Let us consider the following set

$$\mathbb{F}(\varphi)=\{u\in C^1([0,b]),X_\alpha):u(0)=\varphi(0)\}.$$

For $u\in\mathbb{F}(\varphi)$ we define $\widetilde{u}:[-r,b]\to X_\alpha$ by

$$\widetilde{u}(t)=\begin{cases}u(t) \text{ for } t\in[0,b]\\ \varphi(t) \text{ for } t\in[-r,0].\end{cases}$$

Now, we define the operator $\Phi:\mathcal{F}(\varphi)\to\mathcal{F}(\varphi)$ by

$$\Phi(u)(t)=C(t)\varphi(0)+S(t)(\varphi'(0)-g(0,\varphi))+\int_0^t C(t-s)g(s,\widetilde{u}_s)ds+\int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t\in[0,b].$$

We will show that Φ is a strict contraction. Let $u,v\in\mathbb{F}(\varphi)$ and μ be a positive real number such that $\|AC(t)\|\le\mu$ for $t\in[0,b]$. Then we have

$$\Phi(u)(t)-\Phi(v)(t)=\int_0^t C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]ds+\int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds.$$

Then

$$
\begin{aligned}
&|\Phi(u)(t)-\Phi(v)(t)|_\alpha \\
&\leq \Big|\int_0^t C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]ds\Big|_\alpha+\Big|\int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds\Big|_\alpha \\
&\leq \Big|\int_0^t \Big(C(t-s)[g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]\Big)ds\Big|_\alpha+\Big|\int_0^t \Big(\int_0^{t-s} C(\sigma)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]d\sigma\Big)ds\Big|_\alpha \\
&\leq Me^{\omega b}\int_0^t \Big|g(s,\widetilde{u}_s)-g(s,\widetilde{u}_s)]\Big|_\alpha ds+\|(-A)^{\alpha-1}\|\mu b\int_0^t \|f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)\|ds \\
&\leq \Big(Me^{\omega b}Lgb+\|(-A)^{\alpha-1}\|\mu b^2 c_0(r)\Big)\|u-\|_{\mathcal{C}_\alpha},
\end{aligned}
$$

it follows that

$$|\Phi(u)(t)-\Phi(v)(t)|_\alpha \leq \Big(Me^{\omega b}Lgb+\|(-A)^{\alpha-1}\|\mu b^2 c_0(r)\Big)\|u-\|_{\mathcal{C}_\alpha} \tag{3.2}$$

On the other hand, by use of Equation (2.1) and Proposition 2.3, we have

$$
\begin{aligned}
(\phi(u))'(t) = AS(t)\varphi(0)+C(t)(\varphi'(0)-g(0,\varphi))+g(t,u_t)+\int_0^t AS(t-s)g(s,\widetilde{u}_s)ds \\
+\int_0^t C(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds.
\end{aligned}
$$

Using the same reasoning like previously, then we have

$$|(\Phi(u))'(t)-(\Phi(v))'(t)|_\alpha \leq \Big[L_g+\mu L_g b^2+\|(-A)^{\alpha-1}\|\mu c_0(r)b\Big]\|u-v\|_{\mathcal{C}_\alpha}. \tag{3.3}$$

Adding equation (3.2) and equation (3.3), then we have

$$\|\Phi(u)(t)-\Phi(u)(t)\|_{\mathcal{C}_\alpha} \leq \Big[L_g\big(1+(Me^{\omega b}+\mu b)b\big)+\|(-A)^{\alpha-1}\|\mu c_0(r)b(1+b)\Big]\|u-v\|_{\mathcal{C}_\alpha}.$$

This means Φ is a strict contraction.Thus by Banach's fixed point theorem, we deduce that Φ has a unique fixed point in $\mathbb{F}(\varphi)$. Then Equation(1.1) has a unique mild solution on $[0,b]$ ■

4. Existence of strict solutions

Theorem 4.1. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_5)$ *hold and* f *is continuously differentiable. Moreover assume that the partial derivatives* $D_1 f$ *and* $D_2 f$ *are locally lipschitz in classical sens. Let* $\varphi \in C^3([-r,0],D((-A)^\alpha))$ *such that* $\varphi(0),\ \varphi''(0)\in D(A)$ *and* $\varphi'(0)-g(0,\varphi),\ \varphi^{(3)}(0)\in E$ *and*

$$\varphi''(0)-D_t g(0,\varphi)-D_\varphi g(0,\varphi)\varphi' = A\varphi(0)+f(\varphi,\varphi').$$

Then the corresponding of mild solution u *becomes a strict solution of equation* (1.1) *on* $[0,b]$.

Proof Let $\varphi \in C^3([-r,0],D((-A)^\alpha))$ such that $\varphi(0),\ \varphi''(0)\in D(A)$, $\varphi'(0)-g(0,\varphi),\ \varphi^{(3)}(0)\in E$ and

$\varphi''(0) - D_t g(0, \varphi) - D_\varphi g(0, \varphi)\varphi' = A\varphi(0) + f(\varphi, \varphi').$

Let u be the corresponding mild solution of equation (1.1) which is defined on $[0, b]$. Consider

$$\begin{cases} v(t) = C(t)\Big[A\varphi(0) + f(\varphi, \varphi')\Big] + S(t)A(\varphi'(0) - g(0, \varphi)) \\ +[D_1 g(t, u_t) + D_2 g(t, u_t)u_t'] + \int_0^t AC(t-s)g(s, u_s)ds \\ \qquad + \int_0^t C(t-s)[D_1 f(u_s, u_s')u_s' + D_2 f(u_s, u_s')v_s]ds \\ v_0 = \varphi''. \end{cases}$$

Now, we define w by

$$\begin{cases} w(t) = \varphi'(0) + \int_0^t v(s)ds \text{ if } t \in [0, b] \\ w(t) = \varphi'(t) \ \text{ if } \ -r \leq t \leq 0 \\ w'(t) = \varphi''(t) \ \text{ if } \ -r \leq t \leq 0. \end{cases} \tag{4.1}$$

Then we can see that $w_t = \varphi' + \int_0^t v_s ds$ for $t \in [0, b]$.

Consequently the map $t \mapsto w_t$ and $t \mapsto \int_0^t C(t-s)f(u_s, w_s)ds$ are continuously differentiable. Then we have

$$\begin{aligned} \frac{d}{dt}\int_0^t C(t-s)f(u_s, w_s)ds &= \frac{d}{dt}\int_0^t C(s)f(u_{t-s}, w_{t-s})ds \\ &= C(t)f(u_0, w_0) + \int_0^t C(t-s)\Big[D_1 fs(u_s, w_s)u_s' + D_2 f(u_s, w_s)v_s\Big]ds \\ &= C(t)f(\varphi, \varphi') + \int_0^t C(t-s)\Big[D_1 fs(u_s, w_s)u_s' + D_2 f(u_s, w_s)v_s\Big]ds, \end{aligned}$$

it follows that

$$\int_0^t C(s)f(\varphi, \varphi')ds = \int_0^t C(t-s)f(u_s, u_s')ds - \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u_\tau' + D_f(u_\tau, w_\tau)v_\tau\Big]d\tau ds.$$

On other hand by Lemma 2.7 one has

$$\int_0^t \int_0^s AC(s-\tau)g(\tau, u_\tau)d\tau ds = \int_0^t Aq'(s)ds = Aq(t) = \int_0^t AS(t-s)g(s, u_s)ds.$$

Consequently we have

$$w(t) = \varphi'(0) + \int_0^t S(s)A(\varphi'(0) - g(0,\varphi))ds + \int_0^t C(s)A\varphi(0)ds + \int_0^t C(t-s)f(u_s, w_s)ds + g(t, u_t) - g(0,\varphi)$$
$$- \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u'_s + D_2 f(u_\tau, w_\tau)v_\tau\Big]d\tau ds$$
$$+ \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, u_\tau)u'_\tau + D_2 f(u_\tau, u_\tau)v_\tau\Big]d\tau ds.$$

Moreover by Lemma 2.3, we have

$$\int_0^t C(s)A\varphi(0)ds = S(t)A\varphi(0)$$

$$\int_0^t S(s)A(\varphi'(0) - g(0,\varphi))ds = C(t)(\varphi'(0) - g(0,\varphi) - (\varphi'(0) - g(0,\varphi)).$$

It follows that

$$w(t) = \varphi'(0) + C(t)(\varphi'(0) - g(0,\varphi)) + S(t)A\varphi(0) - (\varphi'(0) - g(0,\varphi)) + g(t,u_t) - g(0,\varphi)$$
$$+ \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t C(t-s)f(u_s, w_s)ds$$
$$+ \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, u'_\tau)u'_s + D_2 f(u_\tau, u'_\tau)v_\tau\Big]d\tau ds$$
$$- \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u'_\tau + D_2 f(u_\tau, w_\tau)v_\tau\Big]d\tau ds.$$

Furthermore for $t \geq 0$, we know that

$$u'(t) = AS(t)\varphi(0) + C(t)(\varphi'(0) - g(0,\varphi)) + g(t,u_t) + \int_0^t AS(t-s)g(s,u_s)ds + \int_0^t C(t-s)f(u_s, u'_s)ds,$$

then for $t \in [0,b]$, we have

$$u'(t) - w(t) = \int_0^t C(t-s)[f(u_s, u'_s) - f(u_s, w_s)]ds + \int_0^t \int_0^s C(s-\tau)\Big[(D_1 f(u_\tau, u'_\tau) - D_1 f(u_\tau, u'_\tau))u'_\tau$$
$$+(D_2 f(u_\tau, u'_\tau) - D_2 f(u_\tau, w_\tau))v_\tau d\tau\Big]ds.$$

$$|u'(t) - w(t)|_\alpha$$
$$\leq \int_0^t |C(t-s)[f(u_s, u'_s) - f(s, u_s, w_s)]|_\alpha ds + \int_0^t \int_0^s |C(s-\tau)(D_1 f(u_\tau, u'_\tau) - D_1 f(u_\tau, w_\tau))u'_\tau|_\alpha d\tau ds$$
$$+ \int_0^t \int_0^s |C(s-\tau)(D_2 f(u_\tau, u'_\tau) - D_2 f(u_\tau, w_\tau))v_\tau|_\alpha d\tau ds. \tag{4.2}$$

Let us choose $F = \Big\{u'_s, w_s : s \in [0,b]\Big\}$. Then F is compact set. It follows that $D_1 f$ and $D_2 f$ are globally Lipschitz on F. Let $L_1 > 0$ be such that for $t \in [0,b]$ and $x, y, x', y' \in H$, then we have

$$\|f(x,x')-f(x,y')\| \leq L_1\|x'-y'\|_\alpha$$
$$\|D_1 f(x,x')-D_1 f(x,y')\| \leq L_1\|x'-y'\|_\alpha$$
$$\|D_2 f(x,x')-D_2 f(x,y')\| \leq L_1\|x'-y'\|_\alpha.$$

Consequently, using equation (4.2), we one can find a positive Constance $k(b)$ such that by Gronwall's lemma,

$$\|u(t)-w(t)\|_\alpha \leq k(b)\int_0^t \|u'_s-w_s\|_\alpha ds,$$

then we deduce that $u'=w$. Consequently, we deduce that the mild solution is twice continuous differentiable from $[0,b]$ to X_α. Then functions $t\to g(t,u_t)$ and $t\to f(t,u_t,u'_t)$ are continuously differentiable on $[0,b]$. According to the Theorem 2.5, we conclude that u is a strict solution of equation (1.1) on $[0,b]$. ■

5. Application

For our illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \frac{\partial}{\partial}[z'(t,x)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta] = \frac{\partial^2}{\partial x^2}z(t,x)] \\ +\int_{-r}^0 h(t,\frac{\partial}{\partial x}z(t+\theta,x),\frac{\partial}{\partial x}z'(t+\theta,x))d\theta \text{ for } t\geq 0 \text{ and } x\in[0,\pi] \\ z(t,0)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta = 0 \text{ for } t\geq 0 \\ z(t,\pi)-\int_{-r}^0 k(t,z(t+\theta,x))d\theta = 0 \\ z(\theta,x)=\varphi_0(\theta)(x) \text{ for } \theta\in[-r,0] \text{ and } x\in[0,\pi], \end{cases}$$

where $h:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exists a positive constant L such that for $x,y,x_1,y_1\in\mathbb{R}$,

$$|h(t,x,y)-h(t,x_1,y_2)| \leq L\Big(|x-x_1|+|y-y_1|\Big).$$

we can choose for example

$$h(t,x,y)=e^{-t^2}[\sin(\frac{x}{2})+\sin(\frac{y}{2})] \text{ for } (\theta,x,y)\in\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}.$$

we can observe that

$$|h(t,x_1,y_1)-h(t,x_2,y_2)| \leq \frac{1}{2}\Big(|x_1-x_2|+|y_1-y_2|\Big)$$

and $k:\mathbb{R}^-\times\mathbb{R}\longrightarrow\mathbb{R}$ is Lipschizian with respect to the second argument.
In the order to rewrite equation (5.1) in the abstract form, we introduce the space $X=L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x\in X$,

$$\|x\|_{L^2}=\Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $x \in [0,\pi]$, $n \geq 1$, then $(e_n)_{n\geq 1}$ is an orthogonal base for X.
Let $A : X \to X$ be defined by

$$\begin{cases} Ay = y'' \\ \\ D(A) = \{y \in X : y, y' \text{ are absolutely continuous}, \; y'' \in X, \; y(0) = y(\pi) = 0\} \end{cases}$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} -n^2(y, e_n)e_n, \; y \in D(A),$$

where

$$(u, v) = \int_0^\pi u(s)v(s)ds \;\; \text{for } u, v \in X.$$

It is well known that A is the infinitesimal generator of strongly continuous cosine family $C(t)$, $\in \mathbb{R}$ in X which is given by

$$C(t)y = \sum_{n=1}^{+\infty} \cos nt(y, e_n)e_n, \; y \in X$$

and that the associated sine family is given by

$$S(t)y = \sum_{n=1}^{+\infty} \frac{1}{n}\sin nt(y, e_n)e_n, \; y \in X.$$

If we choose $\alpha = \frac{1}{2}$. then $(\mathbf{H}_0)$ is satisfied since

$$(-A)^{\frac{1}{2}}y = \sum_{n=1}^{+\infty}(y, e_n)e_n, \; y \in D((-A)^{\frac{1}{2}}).$$

and

$$(-A)^{-\frac{1}{2}}y = \sum_{n=1}^{+\infty} \frac{1}{n}(y, e_n)e_n, \; y \in X.$$

From [4], the compactness of A^{-1} follows from Lemma 2.6 and the fact that the eigenvalues of $(-A)^{-\frac{1}{2}}$ are $\lambda_n = \frac{1}{n}$, $\;n = 1, 2\ldots$, the $(\mathbf{H}_3)$ is satisfied.

We define the space

$$\mathcal{C}_{\frac{1}{2}} = C^1([-r, 0], X_{\frac{1}{2}}),$$

where $C^1([-r, 0], X_{\frac{1}{2}})$ is the space of bounded uniformly continuous differentiable from $[-r, 0]$ into $X_{\frac{1}{2}}$, where $X_{\frac{1}{2}}$ is endowed with the norm

$$|\varphi|_{\frac{1}{2}} = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|.$$

Let $f : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}} \times \mathcal{C}_{\frac{1}{2}} \longrightarrow X$ and $g : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}}$ define by

$$f(t, \varphi, \varphi')(x) = \int_{-r}^0 h(t, \frac{\partial}{\partial x}\varphi(\theta)(x), \frac{\partial}{\partial x}\varphi'(\theta)(x))d\theta \text{ for } x \in [0, \pi], t \geq 0, \; \varphi, \in \mathcal{C}_{\frac{1}{2}}$$

and

$$g(t,\varphi,)(x)=\int_{-r}^{0}k(t,\varphi(\theta)(x))d\theta \text{ for } x\in[0,\pi], t\geq 0,\ \varphi,\in\mathcal{C}_{\frac{1}{2}}$$

where $\varphi,\in\mathcal{C}_{\frac{1}{2}}$ define by

$$\varphi(\theta)(x)=\varphi_0(\theta,x)$$

and the norm in $\mathcal{C}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{C}_{\frac{1}{2}}}=\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\big|^2dx\Big)^{\frac{1}{2}}+\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\big|^2dx\Big)^{\frac{1}{2}}.$$

Let us pose $v(t)=z(t,x)$. Then equation (5.1) takes the following abstract form

$$\begin{cases}\dfrac{d}{dt}[v'(t)-g(t,v_t)]=Av(t)+f(t,v_t,v'_t) \text{ for } t\geq 0\\ v_0=\varphi\in\mathcal{C}_{\frac{1}{2}}\\ v'_0=\varphi'\in\mathcal{C}_{\frac{1}{2}}.\end{cases}\tag{5.1}$$

From [4], for all $y\in X_{\frac{1}{2}}$, y is absolutely continuous and $|y|_{\frac{1}{2}}=|y|_{L^2}$ Let $\varphi,\psi\in C^1([-r,0],X_{\frac{1}{2}})$, since $|h(t,x_1,y_1)-h(t,x_2,y_2)|\leq\dfrac{1}{2}\Big(|x_1-x_2|+\|y_1-y_2|\Big)$, we have

$$\begin{aligned}|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}&\leq\Big(\int_0^{\pi}\Big(\int_{-r}^{0}h(t,\frac{\partial}{\partial x}[\varphi(\theta)(x)],\frac{\partial}{\partial x}[\varphi'(\theta)(x)]d\theta\Big)\\&\quad+\Big(\int_0^{\pi}\Big(\int_{-r}^{0}h(t,\frac{\partial}{\partial x}[\psi(\theta)(x)],\frac{\partial}{\partial x}[\psi'(\theta)(x)])d\theta\Big)^2dx\Big)^{\frac{1}{2}}\\&\leq\frac{1}{2}r\Big[\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big].\end{aligned}$$

By Minkowski's Lemma, we have

$$\begin{aligned}|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}&\leq\frac{1}{2}r\Big[\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big]\\&\leq\frac{1}{2}r\Big[\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\\&\quad+\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big],\end{aligned}$$

which implies that

$$|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}\leq\frac{1}{2}r\|\varphi-\psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

Consequently the function f satisfies $(\mathbf{H}_4)$.

$(\mathbf{H}_7)$ $0 < rL_k < 1$.
We claim that g is a contraction function with respect to the second argument with value in $X_{\frac{1}{2}}$. Indeed let $\varphi_1, \varphi_2 \in \mathcal{C}_{\frac{1}{2}}$ and L_k the constant Lipschitz of k. Then we have

$$|g(t,\varphi) - g(t,\psi)|_{\frac{1}{2}} \leq rL_k \|\varphi - \psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

Then, assumption $(\mathbf{H}_7)$ implies that g is a strict contraction. Moreover the boundedness of $(-A)^{-\frac{1}{2}}$ implies that g stays in $X_{\frac{1}{2}}$. Consequently g satisfies $(\mathbf{H}_2)$.

We have the following result.

Proposition 5.1. *Under the above assumptions, equation* (5.1) *has a unique mild solution which is defined for all* $t \geq 0$.

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Conclusion

In this paper we study the existence and regularity of solutions for some nonlinear neutral functional differential equations with finite delay by use of the cosine family theory. Some results of this study when the delay is infinite will be presented in next works.

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