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An approach to a fuzzy problem using the fuzzy Laplace transform under the generalized differentiability

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Abstract
In this paper, the solutions of a second-order fuzzy initial value problem are studied by the fuzzy Laplace transform under the generalized differentiability. An example is solved. Finally, conclusions are given.

Keywords
Second-order fuzzy differential equation, Fuzzy initial value problem, Fuzzy Laplace transform.

AMS Subject Classification
03E72, 34A07, 44A10

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Contents

1. Introduction

There are many approaches to define fuzzy derivative. The first one is Hukuhara derivative[1–3]. But Hukuhara derivative has a drawback. Solution becomes fuzzier as time goes. Thus, fuzzy solution behaves differently from the crips solution. The second one is generalized Hukuhara derivative [4–8]. The third one is Zadeh's extension principle [9]. Another approach is differential inclusion [10].

Fuzzy Laplace transform is useful to solve fuzzy differential equation. It is found the solution of the fuzzy differential equation satisfying the initial condition by fuzzy Laplace transform directly. Allahviranloo and Barkhordari Ahmadi first introduced fuzzy Laplace transform [11]. In many papers, solution of fuzzy differential equation was studied by fuzzy Laplace transform [12–14].

In this paper, we investigate the solutions of the problem

$$\beta u''(t)+\delta u^{'}(t)=[0]^{\alpha}\,,\ t>0 \tag{1.1}$$

$$u(0)=[Q]^{\alpha}\,,\ \ u^{'}(0)=[W]^{\alpha}\,, \tag{1.2}$$

by the fuzzy Laplace transform under the concept of generalized differentiability, where $\beta,\delta>0$, $[0]^{\alpha}=[-1+\alpha,1-\alpha]$, Q, and W are symmetric triangular fuzzy numbers with supports $\left[\underline{q}_{\alpha},\overline{q}_{\alpha}\right]$, $[\underline{w}_{\alpha},\overline{w}_{\alpha}]$ and the $\alpha-$level sets of Q, W are

$$[Q]^{\alpha}=\left[\underline{Q}_{\alpha},\overline{Q}_{\alpha}\right]=\left[\underline{q}+\left(\frac{\overline{q}-\underline{q}}{2}\right)\alpha,\overline{q}-\left(\frac{\overline{q}-\underline{q}}{2}\right)\alpha\right],$$

$$[W]^{\alpha}=[\underline{W}_{\alpha},\overline{W}_{\alpha}]=\left[\underline{w}+\left(\frac{\overline{w}-\underline{w}}{2}\right)\alpha,\overline{w}-\left(\frac{\overline{w}-\underline{w}}{2}\right)\alpha\right].$$

This paper is organized as in section 2 Preliminaries, in section 3 Main Results, in section 4 Conclusion.

2. Preliminaries

Definition 2.1. *[15] A fuzzy number is a mapping $u:\mathbb{R}\to[0,1]$ satisfying the properties $\overline{\{x\in\mathbb{R}\mid u(x)>0\}}$ is compact, u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$.*

Let $\mathbb{R}_F$ show the set of all fuzzy numbers.

Definition 2.2. *[4] Let be $u\in\mathbb{R}_F$. The α-level set of u is*

$$[u]^{\alpha}=[\underline{u}_{\alpha},\overline{u}_{\alpha}]=\{x\in\mathbb{R}\mid u(x)\geq\alpha\}\,,0<\alpha\leq 1.$$

If $\alpha = 0$,

$$[u]^0 = cl\{suppu\} = cl\{x \in \mathbb{R} \mid u(x) > 0\}.$$

Remark 2.3. *[4] The parametric form* $[\underline{u}_\alpha, \overline{u}_\alpha]$ *of a fuzzy number satisfying the following requirements is a valid* α*-level set.*

$\underline{u}_\alpha$ *is left-continuous monotonic increasing (nondecreasing) bounded on* $(0,1]$,

$\overline{u}_\alpha$ *is left-continuous monotonic decreasing (nonincreasing) bounded on* $(0,1]$,

$\underline{u}_\alpha$ *and* $\overline{u}_\alpha$ *are right-continuous for* $\alpha = 0$,

$\underline{u}_\alpha \leq \overline{u}_\alpha, 0 \leq \alpha \leq 1.$

Definition 2.4. *[15] The* $\alpha-$*level set of symmetric triangular fuzzy number* Q *with support* $[\underline{q}, \overline{q}]$ *is*

$$[Q]^\alpha = \left[\underline{Q}_\alpha, \overline{Q}_\alpha\right] = \left[\underline{q} + \left(\frac{\overline{q}-\underline{q}}{2}\right)\alpha, \overline{q} - \left(\frac{\overline{q}-\underline{q}}{2}\right)\alpha\right].$$

Definition 2.5. *[4] Let be* $u, v \in \mathbb{R}_F$ *and* $\lambda \in \mathbb{R}$. $u+v$ *and* λu *are defined by* $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ *and* $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$. $[u]^\alpha + [v]^\alpha$ *and* $\lambda [u]^\alpha$ *mean the usual addition of two intervals (subsets) of* $\mathbb{R}$ *and the usual product between a scalar and a subset of* $\mathbb{R}$*, respectively.*

Definition 2.6. *[4, 16] Let be* $u, v \in \mathbb{R}_F$. *If* $u = v + w$ *such that there exists* $w \in \mathbb{R}_F$, w *is the Hukuhara difference of* u *and* v*, it is denoted as* $w = u \ominus v$.

Definition 2.7. *[4, 17] Let be* $f : [a,b] \to \mathbb{R}_F$ *and* $x_0 \in [a,b]$. *If there exists* $f'(x_0) \in \mathbb{R}_F$ *such that for all* $h > 0$ *sufficiently small,* $\exists f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ *and the limits hold*

$$\lim_{h \to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

f is Hukuhara differentiable at x_0.

Definition 2.8. *[4] Let be* $f : [a,b] \to \mathbb{R}_F$ *and* $x_0 \in [a,b]$. *If there exists* $f'(x_0) \in \mathbb{R}_F$ *such that for all* $h > 0$ *sufficiently small,* $\exists f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ *and the limits hold*

$$\lim_{h \to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

f is (1)-differentiable at x_0. *If there exists* $f'(x_0) \in \mathbb{R}_F$ *such that for all* $h > 0$ *sufficiently small,* $\exists f(x_0) \ominus f(x_0+h)$, $f(x_0-h) \ominus f(x_0)$ *and the limits hold*

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h \to 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0),$$

f is (2)-differentiable.

Theorem 2.9. *[7]Let* $f : [a,b] \to \mathbb{R}_F$ *be fuzzy function and denote* $[f(x)]^\alpha = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$, *for each* $\alpha \in [0,1]$.

(i) *If the function f is (1)-differentiable, the lower function* $\underline{f}_\alpha$ *and the upper function* $\overline{f}_\alpha$ *are differentiable,* $\left[f'(x)\right]^\alpha = \left[\underline{f}'_\alpha(x), \overline{f}'_\alpha(x)\right]$,

(ii) *If the function f is (2)-differentiable, the lower function* $\underline{f}_\alpha$ *and the upper function* $\overline{f}_\alpha$ *are differentiable,* $\left[f'(x)\right]^\alpha = \left[\overline{f}'_\alpha(x), \underline{f}'_\alpha(x)\right]$.

Theorem 2.10. *[7]Let* $f' : [a,b] \to \mathbb{R}_F$ *be fuzzy function and denote* $[f(x)]^\alpha = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$, *for each* $\alpha \in [0,1]$, *the function f is (1)-differentiable or (2)-differentiable.*

(i) *If f and* f' *are (1)-differentiable,* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable,* $\left[f''(x)\right]^\alpha = \left[\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)\right]$,

(ii) *If f is (1)-differentiable and* f' *is (2)-differentiable,* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable,* $\left[f''(x)\right]^\alpha = \left[\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)\right]$,

(iii) *If f is (2)-differentiable and* f' *is (1)-differentiable,* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable,* $\left[f''(x)\right]^\alpha = \left[\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)\right]$,

(iv) *If f and* f' *are (2)-differentiable,* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable,* $\left[f''(x)\right]^\alpha = \left[\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)\right]$.

Definition 2.11. *[12, 13] Let* $f : [a,b] \to \mathbb{R}_F$ *be fuzzy function. The fuzzy Laplace transform of f is*

$$\begin{aligned} F(s) &= L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt \\ &= \left[\lim_{\tau \to \infty} \int_0^\tau e^{-st} \underline{f}(t)\,dt, \lim_{\tau \to \infty} \int_0^\tau e^{-st} \overline{f}(t)\,dt\right]. \end{aligned}$$

$$F(s,\alpha) = L\left((f(t))^\alpha\right) = \left[L\left(\underline{f}_\alpha(t)\right), L\left(\overline{f}_\alpha(t)\right)\right],$$

$$L\left(\underline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \underline{f}_\alpha(t)\,dt = \lim_{\tau \to \infty} \int_0^\tau e^{-st} \underline{f}_\alpha(t)\,dt,$$

$$L\left(\overline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \overline{f}_\alpha(t)\,dt = \lim_{\tau \to \infty} \int_0^\tau e^{-st} \overline{f}_\alpha(t)\,dt.$$

Theorem 2.12. *[12, 13]Suppose that f is continuous fuzzy-valued function on* $[0,\infty)$ *and exponential order* α *and that* f' *is piecewise continuous fuzzy-valued function on* $[0,\infty)$. *If f is (1)-differentiable,*

$$L\left(f'(t)\right) = sL(f(t)) \ominus f(0),$$

if f is (2)-differentiable,

$$L\left(f'(t)\right) = (-f(0)) \ominus (-sL(f(t))).$$

Theorem 2.13. *[12, 13]Suppose that f and f' are continuous fuzzy-valued functions on $[0,\infty)$ and exponential order α and that f'' is piecewise continuous fuzzy-valued function on $[0,\infty)$.*

If f and f' are (1)-differentiable,

$$L\left(f''(t)\right) = s^2 L(f(t)) \ominus sf(0) \ominus f'(0),$$

if f is (1)-differentiable and f' is (2)-differentiable,

$$L\left(f''(t)\right) = -f'(0) \ominus \left(-s^2\right) L(f(t)) - sf(0),$$

if f is (2)-differentiable and f' is (1)-differentiable,

$$L\left(f''(t)\right) = -sf(0) \ominus \left(-s^2\right) L(f(t)) \ominus f'(0),$$

if f and f' are (2)-differentiable,

$$L\left(f''(t)\right) = s^2 L(f(t)) \ominus sf(0) - f'(0).$$

Theorem 2.14. *[11, 13]Let be $f(t)$, $g(t)$ continuous fuzzy-valued functions and c_1 and c_2 constants, then*

$$L(c_1 f(t) + c_2 g(t)) = (c_1 L(f(t))) + (c_2 L(g(t))).$$

3. Main Results

In this section, we research the solutions of the problem (1.1)-(1.2) by fuzzy Laplace transform under the concept of generalized differentiability. In this paper, (i,j) solution means that u is (i)-differentiable, u' is (j)-differentiable, i=1,2.

i) (1,1) solution of the problem:

If u and u' are (1)-differentiable, since

$$\begin{aligned} L\left([0]^\alpha\right) &= \beta\left(s^2 L(u(t)) \ominus su(0) \ominus u'(0)\right) \\ &\quad + \delta\left(sL(u(t)) \ominus u(0)\right), \end{aligned}$$

we have the equations

$$\begin{aligned} L(-1+\alpha) &= \beta s^2 L(\underline{u}_\alpha(t)) - \beta s\underline{u}_\alpha(0) - \beta \underline{u}'_\alpha(0) \\ &\quad + \delta sL(\underline{u}_\alpha(t)) - \delta\underline{u}_\alpha(0), \end{aligned}$$

$$\begin{aligned} L(1-\alpha) &= \beta s^2 L(\overline{u}_\alpha(t)) - \beta s\overline{u}_\alpha(0) - \beta \overline{u}'_\alpha(0) \\ &\quad + \delta sL(\overline{u}_\alpha(t)) - \delta\overline{u}_\alpha(0). \end{aligned}$$

Using the initial values (1.2) , we get

$$L(\underline{u}_\alpha(t))\left(\beta s^2 + \delta s\right) = \frac{-1+\alpha}{s} + \beta\underline{W}_\alpha + (\beta s + \delta)\underline{Q}_\alpha,$$

$$L(\overline{u}_\alpha(t))\left(\beta s^2 + \delta s\right) = \frac{1-\alpha}{s} + \beta\overline{W}_\alpha + (\beta s + \delta)\overline{Q}_\alpha.$$

From this, we obtain

$$L(\underline{u}_\alpha(t)) = \frac{-1+\alpha}{\beta s^3 + \delta s^2} + \frac{\beta\underline{W}_\alpha}{\beta s^2 + \delta s} + \frac{\underline{Q}_\alpha}{s},$$

$$L(\overline{u}_\alpha(t)) = \frac{1-\alpha}{\beta s^3 + \delta s^2} + \frac{\beta\overline{W}_\alpha}{\beta s^2 + \delta s} + \frac{\overline{Q}_\alpha}{s}.$$

Now, taking the inverse Laplace transform of the above equations, (1,1) solution is obtained as

$$\begin{aligned} \underline{u}_\alpha(t) &= \left(\frac{-1+\alpha}{\delta}\right)\left(t + \frac{\beta}{\delta}\left(e^{-\frac{\delta}{\beta}t} - 1\right)\right) \\ &\quad + \frac{\beta\underline{W}_\alpha}{\delta}\left(1 - e^{-\frac{\delta}{\beta}t}\right) + \underline{Q}_\alpha, \end{aligned}$$

$$\begin{aligned} \overline{u}_\alpha(t) &= \left(\frac{1-\alpha}{\delta}\right)\left(t + \frac{\beta}{\delta}\left(e^{-\frac{\delta}{\beta}t} - 1\right)\right) \\ &\quad + \frac{\beta\overline{W}_\alpha}{\delta}\left(1 - e^{-\frac{\delta}{\beta}t}\right) + \overline{Q}_\alpha, \end{aligned}$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)].$$

ii) (1,2) solution of the problem:

If u is (1)-differentiable and u' is (2)-differentiable, we have the equations

$$\begin{aligned} L\left([0]^\alpha\right) &= \beta\left(-u'(0) \ominus \left(-s^2\right) L(u(t)) - su(0)\right) \\ &\quad + \delta\left(sL(u(t)) \ominus u(0)\right), \end{aligned}$$

$$\begin{aligned} L(-1+\alpha) &= -\beta\overline{u}'_\alpha(0) + \beta s^2 L(\overline{u}_\alpha(t)) - \beta s\overline{u}_\alpha(0) \\ &\quad + \delta sL(\underline{u}_\alpha(t)) - \delta\underline{u}_\alpha(0), \end{aligned}$$

$$\begin{aligned} L(1-\alpha) &= -\beta\underline{u}'_\alpha(0) + \beta s^2 L(\underline{u}_\alpha(t)) - \beta s\underline{u}_\alpha(0) \\ &\quad + \delta sL(\overline{u}_\alpha(t)) - \delta\overline{u}_\alpha(0). \end{aligned}$$

From this, we obtain the equations

$$\beta s^2 L(\overline{u}_\alpha(t)) + \delta sL(\underline{u}_\alpha(t)) = \frac{-1+\alpha}{s} + \beta s\overline{Q}_\alpha + \beta\overline{W}_\alpha + \delta\underline{Q}_\alpha \tag{3.1}$$

$$\beta s^2 L(\underline{u}_\alpha(t)) + \delta sL(\overline{u}_\alpha(t)) = \frac{1-\alpha}{s} + \beta s\underline{Q}_\alpha + \beta\underline{W}_\alpha + \delta\overline{Q}_\alpha. \tag{3.2}$$

If $L(\overline{u}_\alpha(t))$ in the equation (3.1) is replaced by the equation (3.2), we have

$$\begin{aligned} L(\underline{u}_\alpha(t)) &= (1-\alpha)\left(\frac{\beta s + \delta}{s^2\left(\beta^2 s^2 - \delta^2\right)}\right) \\ &\quad - \frac{\beta\delta\overline{W}_\alpha}{s\left(\beta^2 s^2 - \delta^2\right)} + \frac{\underline{Q}_\alpha}{s} + \frac{\beta^2\underline{W}_\alpha}{\beta^2 s^2 - \delta^2}. \end{aligned} \tag{3.3}$$

Taking inverse Laplace transform of the equation (3.3), the lower solution is obtained as

$$\begin{aligned}\underline{u}_\alpha(t) &= (1-\alpha)\left(\frac{\beta}{\delta^2}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\right.\\&\left.+\frac{1}{\delta}\left(\frac{\beta\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)}{2\delta}-t\right)\right)\\&-\frac{\beta\overline{W}_\alpha}{\delta}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\\&+\frac{\beta\underline{W}_\alpha}{2\delta}\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)+\underline{Q}_\alpha.\end{aligned}$$

Similarly, the upper solution is obtained as

$$\begin{aligned}\overline{u}_\alpha(t) &= (-1+\alpha)\left(\frac{\beta}{\delta^2}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\right.\\&\left.+\frac{1}{\delta}\left(\frac{\beta\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)}{2\delta}-t\right)\right)\\&-\frac{\beta\underline{W}_\alpha}{\delta}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\\&+\frac{\beta\overline{W}_\alpha}{2\delta}\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)+\overline{Q}_\alpha.\end{aligned}$$

iii) (2,1) solution of the problem:
If u is (2)-differentiable and u' is (1)-differentiable, since

$$\begin{aligned}L\left([0]^\alpha\right) &= \beta\left(-su(0)\ominus\left(-s^2\right)L(u(t))\ominus u'(0)\right)\\&+\delta\left(-u(0)\ominus(-sL(u(t)))\right),\end{aligned}$$

we have the equations

$$\begin{aligned}L(-1+\alpha) &= -\beta s\overline{u}_\alpha(0)+\beta s^2L(\overline{u}_\alpha(t))-\beta\overline{u}'_\alpha(0)\\&-\delta\overline{u}_\alpha(0)+\delta sL(\overline{u}_\alpha(t)),\end{aligned}$$

$$\begin{aligned}L(1-\alpha) &= -\beta s\underline{u}_\alpha(0)+\beta s^2L(\underline{u}_\alpha(t))-\beta\underline{u}'_\alpha(0)\\&-\delta\underline{u}_\alpha(0)+\delta sL(\underline{u}_\alpha(t)).\end{aligned}$$

That is,

$$L(\underline{u}_\alpha(t))=\frac{1-\alpha}{\beta s^3+\delta s^2}+\frac{\beta\overline{W}_\alpha}{\beta s^2+\delta s}+\frac{(\beta s+\delta)\underline{Q}_\alpha}{\beta s^2+\delta s},$$

$$L(\overline{u}_\alpha(t))=\frac{-1+\alpha}{\beta s^3+\delta s^2}+\frac{\beta\underline{W}_\alpha}{\beta s^2+\delta s}+\frac{(\beta s+\delta)\overline{Q}_\alpha}{\beta s^2+\delta s}.$$

From this, (2,1) solution is obtained as

$$\begin{aligned}\underline{u}_\alpha(t) &= \left(\frac{1-\alpha}{\delta}\right)\left(t+\frac{\beta}{\delta}\left(e^{-\frac{\delta}{\beta}t}-1\right)\right)\\&+\frac{\beta\overline{W}_\alpha}{\delta}\left(1-e^{-\frac{\delta}{\beta}t}\right)+\underline{Q}_\alpha,\end{aligned}$$

$$\begin{aligned}\overline{u}_\alpha(t) &= \left(\frac{-1+\alpha}{\delta}\right)\left(t+\frac{\beta}{\delta}\left(e^{-\frac{\delta}{\beta}t}-1\right)\right)\\&+\frac{\beta\underline{W}_\alpha}{\delta}\left(1-e^{-\frac{\delta}{\beta}t}\right)+\overline{Q}_\alpha,\end{aligned}$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)].$$

iv) (2,2) solution of the problem:
If u and u' are (2)-differentiable, since

$$\begin{aligned}L\left([0]^\alpha\right) &= \beta\left(s^2L(u(t))\ominus su(0)-u'(0)\right)\\&+\delta\left(-u(0)\ominus(-sL(u(t)))\right),\end{aligned}$$

the equations

$$\begin{aligned}L(-1+\alpha) &= \beta s^2L(\underline{u}_\alpha(t))-\beta s\underline{u}_\alpha(0)-\beta\underline{u}'_\alpha(0)\\&-\delta\overline{u}_\alpha(0)+\delta sL(\overline{u}_\alpha(t)),\end{aligned}$$

$$\begin{aligned}L(1-\alpha) &= \beta s^2L(\overline{u}_\alpha(t))-\beta s\overline{u}_\alpha(0)-\beta\overline{u}'_\alpha(0)\\&-\delta\underline{u}_\alpha(0)+\delta sL(\underline{u}_\alpha(t))\end{aligned}$$

are obtained. These yield

$$\beta s^2L(\underline{u}_\alpha(t))+\delta sL(\overline{u}_\alpha(t))=\frac{-1+\alpha}{s}+\beta s\underline{Q}_\alpha+\beta\overline{W}_\alpha+\delta\overline{Q}_\alpha, \tag{3.4}$$

$$\beta s^2L(\overline{u}_\alpha(t))+\delta sL(\underline{u}_\alpha(t))=\frac{1-\alpha}{s}+\beta s\overline{Q}_\alpha+\beta\underline{W}_\alpha+\delta\underline{Q}_\alpha. \tag{3.5}$$

If $L(\overline{u}_\alpha(t))$ in the equation (3.5) is replaced by the equation (3.4), we have

$$\begin{aligned}L(\underline{u}_\alpha(t)) &= (-1+\alpha)\left(\frac{\beta s+\delta}{s^2(\beta^2s^2-\delta^2)}\right) \qquad (3.6)\\&-\frac{\beta\delta\underline{W}_\alpha}{s(\beta^2s^2-\delta^2)}+\frac{\underline{Q}_\alpha}{s}+\frac{\beta^2\overline{W}_\alpha}{\beta^2s^2-\delta^2}.\end{aligned}$$

Taking inverse Laplace transform of the equation (3.6), the lower solution is obtained as

$$\begin{aligned}\underline{u}_\alpha(t) &= (-1+\alpha)\left(\frac{\beta}{\delta^2}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\right.\\&\left.+\frac{1}{\delta}\left(\frac{\beta\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)}{2\delta}-t\right)\right)\\&-\frac{\beta\underline{W}_\alpha}{\delta}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\\&+\frac{\beta\overline{W}_\alpha}{2\delta}\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)+\underline{Q}_\alpha.\end{aligned}$$

Similarly, the upper solution is obtained as

$$\begin{aligned}\overline{u}_{\alpha}(t) &= (1-\alpha)\left(\frac{\beta}{\delta^2}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\right.\\ &\quad\left.+\frac{1}{\delta}\left(\frac{\beta\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)}{2\delta}-t\right)\right)\\ &\quad-\frac{\beta\overline{W}_{\alpha}}{\delta}\left(\frac{e^{\frac{\delta}{\beta}t}+e^{-\frac{\delta}{\beta}t}}{2}-1\right)\\ &\quad+\frac{\beta\underline{W}_{\alpha}}{2\delta}\left(e^{\frac{\delta}{\beta}t}-e^{-\frac{\delta}{\beta}t}\right)+\overline{Q}_{\alpha}.\end{aligned}$$

Example 3.1. *Consider the solutions of the problem*

$$u''(t)+u'(t)=[0]^{\alpha},\ t>0,$$

$$u(0)=[0]^{\alpha},\ \ u'(0)=[1]^{\alpha}$$

by fuzzy Laplace transform, where $[0]^{\alpha}=[-1+\alpha,1-\alpha],[1]^{\alpha}=[\alpha,2-\alpha]$.

(1,1) solution is

$$\begin{aligned}\underline{u}_{\alpha}(t) &= (-1+\alpha)\left(t+e^{-t}\right)+\alpha\left(1-e^{-t}\right)\\ &= \alpha(t+1)-t-e^{-t},\end{aligned}$$

$$\begin{aligned}\overline{u}_{\alpha}(t) &= (1-\alpha)\left(t+e^{-t}\right)+(2-\alpha)\left(1-e^{-t}\right)\\ &= t+2-e^{-t}-\alpha(t+1),\end{aligned}$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)].$$

(1,2) solution is

$$\begin{aligned}\underline{u}_{\alpha}(t) &= (1-\alpha)\left(e^{t}-t-2\right)-(2-\alpha)\left(\frac{e^{t}+e^{-t}}{2}-1\right)\\ &\quad+\alpha\left(\frac{e^{t}-e^{-t}}{2}\right)\\ &= \alpha(t+1)-t-e^{-t},\end{aligned}$$

$$\begin{aligned}\overline{u}_{\alpha}(t) &= (-1+\alpha)\left(e^{t}-t-2\right)-\alpha\left(\frac{e^{t}+e^{-t}}{2}-1\right)\\ &\quad+(2-\alpha)\left(\frac{e^{t}-e^{-t}}{2}\right)\\ &= t+2-e^{-t}-\alpha(t+1),\end{aligned}$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)].$$

If

$$\begin{aligned}\frac{\partial\underline{u}_{\alpha}(t)}{\partial\alpha} &\geq 0,\ \frac{\partial\overline{u}_{\alpha}(t)}{\partial\alpha}\leq 0,\ \underline{u}_{\alpha}(t)\leq\overline{u}_{\alpha}(t),\\ \underline{u}'_{\alpha}(t) &\leq \overline{u}'_{\alpha}(t),\ \underline{u}''_{\alpha}(t)\leq\overline{u}''_{\alpha}(t),\end{aligned}$$

(1,1) solution is a valid fuzzy function. If

$$\begin{aligned}\frac{\partial\underline{u}_{\alpha}(t)}{\partial\alpha} &\geq 0,\ \frac{\partial\overline{u}_{\alpha}(t)}{\partial\alpha}\leq 0,\ \underline{u}_{\alpha}(t)\leq\overline{u}_{\alpha}(t),\\ \underline{u}'_{\alpha}(t) &\leq \overline{u}'_{\alpha}(t),\ \overline{u}''_{\alpha}(t)\leq\underline{u}''_{\alpha}(t),\end{aligned}$$

(1,2) solution is a valid fuzzy function. For (1,1) solution, since

$$\frac{\partial\underline{u}_{\alpha}(t)}{\partial\alpha}=t+1>0,\ \frac{\partial\overline{u}_{\alpha}(t)}{\partial\alpha}=-t-1<0,$$

$$\overline{u}_{\alpha}(t)-\underline{u}_{\alpha}(t)=2(1-\alpha)(t+1)\geq 0,$$

$$\overline{u}'_{\alpha}(t)-\underline{u}'_{\alpha}(t)=2(1-\alpha)\geq 0,\ \overline{u}''_{\alpha}(t)-\underline{u}''_{\alpha}(t)=0,$$

(1,1) solution is a valid fuzzy function. Similarly, (1,2) solution is a valid fuzzy function. Also, for (1,1) and (1,2) solutions, since

$$\underline{u}_1(t)=1-e^{-t}=\overline{u}_1(t),$$

$$\underline{u}_1(t)-\underline{u}_{\alpha}(t)=(1-\alpha)(t+1)=\overline{u}_{\alpha}(t)-\overline{u}_1(t),$$

(1,1) and (1,2) solutions are symmetric triangular fuzzy numbers for any $t>0$ time. (2,1) solution is

$$\begin{aligned}\underline{u}_{\alpha}(t) &= (1-\alpha)\left(t+e^{-t}-2\right)+(2-\alpha)\left(1-e^{-t}\right)\\ &= t-e^{-t}+\alpha(1-t),\end{aligned}$$

$$\begin{aligned}\overline{u}_{\alpha}(t) &= (-1+\alpha)\left(t+e^{-t}-2\right)+\alpha\left(1-e^{-t}\right)\\ &= 2-t-e^{-t}+\alpha(t-1),\end{aligned}$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)].$$

(2,2) solution is

$$\begin{aligned}\underline{u}_{\alpha}(t) &= (-1+\alpha)\left(e^{t}-t\right)-\alpha\left(\frac{e^{t}+e^{-t}}{2}-1\right)\\ &\quad+(2-\alpha)\left(\frac{e^{t}-e^{-t}}{2}\right)\\ &= t-e^{-t}+\alpha(1-t),\end{aligned}$$

$$\begin{aligned}\overline{u}_{\alpha}(t) &= (1-\alpha)\left(e^{t}-t\right)-(2-\alpha)\left(\frac{e^{t}+e^{-t}}{2}-1\right)\\ &\quad+\alpha\left(\frac{e^{t}-e^{-t}}{2}\right)\\ &= 2-t-e^{-t}+\alpha(t-1),\end{aligned}$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)].$$

If

$$\begin{aligned}\frac{\partial\underline{u}_{\alpha}(t)}{\partial\alpha} &\geq 0,\ \frac{\partial\overline{u}_{\alpha}(t)}{\partial\alpha}\leq 0,\ \underline{u}_{\alpha}(t)\leq\overline{u}_{\alpha}(t),\\ \overline{u}'_{\alpha}(t) &\leq \underline{u}'_{\alpha}(t),\ \overline{u}''_{\alpha}(t)\leq\underline{u}''_{\alpha}(t),\end{aligned}$$

(2,1) solution is a valid fuzzy function. If

$$\begin{aligned} \frac{\partial \underline{u}_\alpha(t)}{\partial \alpha} &\geq 0, \ \frac{\partial \overline{u}_\alpha(t)}{\partial \alpha} \leq 0, \ \underline{u}_\alpha(t) \leq \overline{u}_\alpha(t), \\ \overline{u}'_\alpha(t) &\leq \underline{u}'_\alpha(t), \ \underline{u}''_\alpha(t) \leq \overline{u}''_\alpha(t), \end{aligned}$$

(2,2) solution is a valid fuzzy function. For (2,1) solution, since

$$\frac{\partial \underline{u}_\alpha(t)}{\partial \alpha} = 1 - t, \ \frac{\partial \overline{u}_\alpha(t)}{\partial \alpha} = t - 1,$$

$$\overline{u}_\alpha(t) - \underline{u}_\alpha(t) = 2(1-\alpha)(1-t),$$

if $t \leq 1$, we have $\frac{\partial \underline{u}_\alpha(t)}{\partial \alpha} \geq 0$, $\frac{\partial \overline{u}_\alpha(t)}{\partial \alpha} \leq 0$, $\underline{u}_\alpha(t) \leq \overline{u}_\alpha(t)$. Also, since

$$\underline{u}'_\alpha(t) - \overline{u}'_\alpha(t) = 2(1-\alpha) \geq 0, \ \underline{u}''_\alpha(t) - \overline{u}''_\alpha(t) = 0,$$

(2,1) solution is a valid fuzzy function for $t \leq 1$. Similarly, (2,2) solution is a valid fuzzy function for $t \leq 1$. In addition, for (2,1) and (2,2) solutions, since

$$\underline{u}_1(t) = 1 - e^{-t} = \overline{u}_1(t),$$

$$\underline{u}_1(t) - \underline{u}_\alpha(t) = (\alpha - 1)(t-1) = \overline{u}_\alpha(t) - \overline{u}_1(t),$$

(2,1) and (2,2) solutions are symmetric triangular fuzzy number for any $t > 0$ time.

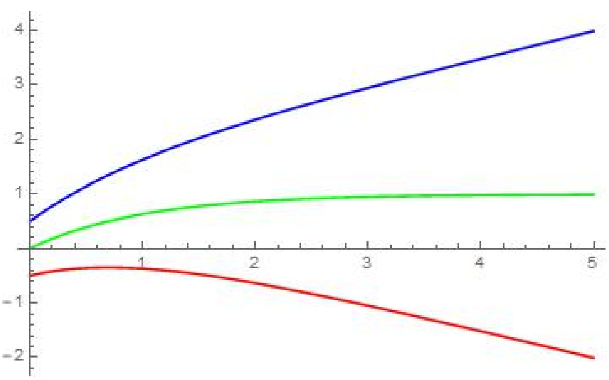

Figure 1. Graphic of (1,1) and (1,2) solutions for $\alpha = 0.5$

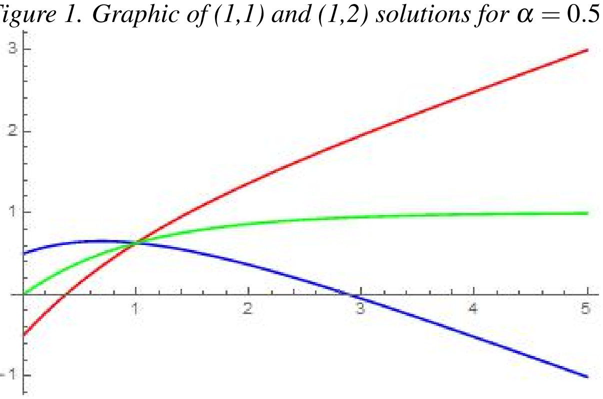

Figure 2. Graphic of (2,1) and (2,2) solutions for $\alpha = 0.5$

Blue $\rightarrow \overline{y}_\alpha(t)$
Red $\rightarrow \underline{y}_\alpha(t)$
Green $\rightarrow \overline{y}_1(t) = \underline{y}_1(t)$

4. Conclusion

In this paper, we study the solutions of a second-order fuzzy initial value problem using the fuzzy Laplace transform under the generalized differentiability. We use symmetric triangular fuzzy number, Hukuhara difference, the properties of fuzzy Laplace transform and fuzzy arithmetic. We solve an example related to the problem. We obtain that (1,1) and (1,2) solutions are valid fuzzy functions and (2,1) and (2,2) solutions are valid fuzzy functions for $t \leq 1$. Also, we obtain that all of the solutions are symmetric triangular fuzzy numbers for any $t > 0$ time.

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