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Nano generalized e-closure and nano generalized e-interior

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Abstract
In this paper we discuss some basic topological properties of generalizations of closure, interior, neighborhood, limit points, derived set, frontier, exterior and border of the sets via nano generalized e (resp. $\mathscr{M}$)-open sets in nano topological spaces.

Keywords
Nano generalized e (resp. $\mathscr{M}$)-neighbourhood, nano generalized e (resp. $\mathscr{M}$)-exterior, nano generalized e (resp. $\mathscr{M}$)-frontier.

AMS Subject Classification
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Contents

1. Introduction and Preliminaries

The notion of Nano topology (in short, $\mathfrak{N}t$) was introduced by Lellis Thivagar [6] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it and also defined Nano closed sets, Nano-interior and Nano-closure.

The class of sets namely, θ-open (resp. δ-open) sets are playing more important role in topological spaces, because of their applications in various fields of Mathematics and other real fields. In [3] Caldas et al. studied various kinds of θ-open sets and their properties in topological spaces. Also, in [10, 11] studied various kinds of δ-open sets. Recently, [1, 5, 8] studied various kinds of generalizations of sets in nano topological spaces. By this motivation, we present the concept of nano generalized e-open sets [7] and study their properties in nano topological spaces. The purpose of this paper is to discuss some basic topological properties of the operators namely closure, interior, neighbourhood, limit points, derived set, frontier, exterior and border by using the sets nano generalized e (resp. $\mathscr{M}$) open sets.

Definition 1.1. [6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. Let $\mathrm{P} \subseteq \mathrm{U}$. Then,

(i) The lower approximation of P with respect to R is the set of all objects, which can be for certain classified as P with respect to R and it is denoted by $\mathscr{L}_{\mathrm{R}}(\mathrm{P})$. That is, $\mathscr{L}_{\mathrm{R}}(\mathrm{P}) = \bigcup_{l \in U} \{\mathrm{R}(l) : \mathrm{R}(l) \subseteq \mathrm{P}\}$, where $\mathrm{R}(l)$ denotes the equivalence class determin- ed by l.

(ii) The upper approximation of P with respect to R is the set of all objects, which can be possibly classified as P with respect to R and it is denoted by $\mathscr{U}_{\mathrm{R}}(\mathrm{P})$. That is, $\mathscr{U}_{\mathrm{R}}(\mathrm{P}) = \bigcup_{l \in U} \{\mathrm{R}(l) : \mathrm{R}(l) \cap \mathrm{P} \neq \phi\}$.

(iii) The boundary region of P with respect to R is the set of all objects, which can be classified neither as P nor as not-P with respect to R and it is denoted by $\mathscr{B}_R(P)$. That is, $\mathscr{B}_R(P) = \mathscr{U}_R(P) - \mathscr{L}_R(P)$.

Proposition 1.2. [6] If (U, R) is an approximation space and $P, Q \subseteq U$, then

(i) $\mathscr{L}_R(P) \subseteq P \subseteq \mathscr{U}_R(P)$.

(ii) $\mathscr{L}_R(\phi) = \mathscr{U}_R(\phi) = \phi$ and $\mathscr{L}_R(U) = \mathscr{U}_R(U) = U$.

(iii) $\mathscr{U}_R(P \cup Q) = \mathscr{U}_R(P) \cup \mathscr{U}_R(Q)$.

(iv) $\mathscr{U}_R(P \cap Q) \subseteq \mathscr{U}_R(P) \cap \mathscr{U}_R(Q)$.

(v) $\mathscr{L}_R(P \cup Q) \supseteq \mathscr{L}_R(P) \cup \mathscr{L}_R(Q)$.

(vi) $\mathscr{L}_R(P \cap Q) = \mathscr{L}_R(P) \cap \mathscr{L}_R(Q)$.

(vii) $\mathscr{L}_R(P) \subseteq \mathscr{L}_R(Q)$ and $\mathscr{U}_R(P) \subseteq \mathscr{U}_R(Q)$, whenever $P \subseteq Q$.

(viii) $\mathscr{U}_R(P^c) = [\mathscr{L}_R(P)^c]$ and $\mathscr{L}_R(P^c) = [\mathscr{U}_R(P)]^c$.

(ix) $\mathscr{U}_R\mathscr{U}_R(P) = \mathscr{L}_R\mathscr{U}_R(P) = \mathscr{U}_R(P)$.

(x) $\mathscr{L}_R\mathscr{L}_R(P) = \mathscr{U}_R\mathscr{L}_R(P) = \mathscr{L}_R(P)$.

Definition 1.3. [6] Let U be an universe, R be an equivalence relation on U and $\tau_R(P) = \{U, \phi, \mathscr{L}_R(P), \mathscr{U}_R(P), \mathscr{B}_R(P)\}$ where $P \subseteq U$, $\tau_R(P)$ satisfies the following axioms:

(i) U and $\phi \in \tau_R(P)$.

(ii) The union of the elements of arbitrary sub collection of $\tau_R(P)$ is in $\tau_R(P)$.

(iii) The intersection of the elements of any finite sub-collection of $\tau_R(P)$ is in $\tau_R(P)$.

That is, $\tau_R(P)$ is a topology on U called the nano topology on U with respect to R and P. We call $(U, \tau_R(P))$ as the nano topological space (briefly, $\mathfrak{N}ts$). The elements of $\tau_R(P)$ are called as nano-open (briefly, $\mathfrak{N}o$) sets and $[\tau_R(P)]^c$ is called as the dual nano topology of $\tau_R(P)$. The elements of $[\tau_R(P)]^c$ are called as nano-closed (briefly, $\mathfrak{N}c$).

Remark 1.4. [6] If $\tau_R(P)$ is the $\mathfrak{N}t$ on U with respect to P, then the set $\mathbb{B} = \{U, \mathscr{L}_R(P), \mathscr{B}_R(P)\}$ is the basis for $\tau_R(P)$.

Definition 1.5. [6] If $(U, \tau_R(P))$ is a $\mathfrak{N}ts$ with respect to P and if $K \subseteq U$, then the nano interior of K is defined as the union of all $\mathfrak{N}o$ subsets of K and it is denoted by $int(K)$. That is, $\mathfrak{N}int(K)$ is the largest $\mathfrak{N}o$ subset of K.

The nano closure of K is defined as the intersection of all $\mathfrak{N}c$ sets containing K and it is denoted by $\mathfrak{N}cl(K)$. That is, $\mathfrak{N}cl(K)$ is the smallest $\mathfrak{N}c$ set containing K.

Definition 1.6. [4] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and let $K \subseteq U$ then the nano θ-interior (resp. nano θ-closure) of K is defined and denoted by $\mathfrak{N}int_\theta(K) = \bigcup\{L : L$ is a nano open set and $\mathfrak{N}cl(L) \subseteq K\}$ (resp. $\mathfrak{N}cl_\theta(K) = \bigcap\{L : L$ is a nano closed set and $\mathfrak{N}int(L) \supseteq K\}$).

Definition 1.7. [4] K subset K of P is said to be nano θ-open (resp. nano θ-closed) (briefly, $\mathfrak{N}\theta o$ (resp. $\mathfrak{N}\theta c$)) set if $K = \mathfrak{N}int_\theta(K)$ (resp. K^c is a nano θ-open set).

Definition 1.8. [6, 12] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and $K \subseteq U$. Then K is said to be nano regular open (briefly, $\mathfrak{N}ro$) if $K = \mathfrak{N}int(\mathfrak{N}cl(K))$.

Definition 1.9. [9] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and let $K \subseteq U$ then the nano δ-interior (resp. nano δ-closure) of K is defined and denoted by $\mathfrak{N}int_\delta(K) = \bigcup\{L : L$ is a $\mathfrak{N}ro$ set and $L \subseteq K\}$ (resp. $\mathfrak{N}cl_\delta(K) = \bigcup\{l \in U : \mathfrak{N}int(\mathfrak{N}cl(L)) \cap K \neq \phi$, L is a $\mathfrak{N}o$ set and $l \in L\}$).

Definition 1.10. [9] A subset K of X is said to be nano δ-open (resp. nano δ-closed) (briefly, $\mathfrak{N}\delta o$ (resp. $\mathfrak{N}\delta c$)) set if $K = \mathfrak{N}int_\delta(K)$ (resp. K^c is a nano δ-open set).

Definition 1.11. [8, 9, 13]Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and $K \subseteq U$. Then K is said to be a nano δ-pre (resp. nano δ-semi, nano e, nano $\mathscr{M}$ and nano θ-semi) open set (briefly $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$, $\mathfrak{N}\mathscr{M}o$ and $\mathfrak{N}\theta\mathscr{S}o$)) if $K \subseteq \mathfrak{N}int(\mathfrak{N}cl_\delta(K))$ (resp. $K \subseteq \mathfrak{N}cl(\mathfrak{N}int_\delta(K))$, $K \subseteq \mathfrak{N}cl(\mathfrak{N}int_\delta(K)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(K))$, $K \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(K)) \cup \mathfrak{N}int(\mathfrak{N}cl_\delta(K))$ and $K \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(K))$).

The complements of the above respective open sets are their respective closed sets.

The family of all $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$, $\mathfrak{N}\mathscr{M}o$ and $\mathfrak{N}\theta\mathscr{S}o$) sets is denoted by $\mathfrak{N}\delta\mathscr{P}O(U, \tau_R(P))$, (resp. $\mathfrak{N}\delta\mathscr{S}O(U, \tau_R(P))$, $\mathfrak{N}eO(U, \tau_R(P))$, $\mathfrak{N}\mathscr{M}O(U, \tau_R(P))$ and $\mathfrak{N}\theta\mathscr{S}O(U, \tau_R(P))$) and the family of all nano δ-pre (resp. nano δ-semi, nano e, nano $\mathscr{M}$ and nano θ-semi) closed (briefly, $\mathfrak{N}\delta\mathscr{P}c$ (resp. $\mathfrak{N}\delta\mathscr{S}c$, $\mathfrak{N}ec$, $\mathfrak{N}\mathscr{M}c$ and $\mathfrak{N}\theta\mathscr{S}c$)) sets is denoted by $\mathfrak{N}\delta\mathscr{P}C(U, \tau_R(P))$, (resp. $\mathfrak{N}\delta\mathscr{S}C(U, \tau_R(P))$, $\mathfrak{N}eC(U, \tau_R(P))$, $\mathfrak{N}\mathscr{M}C(U, \tau_R(P))$ and $\mathfrak{N}\theta\mathscr{S}C(U, \tau_R(P))$).

Definition 1.12. [8, 9, 13] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and let $K \subseteq U$ then the nano δ-pre (resp. nano δ-semi, nano e, nano $\mathscr{M}$ and nano θ-semi) interior of K is the union of all $\mathfrak{N}\delta\mathscr{P}o$ (resp. $\mathfrak{N}\delta\mathscr{S}o$, $\mathfrak{N}eo$, $\mathfrak{N}\mathscr{M}o$ and $\mathfrak{N}\theta\mathscr{S}o$) sets contained in K and denoted by $\mathfrak{N}\mathscr{P}int_\delta(K)$ (resp. $\mathfrak{N}\mathscr{S}int_\delta(K)$, $\mathfrak{N}eint(K)$, $\mathfrak{N}\mathscr{M}int(K)$ and $\mathfrak{N}\mathscr{S}int_\theta(K)$).

Definition 1.13. [8, 9, 13] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and let $K \subseteq U$ then the nano δ-pre (resp. nano δ-semi, nano e, nano $\mathscr{M}$ and nano θ-semi) closure of K is the intersection of all $\mathfrak{N}\delta\mathscr{P}c$ (resp. $\mathfrak{N}\delta\mathscr{S}c$, $\mathfrak{N}ec$, $\mathfrak{N}\mathscr{M}c$ and $\mathfrak{N}\theta\mathscr{S}c$) sets containing K and denoted by $\mathfrak{N}\mathscr{P}cl_\delta(K)$ (resp. $\mathfrak{N}\mathscr{S}cl_\delta(K)$, $\mathfrak{N}ecl(K)$, $\mathfrak{N}\mathscr{M}cl(K)$ and $\mathfrak{N}\mathscr{S}cl_\theta(K)$).

Definition 1.14. [8] Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and $K \subseteq U$. Then K is said to be a nano θ-pre (resp. nano θ-semi) open set (briefly $\mathfrak{N}\theta\mathscr{P}o$ (resp. $\mathfrak{N}\theta\mathscr{S}o$)) if $K \subseteq \mathfrak{N}int(\mathfrak{N}cl_\theta(K))$ (resp. $K \subseteq \mathfrak{N}cl(\mathfrak{N}int_\theta(K))$).

Definition 1.15. A subset K of $(U, \tau_R(P))$ is called nano generalized

(i) closed [2, 4] (briefly, $\mathfrak{N}gc$) set if $\mathfrak{N}cl(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(ii) θ closed [4], (briefly, $\mathfrak{N}g\theta c$) set if $\mathfrak{N}cl_\theta(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(iii) θ semi closed [5] (briefly, $\mathfrak{N}g\theta\mathscr{S}c$) set if $\mathfrak{N}\mathscr{S}cl_\theta(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(iv) δ closed [1], (briefly, $\mathfrak{N}g\delta c$) set if $\mathfrak{N}\delta cl(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(v) δ semi closed [1], (briefly, $\mathfrak{N}g\delta\mathscr{S}c$) set if $\mathfrak{N}\mathscr{S}cl_\delta (K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(vi) δ pre closed [1], (briefly, $\mathfrak{N}g\delta\mathscr{P}c$) set if $\mathfrak{N}\mathscr{P}cl_\delta(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

Definition 1.16. [1, 2, 4, 5] A subset K of a nano generalized (resp. θ, θ semi, δ , δ semi and δ pre) open (briefly, $\mathfrak{N}go$(resp. $\mathfrak{N}g\theta o$, $\mathfrak{N}g\theta\mathscr{S}o$, $\mathfrak{N}g\delta o$, $\mathfrak{N}g\delta\mathscr{S}o$ and $\mathfrak{N}g\delta\mathscr{P}o$)) if its complement K^C is nano generalized (resp. θ, θ semi, δ, δ semi and δ pre) closed (briefly, $\mathfrak{N}gc$(resp. $\mathfrak{N}g\theta c$, $\mathfrak{N}g\theta\mathscr{S}c$, $\mathfrak{N}g\delta c$, $\mathfrak{N}g\delta\mathscr{S}c$ and $\mathfrak{N}g\delta\mathscr{P}c$)).

The collection of all nano generalized (resp. θ, θ semi, δ , δ semi and δ pre) open subsets of $(U, \tau_R(P))$ is denoted by $\mathfrak{N}GO(U,P)$(resp. $\mathfrak{N}G\theta O(U,P)$, $\mathfrak{N}G\theta\mathscr{S}O(U, P)$, $\mathfrak{N}G\delta O(U,P)$, $\mathfrak{N}G\delta\mathscr{S}O(U,P)$ and $\mathfrak{N}G\delta\mathscr{P}O(U,P)$).

Definition 1.17. [7] A subset K of $(U, \tau_R(P))$ is called nano generalized

(i) e closed, (briefly, $\mathfrak{N}gec$) set if $\mathfrak{N}ecl(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

(ii) M closed, (briefly, $\mathfrak{N}g\mathscr{M}c$) set if $\mathfrak{N}\mathscr{M}cl(K) \subseteq V$ whenever $K \subseteq V$ and V is $\mathfrak{N}o$ in $(U, \tau_R(P))$.

The collection of all $\mathfrak{N}gec$ (resp. $\mathfrak{N}g\mathscr{M}c$) subsets of U is denoted by $\mathfrak{N}GeC(U,P)$ (resp. $\mathfrak{N}G\mathscr{M}C(U,P)$).

Definition 1.18. Let K be a subset of a $\mathfrak{N}ts$ $(U, \tau_R(P))$, the intersection of all $\mathfrak{N}o$ subsets of U containing K is called nano kernel[8] of K and is denoted by $\mathfrak{N}ker(K)$.

Throughout this paper, $(\mathrm{U}, \tau_\mathrm{R}(\mathrm{P}))$ is a $\mathfrak{N}ts$ with respect to P where $\mathrm{P} \subseteq \mathrm{U}$, R is an equivalence relation on U. Then U/R denotes the family of equivalence classes of U by R.

2. Nano generalized e neighbourhoods

Definition 2.1. Let K be subset of a $\mathfrak{N}ts$ $(U, \tau_R(P))$ and $l \in U$ then a set K is called nano generalized e (resp. nano generalized $\mathscr{M}$) neighbourhood (briefly, $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$)) of $l \in U$, if there is a $\mathfrak{N}geo$ (resp. $\mathfrak{N}g\mathscr{M}o$) set $G \subset K$ with $l \in G$.

Definition 2.2. Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$ and K be subset of U. A subset S of U is said to be nano generalized e (resp. nano generalized $\mathscr{M}$) neighbourhood (briefly, $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$)) of K, if there exist a $\mathfrak{N}geo$ (resp. $\mathfrak{N}g\mathscr{M}o$) set G such that $K \subset G \subset S$.

The collection of all $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$) of $l \in U$ is called $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$) system of l and is denoted by $\mathfrak{N}GeNbdS(l)$ (resp. $\mathfrak{N}G\mathscr{M}NbdS(l)$).

Example 2.3. Let $\mathrm{U} = \{\mathrm{l,m,n,o}\}$ with $\mathrm{U/R} = \{\{\mathrm{l}\}, \{\mathrm{o}\}, \{\mathrm{m,n}\}\}$ and $\mathrm{P} = \{\mathrm{m,o}\}$. Then, the $\mathfrak{N}t$ is defined as $\tau_\mathrm{R}(\mathrm{P}) = \{U, \phi, \{o\}, \{m,n\}, \{m,n,o\}\}$ and $\mathfrak{N}GeO(U,P) = \{U, \phi, \{m\}, \{n\}, \{o\}, \{l,m\}, \{l,o\}, \{m,n\}, \{m,o\}, \{n,o\}, \{l,m,n\}, \{l,m,o\}, \{l,n,o\}, \{m,n,o\}\} = \mathfrak{N}G\mathscr{M}O(U,P)$. Now, $\mathfrak{N}ge\ NbdS(l) = \{U, \{l,m\}, \{l,o\}, \{l,m,n\}, \{l,m,o\}, \{l,n,o\}\}$, $\mathfrak{N}geNbd\ S(m) = \{U, \{m\}, \{l,m\}, \{m,n\}, \{m,o\}, \{l,m,n\}, \{l,m,o\}, \{m,n,o\}\}$, $\mathfrak{N}geNbdS(n) = \{U, \{n\}, \{m,n\}, \{n,o\}, \{l,m,n\}, \{l, n,o\}, \{m,n,o\}\}$ and $\mathfrak{N}geNbdS(o) = \{U, \{o\}, \{l,o\}, \{m,o\}, \{n,o\}, \{l,m,o\}, \{l,n,o\}, \{m,n,o\}\}$.

Remark 2.4. Every $\mathfrak{N}Nbd$ of l in U is $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$) of l, because every $\mathfrak{N}o$ set is $\mathfrak{N}geo$ (resp. $\mathfrak{N}g\mathscr{M}o$). But converse need not be true as seen from the following example.

Example 2.5. In Example 2.3, For $l \in U$, $\mathfrak{N}NbdS(l) = \{U\}$ and $\mathfrak{N}geNbdS(l) = \{U, \{l,m\}, \{l,o\}, \{l,m,n\}, \{l,m,o\}, \{l,n,o\}\}$. Clearly the sets $\{l,m\}, \{l,o\}, \{l,m,n\}, \{l,m,o\}$ and $\{l,n,o\}$ are $\mathfrak{N}geNbdS(l)$ but not $\mathfrak{N}NbdS(l)$.

Lemma 2.6. An arbitrary union of $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\ \mathscr{M}Nbd$) of a point $l \in U$ is again $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}N\ bd$) of a point $l \in U$, if $\mathfrak{N}GeO(U,P)$ (resp. $\mathfrak{N}G\mathscr{M}O(U,P)$) is closed under arbitrary union.

Proof. Let $\{K_i : i \in I\}$ be an arbitrary collection of $\mathfrak{N}geNbd$ of $l \in U$. Since for each $i \in I$, K_i is $\mathfrak{N}geNbd$ of l, there exist $\mathfrak{N}geo$ set G_i such that $l \in G_i \subseteq K_i$. But for each $i \in I$, $K_i \subseteq \cup K_i$, therefore $l \in G_i \subseteq \cup K_i$, which implies $\cup K_i$ is again $\mathfrak{N}geNbd$ of l.

The other case is similar.

Remark 2.7. But the intersection of $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}Nbd$) of a point $l \in U$ is not a $\mathfrak{N}geNbd$ (resp. $\mathfrak{N}g\mathscr{M}\ Nbd$) of the point $l \in U$ in general.

Example 2.8. In example 2.3, the sets $\{l,m\}$ and $\{l,o\}$ are $\mathfrak{N}geNbdS(l)$ but their intersection $\{l\}$ is not a $\mathfrak{N}geN\ bdS(l)$.

Theorem 2.9. Let $(U, \tau_R(P))$ be a $\mathfrak{N}ts$. Then,

(i) Every $\mathfrak{N}eNbd$ of $l \in U$ is $\mathfrak{N}geNbd$ of $l \in U$.

(ii) Every $\mathfrak{N}\mathscr{M}Nbd$ of $l \in U$ is $\mathfrak{N}g\mathscr{M}Nbd$ of $l \in U$.

(iii) Every $\mathfrak{N}g\delta\mathscr{S}Nbd$ of $l \in U$ is $\mathfrak{N}geNbd$ of $l \in U$.

(iv) Every $\mathfrak{N}g\delta\mathscr{P}Nbd$ of $l \in U$ is $\mathfrak{N}geNbd$ of $l \in U$.

(v) Every $\mathfrak{N}g\theta\mathscr{S}Nbd$ of $l \in U$ is $\mathfrak{N}g\mathscr{M}Nbd$ of $l \in U$.

(vi) Every $\mathfrak{N}g\delta\mathscr{P}Nbd$ of $l \in U$ is $\mathfrak{N}g\mathscr{M}Nbd$ of $l \in U$.

Proof. (v) Let K be an arbitrary $\mathfrak{N}g\theta\mathscr{S}Nbd$ of $l \in U$ then there exists an $\mathfrak{N}g\theta\mathscr{S}o$ set L such that $l \in G \subseteq K$. By [7], every $\mathfrak{N}g\theta\mathscr{S}o$ set is $\mathfrak{N}g\mathscr{M}o$ set, L is $\mathfrak{N}g\mathscr{M}o$ set such that $l \in L \subseteq K$. Thus K is $\mathfrak{N}g\mathscr{M}Nbd$ of l.

The other cases are similar.

Remark 2.10. The converse of the Theorem 2.9 is need not be true. It can be verified by the forthcoming example.

Example 2.11. In the Example 2.3, the set

(i) $\{n,o\}$ is a $\mathfrak{N}geNbd(n)$ (resp. $\mathfrak{N}g\mathscr{M}Nbd(n)$) but not a $\mathfrak{N}eNbd(n)$ (resp. $\mathfrak{N}\mathscr{M}Nbd(n)$).

(ii) $\{l,m\}$ is a $\mathfrak{N}geNbd(l)$ but not a $\mathfrak{N}g\delta\mathscr{S}Nbd(l)$.

(iii) $\{l,o\}$ is a $\mathfrak{N}geNbd(l)$ but not a $\mathfrak{N}g\delta\mathscr{P}Nbd(l)$.

(iv) $\{l,m\}$ is a $\mathfrak{N}g\mathscr{M}Nbd(l)$ but not a $\mathfrak{N}g\theta\mathscr{S}Nbd(l)$.

(v) $\{l,o\}$ is a (resp. $\mathfrak{N}g\mathscr{M}Nbd(l)$) but not a $\mathfrak{N}g\delta\mathscr{P}\,Nbd(l)$.

(vi) $\{l,o\}$ is $\mathfrak{N}gNbd(l)$ but not $\mathfrak{N}g\delta\mathscr{P}Nbd(l)$.

(vii) $\{l,m\}$ is $\mathfrak{N}g\delta\mathscr{P}Nbd(l)$ but not $\mathfrak{N}gNbd(l)$.

Theorem 2.12. Let l be any arbitrary point of a $\mathfrak{N}ts$ $(U,\tau_R(P))$. Then $\mathfrak{N}geNbdS(l)$ satisfies the following properties.

(i) $\mathfrak{N}geNbdS(l) \neq \phi$

(ii) if $F \in \mathfrak{N}geNbdS(l)$, then $l \in F$

(iii) if $F \in \mathfrak{N}geNbdS(l)$ & $F \subseteq H$, then $H \in \mathfrak{N}geNbdS(l)$.

Proof. (i) Since for each $l \in U$, U is a $\mathfrak{N}geo$ set. Therefore, $l \in U \subseteq U$, implies U is $\mathfrak{N}geNbdS(l)$. Hence, $U \in \mathfrak{N}geNbdS(l)$. Therefore $U \in \mathfrak{N}geNbdS(l) \neq \phi$.

(ii) Given $F \in \mathfrak{N}geNbdS(l)$ implies F is $\mathfrak{N}geNbdS(l)$, which implies there exists a $\mathfrak{N}geo$ set G such that $l \in G \subseteq F$. This implies $l \in F$.

(iii) Given $F \in \mathfrak{N}geNbdS(l)$ implies there exists a $\mathfrak{N}geo$ set G such that $l \in G \subseteq F$ and $F \subseteq H$ which implies $l \in G \subseteq F \subseteq H$. This shows that $H \in \mathfrak{N}geNbdS(l)$.

The system $\mathfrak{N}g\mathscr{M}NbdS(l)$ is also satisfy the Theorem 2.12

Theorem 2.13. Let K be a subset of a $\mathfrak{N}ts$ $(U,\tau_R(P))$ and $\mathfrak{N}GeO(U,P)$ is closed under arbitrary union, then K is a $\mathfrak{N}geo$ set iff K is $\mathfrak{N}geNbd$ of each of its points.

Proof. Let K be any $\mathfrak{N}geo$ set of U. Then for each $l \in K \subseteq K$, implies K is $\mathfrak{N}geNbdS(l)$. Since l is arbitrary point of K, implies K is $\mathfrak{N}geNbd$ of each of its points.

Conversely, K is $\mathfrak{N}geNbd$ of each of its points which implies for each $l \in K$, there exists $\mathfrak{N}geo$ set G, such that $l \in G_l \subseteq K$. Suppose if $l \in K$, there exists atleast one $\mathfrak{N}geo$ set G_l such that $l \in G_l \subseteq \bigcup_{l\in K} G_l$. Therefore $K \subseteq \bigcup_{l\in K} G_l \subseteq K$. Thus it follows that $K = \bigcup_{l\in K} G_l$. As each G_l is $\mathfrak{N}geo$ set, K is also $\mathfrak{N}geo$ set.

The system $\mathfrak{N}g\mathscr{M}O(U,P)$ is also satisfy the Theorem 2.13

Theorem 2.14. Let $\mathfrak{N}GeO(U,P)$ is closed under finite intersection, if K is $\mathfrak{N}gec$ subset of U and $l \in U-K$ then there exists a $\mathfrak{N}geNbd$ H of l such that $H \cap K = \phi$.

Proof. Given K is $\mathfrak{N}gec$ set, $U-K$ is $\mathfrak{N}geo$ set. By Theorem 2.13, $U-K$ is $\mathfrak{N}geNbd$ of each of its points. Let $l \in U-K$, implies there exists a $\mathfrak{N}geo$ set H such that $l \in H \subseteq U-K$ which implies $H \cap K = \phi$.

The system $\mathfrak{N}G\mathscr{M}O(U,P)$ is also satisfy the Theorem 2.14

Definition 2.15. A point $l \in U$ is said to be nano generalized e (resp. nano generalized $\mathscr{M}$) limit point of a set K if for each $T \in \mathfrak{N}GeO(U,\tau_R(P))$ (resp. $\mathfrak{N}G\mathscr{M}O(U,\tau_R(P))$) containing l, satisfies $T \cap (K-\{l\}) \neq \phi$.

Definition 2.16. Let $(U,\tau_R(P))$ be $\mathfrak{N}ts$ and $K \subset U$, the set of all nano generalized e (resp. nano generalized $\mathscr{M}$) limit points of K is said to be nano generalized e (resp. nano generalized $\mathscr{M}$) derived set of K and is denoted by $\mathfrak{N}g\mathfrak{D}_e(K)$ (resp. $\mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)$).

Theorem 2.17. Let K and L are the subsets of a $\mathfrak{N}ts$ $(U,\tau_R(P))$. Then the following properties hold.

(i) $\mathfrak{N}g\mathfrak{D}_e(\phi) = \phi$.

(ii) if $K \subseteq L$ then $\mathfrak{N}g\mathfrak{D}_e(K) \subseteq \mathfrak{N}g\mathfrak{D}_e(L)$.

(iii) if $l \in \mathfrak{N}g\mathfrak{D}_e(K)$ then $l \in \mathfrak{N}g\mathfrak{D}_e(K-\{l\})$.

(iv) $\mathfrak{N}g\mathfrak{D}_e(K) \cup \mathfrak{N}g\mathfrak{D}_e(L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K\cup L)$.

(v) $\mathfrak{N}g\mathfrak{D}_e(K\cap L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K) \cap \mathfrak{N}g\mathfrak{D}_e(L)$.

Proof. (i) Let $l \in U$ and G be a $\mathfrak{N}geo$ set containing l. Then $(G-\{l\}) \cap \phi = \phi$. This implies $l \notin \mathfrak{N}g\mathfrak{D}_e(\phi)$. Therefore for any $l \in U$, l is not $\mathfrak{N}ge$ limit point of ϕ. Hence $\mathfrak{N}g\mathfrak{D}_e(\phi) = \phi$.

(ii) Let $l \in \mathfrak{N}g\mathfrak{D}_e(K)$. Then $G \cap (K-\{l\}) \neq \phi$, for every $\mathfrak{N}geo$ set G containing l. Since $K \subseteq L$ implies $G \cap (L-\{l\}) \neq \phi$. This implies $l \in \mathfrak{N}g\mathfrak{D}_e(L)$. Thus $l \in \mathfrak{N}g\mathfrak{D}_e(K)$ implies $l \in \mathfrak{N}g\mathfrak{D}_e(L)$. Therefore $\mathfrak{N}g\mathfrak{D}_e(K) \subseteq \mathfrak{N}g\mathfrak{D}_e(L)$.

(iii) Let $l \in \mathfrak{N}g\mathfrak{D}_e(K)$. Then $G \cap (K-\{l\}) \neq \phi$, for every $\mathfrak{N}geo$ set G containing l. This implies every $\mathfrak{N}geo$ set G containing l, contains atleast one point other than l of $K-\{l\}$. Therefore $l \in \mathfrak{N}g\mathfrak{D}_e(K-\{l\})$.

(iv) Since $K \subseteq K\cup L$ and $L \subseteq K \cup L$, by (ii), $\mathfrak{N}g\mathfrak{D}_e(K) \subseteq \mathfrak{N}g\mathfrak{D}_e(K\cup L)$ and $\mathfrak{N}g\mathfrak{D}_e(L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K\cup L)$. Hence $\mathfrak{N}g\mathfrak{D}_e(K) \cup \mathfrak{N}g\mathfrak{D}_e(L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K\cup L)$.

(v) Since $K\cap L \subseteq K$ and $K \cap L \subseteq L$, by (ii), $\mathfrak{N}g\mathfrak{D}_e(K\cap L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K)$ and $\mathfrak{N}g\mathfrak{D}_e(K\cap L) \subseteq \mathfrak{N}g\mathfrak{D}_e(L)$. Hence $\mathfrak{N}g\mathfrak{D}_e(K\cap L) \subseteq \mathfrak{N}g\mathfrak{D}_e(K) \cap \mathfrak{N}g\mathfrak{D}_e(L)$.

The derived set $\mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(\cdot)$ is also satisfies the Theorem 2.17

Theorem 2.18. Let $\mathfrak{N}GeC(U,P)$ (resp. $\mathfrak{N}G\mathscr{M}C(U,P)$) is closed under arbitrary union and if K is a subset of $\mathfrak{N}ts$ $(U,\tau_R(P))$ then $K \cup \mathfrak{N}g\mathfrak{D}_e(K)$ (resp. $K \cup \mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)$) is a $\mathfrak{N}gec$ (resp. $\mathfrak{N}g\mathscr{M}c$) set.

Proof. To prove $K \cup \mathfrak{N}g\mathfrak{D}_e(K)$ is a $\mathfrak{N}gec$ set, it is sufficient to prove $U-(K\cup \mathfrak{N}g\mathfrak{D}_e(K))$ is $\mathfrak{N}geo$ set.

Case 1: Let $U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))=\phi$, the result is obvious.

Case 2: Let $U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))\neq\phi$ and $l\in U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))$, implies $l\notin K\cup\mathfrak{N}g\mathfrak{D}_e(K)$. This implies $l\notin K$ and $l\notin\mathfrak{N}g\mathfrak{D}_e(K)$. Now $l\notin\mathfrak{N}g\mathfrak{D}_e(K)$ implies l is not $\mathfrak{N}ge$ limit point of K. Therefore there exist a $\mathfrak{N}geo$ set G such that $G\cap(K-\{l\})=\phi$. Since $l\notin K$, implies $G\cap K=\phi$. This implies $l\in G\subseteq U-K$. Again G is $\mathfrak{N}geo$ set and $G\cap K=\phi$ implies no point of G can be $\mathfrak{N}ge$ limit point of K. This follows $G\cap\mathfrak{N}g\mathfrak{D}_e(K)=\phi$, implies $l\in G\subseteq U-\mathfrak{N}g\mathfrak{D}_e(K)$. This shows, $l\in G\subseteq(U-K)\cap(U-\mathfrak{N}g\mathfrak{D}_e(K))=U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))$. That is $l\in G\subseteq U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))$. This implies $U-(K\cup\mathfrak{N}g\mathfrak{D}_e(K))$ is $\mathfrak{N}geNbd$ of each of its points. By Theorem 2.13, $U-K\cup\mathfrak{N}g\mathfrak{D}_e(K)$ is $\mathfrak{N}geo$ set and hence $K\cup\mathfrak{N}g\mathfrak{D}_e(K)$ is $\mathfrak{N}gec$ set.

The other case is similar.

Theorem 2.19. Let $(U,\tau_R(P))$ be a $\mathfrak{N}ts$ and $K\subseteq U$. Then K is a $\mathfrak{N}gec$ (resp. $\mathfrak{N}g\mathscr{M}c$) set iff $\mathfrak{N}g\mathfrak{D}_e(K)\subseteq K$ (resp. $\mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)\subseteq K$).

Proof. Suppose K is a $\mathfrak{N}gec$ set.

Case 1: If $\mathfrak{N}g\mathfrak{D}_e(K)=\phi$, then the result is obvious.

Case 2: If $\mathfrak{N}g\mathfrak{D}_e(K)\neq\phi$ then let $l\in\mathfrak{N}g\mathfrak{D}_e(K)$, implies $G\cap(K-\{l\})\neq\phi$ for every $\mathfrak{N}geo$ set G containing l. If $l\notin K$, then $l\in U-K$. Since K is $\mathfrak{N}gec$ set and $U-K$ is $\mathfrak{N}geo$ set containing l and not containing any other point of K which is a contradiction to $l\in\mathfrak{N}g\mathfrak{D}_e(K)$. Therefore $l\in K$. Thus $l\in\mathfrak{N}g\mathfrak{D}_e(K)$ implies $l\in K$. Hence $\mathfrak{N}g\mathfrak{D}_e(K)\subseteq K$.

Conversely, let $\mathfrak{N}g\mathfrak{D}_e(K)\subseteq K$. Let $l\in U-K$ implies $l\notin K$. Since $\mathfrak{N}g\mathfrak{D}_e(K)\subseteq K$, $l\notin\mathfrak{N}g\mathfrak{D}_e(K)$, there exists a $\mathfrak{N}geo$ set G containing l such that $G\cap(K-\{l\})=\phi$. That is $G\cap K=\phi$ as $l\notin K$, implies $l\in G\subseteq U-K$. Therefore $U-K$ is $\mathfrak{N}geNbdS(l)$. Since l is arbitrary, $U-K$ is $\mathfrak{N}geNbd$ of each of its points. By Theorem 2.13, $U-K$ is $\mathfrak{N}geo$ set. Hence K is $\mathfrak{N}gec$ set.

The other case is similar.

Definition 2.20. Let K be a subset of a $\mathfrak{N}ts$ $(U,\tau_R(P))$, then nano generalized

(i) e closure of K (briefly, $\mathfrak{N}gecl(K)$) is defined as the intersection of all $\mathfrak{N}gec$ sets of U containing K.

(ii) $\mathscr{M}$ closure of K (briefly, $\mathfrak{N}g\mathscr{M}cl(K)$) is defined as the intersection of all $\mathfrak{N}g\mathscr{M}c$ sets of U containing K.

Theorem 2.21. Let $\mathfrak{N}GeC(U,P)$ (resp. $\mathfrak{N}g\mathscr{M}C(U,P)$) be closed under arbitrary intersection and let K be any subset of a $\mathfrak{N}ts$, $(U,\tau_R(P))$. Then the following holds.

(i) $\mathfrak{N}gecl(K)$(resp. $\mathfrak{N}g\mathscr{M}cl(K)$) is the smallest $\mathfrak{N}gec$ (resp. $\mathfrak{N}g\mathscr{M}c$) superset of K.

(ii) K is $\mathfrak{N}gec$ (resp. $\mathfrak{N}g\mathscr{M}c$) iff $\mathfrak{N}gecl(K)=K$ (resp. $\mathfrak{N}g\mathscr{M}cl(K)=K$).

(iii) $\mathfrak{N}gecl(K)=K\cup\mathfrak{N}g\mathfrak{D}_e(K)$ (resp. $\mathfrak{N}g\mathscr{M}cl(K)=K\cup\mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)$).

Proof. (i) Let $\{F_i:i\in I\}$ be the collection of all $\mathfrak{N}gec$ subsets of U containing the set K. Therefore $\mathfrak{N}gecl(K)=\cap\{F_i:i\in I\}$, by the definition of the $\mathfrak{N}gecl(K)$. Since the intersection of an arbitrary collection of $\mathfrak{N}gec$ sets is $\mathfrak{N}gec$, implies $\cap\{F_i:i\in I\}$ is $\mathfrak{N}gec$. Therefore $\mathfrak{N}gecl(K)$ is a $\mathfrak{N}gec$ set. Also since $K\subseteq F_i$, for each $i\in I$, implies $K\subseteq\cap\{F_i:i\in I\}=\mathfrak{N}gecl(K)$. Thus $\mathfrak{N}gecl(K)$ is a $\mathfrak{N}gec$ set containing the set K. Since $\mathfrak{N}gecl(K)=\cap\{F_i:i\in I\}$, implies $\mathfrak{N}gecl(K)\subseteq F$, for each $i\in I$. Consequently, $\mathfrak{N}gecl(K)$ is the smallest $\mathfrak{N}gec$ superset of K.

(ii) If K is a $\mathfrak{N}gec$ set, then obviously it is the smallest $\mathfrak{N}gec$ superset of K, therefore it must coincide with $\mathfrak{N}gecl(K)$. Hence K is $\mathfrak{N}gec$ implies $\mathfrak{N}gecl(K)=K$. Again if $\mathfrak{N}gecl(K)=K$, then $\mathfrak{N}gecl(K)$ is $\mathfrak{N}gec$ set, so K is a $\mathfrak{N}gec$ set. Hence K is $\mathfrak{N}gec$ iff $\mathfrak{N}gecl(K)=K$.

(iii) By Theorem 2.18, $K\cup\mathfrak{N}g\mathfrak{D}_e(K)$ is a $\mathfrak{N}gec$ set. Therefore $K\cup\mathfrak{N}g\mathfrak{D}_e(K)$ is $\mathfrak{N}gec$ set containing K. Therefore $\mathfrak{N}gecl(K)\subseteq K\cup\mathfrak{N}g\mathfrak{D}_e(K)$. Again $K\subseteq\mathfrak{N}gecl(K)$, implies $\mathfrak{N}g\mathfrak{D}_e(K)\subseteq\mathfrak{N}g\mathfrak{D}_e(\mathfrak{N}gecl(K))\subseteq\mathfrak{N}gecl(K)$, because $\mathfrak{N}gecl(K)$ is $\mathfrak{N}gec$ set and by Theorem 2.19. Hence $K\cup\mathfrak{N}g\mathfrak{D}_e(K)\subseteq\mathfrak{N}gecl(K)$. Thus $\mathfrak{N}gecl(K)=K\cup\mathfrak{N}g\mathfrak{D}_e(K)$.

The other case is similar.

Theorem 2.22. Let K and L be two subsets of a $\mathfrak{N}ts$, $(U,\tau_R(P))$, then the following properties hold.

(i) $\mathfrak{N}gecl(\phi)=\phi$, $\mathfrak{N}gecl(U)=U$ and $\mathfrak{N}gecl(\mathfrak{N}gecl(K))=\mathfrak{N}gecl(K)$.

(ii) If $K\subset L$ then $\mathfrak{N}gecl(K)\subset\mathfrak{N}gecl(L)$.

(iii) $\mathfrak{N}gecl(K)\cup\mathfrak{N}gecl(L)\subset\mathfrak{N}gecl(K\cup L)$.

(iv) $\mathfrak{N}gecl(K\cap L)\subset\mathfrak{N}gecl(K)\cap\mathfrak{N}gecl(L)$.

Proof. (i) Since each one of the sets ϕ, U and $\mathfrak{N}gecl(K)$ being $\mathfrak{N}gec$ sets. By Theorem 2.21 (ii), $\mathfrak{N}gecl\ (\phi)=\phi$, $\mathfrak{N}gecl(U)=U$ and $\mathfrak{N}gecl(\mathfrak{N}gecl(K))=\mathfrak{N}gecl(K)$.

(ii) Let $K\subset L$, then $K\subset L\subset\mathfrak{N}gecl(L)$. This implies $\mathfrak{N}gecl(L)$ is a $\mathfrak{N}gec$ superset of K. But $\mathfrak{N}gecl(K)$ is the smallest $\mathfrak{N}gec$ superset of K. Therefore $\mathfrak{N}gecl(K)\subset\mathfrak{N}gecl(L)$.

(iii) Since $K\subset K\cup L$ and $L\subset K\cup L$ from (ii), $\mathfrak{N}gecl(K)\subset\mathfrak{N}gecl(K\cup L)$ and $\mathfrak{N}gecl(L)\subset\mathfrak{N}gecl(K\cup L)$. Therefore $\mathfrak{N}gecl(K)\cup\mathfrak{N}gecl(L)\subset\mathfrak{N}gecl(K\cup L)$.

(iv) Since $K\cap L\subset K$ and $K\cap L\subset L$ from (ii), $\mathfrak{N}gecl(K\cap L)\subset\mathfrak{N}gecl(K)$ and $\mathfrak{N}gecl(K\cap L)\subset\mathfrak{N}gecl(L)$. Therefore $\mathfrak{N}gecl(K\cap L)\subset\mathfrak{N}gecl(K)\cap\mathfrak{N}gecl(L)$.

The operator $\mathfrak{N}g\mathscr{M}cl(\cdot)$ is also satisfy the Theorem 2.22.

Remark 2.23. Equality does not hold in result of (ii), (iii) and (iv) of the Theorem 2.22 as seen from the following example.

Example 2.24. In Example 2.3,

(i) the sets $K=\{o\}$ and $L=\{l,n,o\}$, clearly $K\subset L$. Also $\mathfrak{N}gecl(K)=\{o\}$ and $\mathfrak{N}gecl(L)=\{l,n,o\}$. The- refore $\mathfrak{N}gecl(K)\neq\mathfrak{N}gecl(L)$.

(ii) the sets $K = \{m\}$ and $L = \{n,o\}$, $K \cup L = \{m,n,o\}$. Also $\mathfrak{N}gecl(K) = \{m\}$, $\mathfrak{N}gecl(L) = \{n,o\}$ and $\mathfrak{N}gecl(K \cup L) = U$. Clearly, $\mathfrak{N}gecl(K) \cup \mathfrak{N}gecl(L) \neq \mathfrak{N}gecl(K \cup L)$.

(iii) the sets $K = \{l,o\}$ and $L = \{m,n,o\}$, $K \cap L = \{o\}$. Also $\mathfrak{N}gecl(K) = \{l,o\}$, $\mathfrak{N}gecl(L) = U$ and $\mathfrak{N}gecl(K \cap L) = \{o\}$. Clearly, $\mathfrak{N}gecl(K \cap L) \neq \mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(L)$.

Theorem 2.25. Let K be a subset of a $\mathfrak{N}ts$, $(U, \tau_R(P))$. Then $l \in \mathfrak{N}gecl(K)$ (resp. $\mathfrak{N}g\mathscr{M}cl(K)$) iff $G \cap K \neq \phi$ for every $\mathfrak{N}geo$ (resp. $\mathfrak{N}g\mathscr{M}o$) set containing l.

Proof. Let $l \in \mathfrak{N}gecl(K)$. Suppose there exists $\mathfrak{N}geo$ set G containing l such that $G \cap K = \phi$. Then $K \subseteq U - G$. Now $U - G$ is a $\mathfrak{N}gec$ set containing K implies $\mathfrak{N}gecl(K) \subseteq U - G$ and $l \notin U - G$ implies $l \notin \mathfrak{N}gecl(K)$. This is contradiction to hypothesis. Hence $G \cap K \neq \phi$.

Conversely, let $G \cap K \neq \phi$ for every $\mathfrak{N}geo$ set G containing l. Suppose $l \notin \mathfrak{N}gecl(K)$, there exists a $\mathfrak{N}gec$ set F containing K such that $l \notin F$. This implies $K \cap (U - F) = \phi$ and $U - F$ is $\mathfrak{N}geo$ set containing l. This is contradiction to the hypothesis. Therefore $l \in \mathfrak{N}gecl(K)$.

The other case is similar.

Proposition 2.26. Let K and L be two subsets of a $\mathfrak{N}ts$, $(U, \tau_R(P))$, then the following properties hold.

(i) If $\mathfrak{N}GeC(U,P)$ is closed under finite union, then $\mathfrak{N}gecl(K \cup L) = \mathfrak{N}gecl(K) \cup \mathfrak{N}gecl(L)$ for every K, $L \in \mathfrak{N}GeC(U,P)$.

(ii) If $\mathfrak{N}G\mathscr{M}C(U,P)$ is closed under finite union, then $\mathfrak{N}g\mathscr{M}cl(K \cup L) = \mathfrak{N}g\mathscr{M}cl(K) \cup \mathfrak{N}g\mathscr{M}cl(L)$ for every K, $L \in \mathfrak{N}G\mathscr{M}C(U,P)$.

Proof. Let K and L be $\mathfrak{N}gec$ sets in U. By hypothesis, $K \cup L$ is $\mathfrak{N}gec$. Then $\mathfrak{N}gecl(K \cup L) = K \cup L = \mathfrak{N}gecl(K) \cup \mathfrak{N}gecl(L)$.

The other case is similar.

Theorem 2.27. Let K and L be two subsets of a $\mathfrak{N}ts$, $(U, \tau_R(P))$, if $\mathfrak{N}eC(U,P)$ (resp. $\mathfrak{N}\mathscr{M}C(U,P)$) is closed under finite union, then $\mathfrak{N}GeC(U,P)$ (resp. $\mathfrak{N}G\mathscr{M}C(U,P)$) is closed under finite union.

Proof. Let $\mathfrak{N}eC(U,P)$ is closed under finite unions. Suppose K, $L \in \mathfrak{N}GeC(U,P)$ and let $K \cup L \subseteq G$ where G is $\mathfrak{N}o$ in U. Then $K \subseteq G$ and $L \subseteq G$. Hence $\mathfrak{N}ecl(K) \subseteq G$ and $\mathfrak{N}ecl(L) \subseteq G$. This implies $\mathfrak{N}ecl(K) \cup \mathfrak{N}ecl(L) \subseteq G$. By hypothesis, $\mathfrak{N}ecl(K \cup L) \subseteq G$. That is, $K \cup L \in \mathfrak{N}GeC(U,P)$.

The other case is similar.

Lemma 2.28. For any subset K of U, if $\mathfrak{N}\mathfrak{D}(K) \subseteq \mathfrak{N}g\mathfrak{D}_e(K)$ (resp. $\mathfrak{N}\mathfrak{D}(K) \subseteq \mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)$), then $\mathfrak{N}cl(K) = \mathfrak{N}gecl(K)$ (resp. $\mathfrak{N}cl(K) = \mathfrak{N}g\mathscr{M}cl(K)$).

Proof. For any subset K of U, $\mathfrak{N}\mathfrak{D}(K) \subset \mathfrak{N}g\mathfrak{D}_e(K)$ is always true. By hypothesis, $\mathfrak{N}\mathfrak{D}(K) \subset \mathfrak{N}g\mathfrak{D}_e(K)$. Therefore, $\mathfrak{N}\mathfrak{D}(K) = \mathfrak{N}g\mathfrak{D}_e(K)$. By Theorem 2.21 (iii), that is $K \cup \mathfrak{N}cl(K) = A \cup \mathfrak{N}gecl(K)$, which implies $\mathfrak{N}cl(K) = \mathfrak{N}gecl(K)$.

The other case is similar.

Theorem 2.29. (i) Let $\mathfrak{N}GeC(U,P)$ is closed under finite union and if K and L are $\mathfrak{N}gec$ sets such that $\mathfrak{N}\mathfrak{D}(K) \subset \mathfrak{N}g\mathfrak{D}_e(K)$ and $\mathfrak{N}\mathfrak{D}(L) \subset \mathfrak{N}g\mathfrak{D}_e(L)$. Then $K \cup L$ is a $\mathfrak{N}gec$ set in U.

(ii) Let $\mathfrak{N}G\mathscr{M}C(U,P)$ is closed under finite union and if K and L are $\mathfrak{N}g\mathscr{M}c$ sets such that $\mathfrak{N}\mathfrak{D}(K) \subset \mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(K)$ and $\mathfrak{N}\mathfrak{D}(L) \subset \mathfrak{N}g\mathfrak{D}_{\mathscr{M}}(L)$. Then $K \cup L$ is a $\mathfrak{N}g\mathscr{M}c$ set in U.

Proof. Let K and L are $\mathfrak{N}gec$ sets such that $\mathfrak{N}\mathfrak{D}(K) \subset \mathfrak{N}g\mathfrak{D}_e(K)$ and $\mathfrak{N}\mathfrak{D}(L) \subset \mathfrak{N}g\mathfrak{D}_e(L)$. Therefore by Lemma 2.28, $\mathfrak{N}cl(K) = \mathfrak{N}gecl(K)$ and $\mathfrak{N}cl(L) = \mathfrak{N}gecl(L)$. Let G be $\mathfrak{N}o$ set such that $K \cup L \subset G$, then $K \subset G$ and $L \subset G$. Since K and L are $\mathfrak{N}gec$ sets, $\mathfrak{N}gecl(K) \subset G$ and $\mathfrak{N}gecl(L) \subset G$. Since $\mathfrak{N}gecl(K \cup L) \subset \mathfrak{N}cl(K \cup L) = \mathfrak{N}cl(K) \cup \mathfrak{N}cl(L) = \mathfrak{N}gecl(K) \cup \mathfrak{N}gecl(L) \subset G \cup G = G$. Thus $\mathfrak{N}gecl\ (K \cup L) \subset G$. This shows that, $K \cup L$ is $\mathfrak{N}gec$ set in U.

The other case is similar.

Definition 2.30. Let K be a subset of a $\mathfrak{N}ts$ $(U, \tau_R(P))$, then nano generalized

(i) e interior of K (briefly, $\mathfrak{N}geint(K)$) is defined as the union of all $\mathfrak{N}geo$ sets of U contained in K.

(ii) $\mathscr{M}$ interior of K (briefly, $\mathfrak{N}g\mathscr{M}int(K)$) is defined as the union of all $\mathfrak{N}g\mathscr{M}o$ sets of U contained in K.

Remark 2.31. Every $\mathfrak{N}o$ is a $\mathfrak{N}geo$ set which implies every $\mathfrak{N}int$ point of K is a $\mathfrak{N}geint$ point of K. Therefore $\mathfrak{N}int(K) \subset \mathfrak{N}geint(K)$ for any $K \subset U$. But the converse of this result is not true as seen from the following example.

Example 2.32. Let $U = \{l,m,n\}$ with $U/R = \{\{l\}, \{m,\ n\}\}$ and $P = \{l,n\}$. Then, the $\mathfrak{N}t$ is defined as $\tau_{\mathrm{R}}(\mathrm{P}) = \{U, \phi, \{l\}, \{m,n\}\}$. For $K = \{l,m\}$, $\mathfrak{N}geint(K) = \{l,m\}$ and $\mathfrak{N}int(K) = \{l\}$. Clearly $m \in \mathfrak{N}geint(K)$ and $m \notin \mathfrak{N}int\ (K)$, implies $\mathfrak{N}geint(K) \not\subset \mathfrak{N}int(K)$.

Theorem 2.33. Let K and L be two subsets of $\mathfrak{N}ts$, $(U, \tau_R(P))$, the following properties hold.

(i) $\mathfrak{N}geint(K)$ is largest $\mathfrak{N}geo$ set contained in K.

(ii) K is $\mathfrak{N}geo$ set iff $K = \mathfrak{N}geint(K)$.

(iii) $\mathfrak{N}geint(\phi) = \phi$ and $\mathfrak{N}geint(U) = U$.

(iv) If $K \subset L$ then $\mathfrak{N}geint(K) \subset \mathfrak{N}geint(L)$.

(v) $\mathfrak{N}geint(K) \cup \mathfrak{N}geint(L) \subset \mathfrak{N}geint(K \cup L)$.

(vi) $\mathfrak{N}geint(K \cap L) \subset \mathfrak{N}geint(K) \cap \mathfrak{N}geint(L)$.

(vii) $\mathfrak{N}geint(\mathfrak{N}geint(K)) = \mathfrak{N}geint(K)$.

Proof. (i) Let G be any $\mathfrak{N}geo$ subset of K and if $l \in G$, then $l \in G \subseteq K$. Since G being $\mathfrak{N}geo$, K is $\mathfrak{N}geNbd$ of l. Therefore l is $\mathfrak{N}geint$ point of K. Thus $l \in G$ implies $l \in \mathfrak{N}geint(K)$. This implies every $\mathfrak{N}geo$ subset of K is contained in $\mathfrak{N}geint(K)$. Therefore $\mathfrak{N}geint(K)$ is the largest $\mathfrak{N}geo$ set contained in K.

(ii) Let K be a $\mathfrak{N}geo$ set. Since $K \subseteq K$, K is identical with largest $\mathfrak{N}geo$ subset of K. By (i), $\mathfrak{N}geint(K)$ is the largest $\mathfrak{N}geo$ subset of K. Therefore $K = \mathfrak{N}geint(K)$.

(iii) Since ϕ and U are $\mathfrak{N}geo$ sets, by (ii), $\mathfrak{N}geint(\phi) = \phi$ and $\mathfrak{N}geint(U) = U$.

(iv) Let $K \subset L$, then $l \in \mathfrak{N}geint(K)$, implies there exists $\mathfrak{N}geo$ set G such that $l \in G \subset K$, which implies $l \in G \subset K \subset L$. That is $l \in G \subset L$. Therefore $l \in \mathfrak{N}geint(L)$. Thus $l \in \mathfrak{N}geint(K)$ implies $l \in \mathfrak{N}geint(L)$. Therefore. $\mathfrak{N}geint(K) \subset \mathfrak{N}geint(L)$.

(v) Since $K \subset K \cup L$ and $L \subset K \cup L$, from (iv), $\mathfrak{N}geint(K) \subset \mathfrak{N}geint(K \cup L)$ and $\mathfrak{N}geint(L) \subset \mathfrak{N}geint(K \cup L)$. Therefore $\mathfrak{N}geint(K) \cup \mathfrak{N}geint(L) \subset \mathfrak{N}geint(K \cup L)$.

(vi) Since $K \cap L \subset K$ and $K \cap L \subset L$, from (iv), $\mathfrak{N}geint(K \cap L) \subset \mathfrak{N}geint(K)$ and $\mathfrak{N}geint(K \cap L) \subset \mathfrak{N}geint(L)$. Therefore $\mathfrak{N}geint(K \cap L) \subset \mathfrak{N}geint(K) \cap \mathfrak{N}geint(L)$.

(vii) By (ii), K is $\mathfrak{N}geo$ iff $K = \mathfrak{N}geint(K)$ and by (i), $\mathfrak{N}geint(K)$ is the largest $\mathfrak{N}geo$ set contained in K. Therefore $\mathfrak{N}geint(\mathfrak{N}geint(K)) = \mathfrak{N}geint(K)$.

The operator $\mathfrak{N}g\mathscr{M}int(\cdot)$ is also satisfy the Theorem 2.33 for respective open sets.

Remark 2.34. Equality does not hold in results (iv), (v) and (vi) of the Theorem 2.33 as seen from the following example.

Example 2.35. In Example 2.3,

(i) the sets $K = \{o\}$ and $L = \{l,n,o\}$, clearly $K \subset L$. Also $\mathfrak{N}geint(K) = \{o\}$ and $\mathfrak{N}geint(L) = \{l,n,o\}$. Therefore $\mathfrak{N}geint(K) \neq \mathfrak{N}geint(L)$.

(ii) the sets $K = \{l\}$ and $L = \{n,o\}$, $K \cup L = \{l,n,o\}$. Also $\mathfrak{N}geint(K) = \{\phi\}$, $\mathfrak{N}geint(L) = \{n,o\}$ and $\mathfrak{N}ge\,int(K \cup L) = \{l,n,o\}$. Clearly, $\mathfrak{N}geint(K) \cup \mathfrak{N}geint(L) \neq \mathfrak{N}geint(K \cup L)$.

(iii) the sets $K = \{l,m\}$ and $L = \{l,o\}$, $K \cap L = \{l\}$. Also $\mathfrak{N}geint(K) = \{l,m\}$, $\mathfrak{N}geint(L) = \{l,o\}$ and $\mathfrak{N}geint(K \cap L) = \{\phi\}$. Clearly, $\mathfrak{N}geint(K \cap L) \neq \mathfrak{N}geint(K) \cap \mathfrak{N}geint(L)$.

Theorem 2.36. If K is a subset of a $\mathfrak{N}ts$, $(U, \tau_R(P))$, then $\mathfrak{N}geint(K)$ equals to the set of all points of K which are not $\mathfrak{N}ge$ limit points of $(U-K)$. That is $\mathfrak{N}geint(K) = K - \mathfrak{N}g\mathfrak{D}_e(U-K)$.

Proof. Let $l \in K - \mathfrak{N}g\mathfrak{D}_e(U-K)$, implies $l \in K$ and $l \notin \mathfrak{N}g\mathfrak{D}_e(U-K)$. This implies l is not $\mathfrak{N}ge$ limit point of $(U-K)$, therefore there exists $\mathfrak{N}geo$ set G containing l but not contains the points of $(U-K)$. That is $G \cap (U-K) = \phi$. This implies $G \subseteq K$. Thus $l \in G \subseteq K$ implies $l \in \mathfrak{N}geint(K)$. Therefore $K - \mathfrak{N}g\mathfrak{D}_e(U-K) \subseteq \mathfrak{N}geint(K)$.

On the other hand, if $l \in \mathfrak{N}geint(K)$ then $l \in K$ as $\mathfrak{N}geint(K) \subseteq K$ and also $\mathfrak{N}geint(K)$ is $\mathfrak{N}geo$ set containing l and not containing any other points of $(U-K)$, implies l is not $\mathfrak{N}ge$ limit point of $(U-K)$. Since l is arbitrary, every point of $\mathfrak{N}geint(K)$ is point of K but not a limit point of $(U-K)$. This shows that $l \in K - \mathfrak{N}g\mathfrak{D}_e(U-K)$. Therefore $\mathfrak{N}geint(K) \subseteq K - \mathfrak{N}g\mathfrak{D}_e(U-K)$. Hence $\mathfrak{N}geint(K) = K - \mathfrak{N}g\mathfrak{D}_e(U-K)$. Thus $\mathfrak{N}geint(K)$ equals to the set of all points of K which are not $\mathfrak{N}ge$ limit points of $(U-K)$.

The operator $\mathfrak{N}g\mathscr{M}int(\cdot)$ is also satisfy the Theorem 2.36 for respective derived sets.

Theorem 2.37. For the subsets K and L of space U, the following statements are true.

(i) $\mathfrak{N}geint(U-K) \subseteq U - \mathfrak{N}geint(K)$.

(ii) $\mathfrak{N}geint(K-L) \subseteq \mathfrak{N}geint(K) - \mathfrak{N}geint(L)$.

Proof. (i) Let $l \in \mathfrak{N}geint(U-K)$. Since $\mathfrak{N}geint(U-K) \subseteq (U-K)$ implies $l \notin K$ and hence $l \notin \mathfrak{N}geint(K)$. This implies $l \in U - \mathfrak{N}geint(K)$. Therefore $\mathfrak{N}geint(U-K) \subseteq U - \mathfrak{N}geint(K)$.

(ii) Let $\mathfrak{N}geint(K-L) = \mathfrak{N}geint(K \cap (U-L)) \subseteq \mathfrak{N}geint(K) \cap \mathfrak{N}geint(U-L) \subseteq \mathfrak{N}geint(K) \cap (U - \mathfrak{N}geint(L)) = \mathfrak{N}ge\,int(K) - \mathfrak{N}geint(L)$.

The operator $\mathfrak{N}g\mathscr{M}int(\cdot)$ is also satisfy the Theorem 2.37.

3. Nano generalized e (resp. $\mathscr{M}$) exterior

Definition 3.1. For a subset $K \subseteq U$, the nano generalized e (resp. $\mathscr{M}$) exterior (briefly, $\mathfrak{N}ge\mathfrak{E}r$ (resp. $\mathfrak{N}g\mathscr{M}\,\mathfrak{E}r$)) of K is defined as $\mathfrak{N}ge\mathfrak{E}r(K) = \mathfrak{N}geint(U-K)$ (resp. $\mathfrak{N}g\mathscr{M}\mathfrak{E}r(K) = \mathfrak{N}g\mathscr{M}int(U-K)$).

Definition 3.2. For a subset $K \subseteq U$, the nano generalized e (resp. $\mathscr{M}$) border (briefly, $\mathfrak{N}ge\mathfrak{B}r$ (resp. $\mathfrak{N}g\mathscr{M}\,\mathfrak{B}r$)) of K is defined as $\mathfrak{N}ge\mathfrak{B}r(K) = K - \mathfrak{N}geint(K)$ (resp. $\mathfrak{N}g\mathscr{M}\mathfrak{B}r(K) = K - \mathfrak{N}g\mathscr{M}int(K)$).

Theorem 3.3. Let K and L be two subsets of $\mathfrak{N}ts$, $(U, \tau_R(X))$. Then the following holds.

(i) $\mathfrak{N}\mathfrak{E}r(K) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$.

(ii) $\mathfrak{N}ge\mathfrak{E}r(K)$ is $\mathfrak{N}geo$ set.

(iii) $\mathfrak{N}ge\mathfrak{E}r(K) = U - \mathfrak{N}gecl(K)$.

(iv) $\mathfrak{N}ge\mathfrak{E}r[\mathfrak{N}ge\mathfrak{E}r(K)] = \mathfrak{N}geint[\mathfrak{N}gecl(K)]$.

(v) If $K \subseteq L$, then $\mathfrak{N}ge\mathfrak{E}r(L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$.

(vi) $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K) \cup \mathfrak{N}ge\mathfrak{E}r(L)$.

(vii) $\mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K \cap L)$.

(viii) $\mathfrak{N}ge\mathfrak{E}r(K \cup L) = \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L)$.

(ix) $\mathfrak{N}ge\mathfrak{E}r(U) = \phi$ and $\mathfrak{N}ge\mathfrak{E}r(\phi) = U$.

(x) $\mathfrak{N}ge\mathfrak{E}r(K) = \mathfrak{N}ge\mathfrak{E}r[U - \mathfrak{N}ge\mathfrak{E}r(K)]$.

(xi) $\mathfrak{N}geint(K) \subseteq \mathfrak{N}ge\mathfrak{E}r[\mathfrak{N}ge\mathfrak{E}r(K)]$.

(xii) $\mathfrak{N}geint(K)$ and $\mathfrak{N}ge\mathfrak{E}r(K)$ are mutually disjoint and $U = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{E}r(K)$.

(xiv) $K \cap \mathfrak{N}ge\mathfrak{E}r(K) = \phi$.

(xv) $\mathfrak{N}ge\mathfrak{E}r(K) \subseteq U - K$.

Proof. (i) For any subset L of U, $\mathfrak{N}int(L) \subseteq \mathfrak{N}geint(L)$. Put $L = U - K$ then $\mathfrak{N}int(U-K) \subseteq \mathfrak{N}geint(U-K)$. This implies $\mathfrak{N}\mathfrak{E}r(K) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$.

(ii) By definition, $\mathfrak{N}ge\mathfrak{E}r(K) = \mathfrak{N}geint(U-K)$ and $\mathfrak{N}geint(K)$ is $\mathfrak{N}geo$ set. Therefore $\mathfrak{N}ge\mathfrak{E}r(K)$ is $\mathfrak{N}geo$ set.

(iii) By definition, $\mathfrak{N}ge\mathfrak{E}r(K) = \mathfrak{N}geint(U-K) = U - \mathfrak{N}gecl(K)$.

(iv) Consider $\mathfrak{N}ge\mathfrak{E}r(\mathfrak{N}ge\mathfrak{E}r(K)) = \mathfrak{N}ge\mathfrak{E}r(\mathfrak{N}geint(U - K)) = \mathfrak{N}ge\mathfrak{E}r(U - \mathfrak{N}gecl(K)) = \mathfrak{N}geint(U - (U - \mathfrak{N}gecl(K))) = \mathfrak{N}geint(\mathfrak{N}gecl(K))$.

(v) If $K \subseteq L$ then $U - L \subseteq U - K$. This implies $\mathfrak{N}geint(U - L) \subseteq \mathfrak{N}geint(U-K)$. Therefore $\mathfrak{N}ge\mathfrak{E}r(L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$.

(vi) Since $K \subseteq K \cup L$ and $L \subseteq K \cup L$. By (v), $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$ and $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(L)$. Hence $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K) \cup \mathfrak{N}ge\mathfrak{E}r(L)$.

(vii) Since $K \cap L \subseteq K$ and $K \cap L \subseteq L$. By (v), $\mathfrak{N}ge\mathfrak{E}r(K) \subseteq \mathfrak{N}ge\mathfrak{E}r(K \cap L)$ and $\mathfrak{N}ge\mathfrak{E}r(L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K \cap L)$. Hence $\mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K \cap L)$.

(viii) Consider $\mathfrak{N}ge\mathfrak{E}r(K \cup L) = \mathfrak{N}geint(U - (K \cup L)) = \mathfrak{N}geint[(U-K) \cap (U-L)] \supseteq \mathfrak{N}geint(U-K) \cap \mathfrak{N}geint(U - L) = \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L)$. That is,

$$\mathfrak{N}ge\mathfrak{E}r(K \cup L) \supseteq \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L). \tag{3.1}$$

Also, we have $K \subseteq K \cup L$ and $L \subseteq K \cup L$. By (v), $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K)$ and $\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(L)$. Hence,

$$\mathfrak{N}ge\mathfrak{E}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L). \tag{3.2}$$

From, (3.1) and (3.2), we have $\mathfrak{N}ge\mathfrak{E}r(K \cup L) = \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}ge\mathfrak{E}r(L)$.

(ix) By definition, $\mathfrak{N}ge\mathfrak{E}r(U) = \mathfrak{N}eint(U-U) = \mathfrak{N}eint(\phi) = \phi$ and $\mathfrak{N}ge\mathfrak{E}r(\phi) = \mathfrak{N}eint(U - \phi) = \mathfrak{N}eint(U) = U$.

(x) Consider $\mathfrak{N}ge\mathfrak{E}r(U - \mathfrak{N}ge\mathfrak{E}r(K)) = \mathfrak{N}geint(U - (U - \mathfrak{N}ge\mathfrak{E}r(K))) = \mathfrak{N}geint(\mathfrak{N}ge\mathfrak{E}r(K)) = \mathfrak{N}geint(\mathfrak{N}geint(U-K)) = \mathfrak{N}geint(U-K) = \mathfrak{N}ge\mathfrak{E}r(K)$.

(xi) Since $K \subseteq \mathfrak{N}gecl(K)$, implies $\mathfrak{N}geint(K) \subseteq \mathfrak{N}geint(\mathfrak{N}gecl(K)) = \mathfrak{N}geint(U - \mathfrak{N}geint(U-K)) = \mathfrak{N}ge\mathfrak{E}r(\mathfrak{N}geint(U-K)) = \mathfrak{N}ge\mathfrak{E}r(\mathfrak{N}ge\mathfrak{E}r(K))$. Thus $\mathfrak{N}geint(K) \subseteq \mathfrak{N}ge\mathfrak{E}r(\mathfrak{N}ge\mathfrak{E}r(K))$.

(xii) Let us assume that $\mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}geint(K) \neq \phi$, then, there exists $l \in \mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}geint(K)$. Therefore $l \in \mathfrak{N}ge\mathfrak{E}r(K)$ and $l \in \mathfrak{N}geint(K)$ implies $l \in U - K$ and $l \in K$, which is not possible. Therefore our assumption is wrong. Hence $\mathfrak{N}ge\mathfrak{E}r(K) \cap \mathfrak{N}geint(K) = \phi$. Similarly other cases can be obtained. Now we consider $\mathfrak{N}ge\mathfrak{E}r(K) = U - \mathfrak{N}gecl(K)$.

(xiii) $K \cap \mathfrak{N}ge\mathfrak{E}r(K) = K \cap \mathfrak{N}geint(U-K) \subseteq K \cap (U - K) = \phi$ by (ii). Therefore $K \cap \mathfrak{N}ge\mathfrak{E}r(K) = \phi$.

(xiv) By definition, $\mathfrak{N}ge\mathfrak{E}r(K) = \mathfrak{N}geint(U-K) \subseteq U - K$.

The Theorem 3.3 is satisfied by $\mathfrak{N}g\mathscr{M}\mathfrak{E}r$ of a set K.

Theorem 3.4. Let K and L be two subsets of $\mathfrak{N}ts$, $(U, \tau_R(P))$. Then the following holds.

(i) $\mathfrak{N}ge\mathfrak{B}r(K) \subseteq \mathfrak{N}\mathfrak{B}r(K)$.

(ii) $\mathfrak{N}geint(K) \cap \mathfrak{N}ge\mathfrak{B}r(K) = \phi$.

(iii) K is $\mathfrak{N}geo$ set iff $\mathfrak{N}ge\mathfrak{B}r(K) = \phi$.

(iv) $\mathfrak{N}geint(\mathfrak{N}ge\mathfrak{B}r(K)) = \phi$.

(v) $\mathfrak{N}ge\mathfrak{B}r(\mathfrak{N}geint(K)) = \phi$.

(vi) $\mathfrak{N}ge\mathfrak{B}r(\mathfrak{N}ge\mathfrak{B}r(K)) = \mathfrak{N}ge\mathfrak{B}r(K)$.

(vii) $\mathfrak{N}ge\mathfrak{B}r(K) = K - \mathfrak{N}geint(K) = K \cap \mathfrak{N}gecl(U-K)$.

(viii) $K \subseteq L$ then $\mathfrak{N}ge\mathfrak{B}r(L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K)$.

(ix) $\mathfrak{N}ge\mathfrak{B}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}ge\mathfrak{B}r(L)$.

(x) $\mathfrak{N}ge\mathfrak{B}r(K) \cap \mathfrak{N}ge\mathfrak{B}r(L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K \cap L)$.

(xi) $\mathfrak{N}ge\mathfrak{B}r(K) = \mathfrak{N}g\mathfrak{D}_e(U-K)$ & $\mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}ge\mathfrak{B}r(U-K)$.

(xii) $K = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{B}r(K)$.

Proof. (i) Since $\mathfrak{N}int(K) \subseteq \mathfrak{N}geint(K) \Rightarrow U - \mathfrak{N}geint(K) \subseteq U - \mathfrak{N}int(K) \Rightarrow K \cap (U - \mathfrak{N}geint(K)) \subseteq K \cap (U - \mathfrak{N}int(K)) \Rightarrow K - \mathfrak{N}geint(K) \subseteq K - \mathfrak{N}int(K)$. Therefore $\mathfrak{N}ge\mathfrak{B}r(K) \subseteq \mathfrak{N}\mathfrak{B}r(K)$.

(ii) Consider $\mathfrak{N}geint(K) \cap \mathfrak{N}ge\mathfrak{B}r(K) = \mathfrak{N}geint(K) \cap (K - \mathfrak{N}geint(K)) = \mathfrak{N}geint(K) \cap (K \cap (U - \mathfrak{N}geint(K))) = \mathfrak{N}geint(K) \cap (U - \mathfrak{N}geint(K)) \cap K = \phi \cap K = \phi$.

(iii) Any subset K of space U is $\mathfrak{N}geo$ set iff $K = \mathfrak{N}geint(K) \Leftrightarrow K - \mathfrak{N}geint(K) = \phi \Leftrightarrow \mathfrak{N}ge\mathfrak{B}r(K) = \phi$.

(iv) Consider $\mathfrak{N}geint(\mathfrak{N}ge\mathfrak{B}r(K)) = \mathfrak{N}geint(K - \mathfrak{N}geint(K)) = \mathfrak{N}geint(K \cap (U - \mathfrak{N}geint(K))) \subseteq \mathfrak{N}geint(K) \cap \mathfrak{N}geint(U - \mathfrak{N}geint(K)) \subseteq \mathfrak{N}geint(K) \cap (U - \mathfrak{N}geint(K)) = \phi$ as $\mathfrak{N}geint(K) \subseteq K$. Therefore $\mathfrak{N}geint(\mathfrak{N}ge\mathfrak{B}r(K)) = \phi$.

(v) Consider $\mathfrak{N}ge\mathfrak{B}r(\mathfrak{N}geint(K)) = \mathfrak{N}geint(K) - \mathfrak{N}geint(\mathfrak{N}geint(K)) = \mathfrak{N}geint(K) - \mathfrak{N}geint(K) = \phi$.

(vi) Consider $\mathfrak{N}ge\mathfrak{B}r(\mathfrak{N}ge\mathfrak{B}r(K)) = \mathfrak{N}ge\mathfrak{B}r(K) - \mathfrak{N}geint(\mathfrak{N}ge\mathfrak{B}r(K)) = \mathfrak{N}ge\mathfrak{B}r(K)$ by (iv).

(vii) $\mathfrak{N}ge\mathfrak{B}r(K) = K - \mathfrak{N}geint(K) = K \cap (U - \mathfrak{N}geint(K)) = K \cap \mathfrak{N}gecl(U-K)$.

(viii) If $K \subseteq L$ then $\mathfrak{N}geint(K) \subseteq \mathfrak{N}eint(L) \Rightarrow U - \mathfrak{N}geint(L) \subseteq U - \mathfrak{N}geint(K) \Rightarrow K \cap (U - \mathfrak{N}eint(L)) \subseteq K \cap (U - \mathfrak{N}geint(K)) \Rightarrow L - \mathfrak{N}geint(L) \subseteq K - \mathfrak{N}geint(K) \Rightarrow \mathfrak{N}ge\mathfrak{B}r(L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K)$.

(ix) Since $K \subseteq K \cup L$ and $L \subseteq K \cup L$ by (viii) $\mathfrak{N}ge\mathfrak{B}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K)$ and $\mathfrak{N}ge\mathfrak{B}r(L) \supseteq \mathfrak{N}ge\mathfrak{B}r(K \cup L)$. Hence $\mathfrak{N}ge\mathfrak{B}r(K \cup L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}ge\mathfrak{B}r(L)$.

(x) Since $K \cap L \subseteq K$ and $K \cap L \subseteq L$ by (viii) $\mathfrak{N}ge\mathfrak{B}r(K) \subseteq \mathfrak{N}ge\mathfrak{B}r(K \cap L)$ and $\mathfrak{N}ge\mathfrak{B}r(L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K \cap L)$. Hence $\mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}ge\mathfrak{B}r(L) \subseteq \mathfrak{N}ge\mathfrak{B}r(K \cap L)$.

(xi) By definition $\mathfrak{N}ge\mathfrak{B}r(K) = K - \mathfrak{N}geint(K) = K - (K - \mathfrak{N}g\mathfrak{D}_e(U - K)) = \mathfrak{N}g\mathfrak{D}_e(U - K)$ by Theorem 2.36 and $\mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}ge\mathfrak{B}r(U - K)$ is obtained by replacing K by $U - K$.

(xii) $\mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{B}r(K) = \mathfrak{N}geint(K) \cup (K - \mathfrak{N}geint(K)) = \mathfrak{N}geint(K) \cup (K \cap (U - \mathfrak{N}geint(K))) = (\mathfrak{N}geint(K) \cup K) \cap (\mathfrak{N}geint(K) \cup (U - \mathfrak{N}geint(K))) = K \cap U = K$. Therefore $K = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{B}r(K)$.

The Theorem 3.4 is satisfied by $\mathfrak{N}g\mathscr{M}\mathfrak{B}r$ of a set K.

4. Nano generalized e (resp. $\mathscr{M}$) frontier

Definition 4.1. For a subset $K \subseteq U$, the nano generalized e (resp. $\mathscr{M}$) frontier (briefly, $\mathfrak{N}ge\mathfrak{F}r$ (resp. $\mathfrak{N}g\mathscr{M}\mathfrak{F}r$)) of K is defined as $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}gecl(K) - \mathfrak{N}geint(K)$ (resp. $\mathfrak{N}g\mathscr{M}\mathfrak{F}r(K) = \mathfrak{N}g\mathscr{M}cl(K) - \mathfrak{N}g\mathscr{M}int(K)$).

Theorem 4.2. Let K and L be two subsets of $\mathfrak{N}ts$, $(U, \tau_R(P))$. Then the following holds.

(i) $\mathfrak{N}ge\mathfrak{F}r(K) \subseteq \mathfrak{N}ge\mathfrak{F}r_\delta(K)$.

(ii) $\mathfrak{N}ge\mathfrak{B}r(K) \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(iii) $\mathfrak{N}gecl(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$.

(iv) $\mathfrak{N}geint(K) \cap \mathfrak{N}ge\mathfrak{F}r(K) = \phi$.

(v) $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}g\mathfrak{D}_e(K)$.

(vi) K is $\mathfrak{N}geo$ set iff $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}g\mathfrak{D}_e(K)$.

(vii) $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - K)$.

(viii) $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}ge\mathfrak{F}r(U - K)$.

(ix) $\mathfrak{N}ge\mathfrak{F}r(K)$ is a $\mathfrak{N}gec$ set.

(x) $\mathfrak{N}geint(K) = K - \mathfrak{N}ge\mathfrak{F}r(K)$.

(xi) $\mathfrak{N}ge\mathfrak{F}r(K) = \phi$ iff K is $\mathfrak{N}geo$ as well as $\mathfrak{N}gec$ set.

(xii) $\mathfrak{N}ge\mathfrak{F}r[\mathfrak{N}geint(K)] \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(xiii) $U - \mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}geint(U - K)$.

(xiv) $\mathfrak{N}ge\mathfrak{F}r[\mathfrak{N}gecl(K)] \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(xv) $\mathfrak{N}gecl(K) = K \cup \mathfrak{N}ge\mathfrak{F}r(K)$.

Proof. (i) Since $\mathfrak{N}int(K) \subseteq \mathfrak{N}geint(K)$ implies $U - \mathfrak{N}geint(K) \subseteq U - \mathfrak{N}int(K)$. Also $\mathfrak{N}gecl(K) \subseteq \mathfrak{N}cl(K)$. The- refore $\mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K)) \subseteq \mathfrak{N}cl(K) \cap (U - \mathfrak{N}int\ (K))$. This implies $\mathfrak{N}gecl(K) - \mathfrak{N}geint(K) \subseteq \mathfrak{N}cl(K) - \mathfrak{N}\ int(K)$. Hence $\mathfrak{N}ge\mathfrak{F}r(K) \subseteq \mathfrak{N}\mathfrak{F}r(K)$.

(ii) Since $K \subseteq \mathfrak{N}gecl(K)$ implies $K \cap (U - \mathfrak{N}geint(K)) \subseteq \mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K))$. This implies $K - \mathfrak{N}geint(K) \subseteq \mathfrak{N}gecl(K) - \mathfrak{N}geint(K)$. This shows $\mathfrak{N}ge\mathfrak{B}r(K) \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(iii) $\mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}geint(K) \cup (\mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K))) = (\mathfrak{N}geint(K) \cup \mathfrak{N}gecl(K)) \cap (\mathfrak{N}geint(K) \cup (U - \mathfrak{N}geint(K))) = \mathfrak{N}gecl(K) \cap U = \mathfrak{N}gecl(K)$.

(iv) $\mathfrak{N}geint(K) \cap \mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}geint(K) \cap (\mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K))) = (\mathfrak{N}geint(K) \cap \mathfrak{N}gecl(K)) \cap (\mathfrak{N}geint(K) \cap (U - \mathfrak{N}geint(K))) = \mathfrak{N}geint(K) \cap \phi = \phi$.

(v) From (iii), $\mathfrak{N}gecl(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$. This implies $K \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$ by Theorem 2.21 (iii). But $K = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{B}r(K)$ by Theorem 3.4 Therefore $\mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$. Hence $\mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}\mathfrak{D}(K) = \mathfrak{N}ge\mathfrak{F}r(K)$.

(vi) Suppose K is $\mathfrak{N}geo$ set and by Theorem 3.4 (iv), $\mathfrak{N}ge\mathfrak{B}r(K) = \phi$. From (v) $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}ge\mathfrak{B}r(K) \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}g\mathfrak{D}_e(K)$. Therefore if K is $\mathfrak{N}geo$ set, $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}g\mathfrak{D}_e(K)$. Conversely, suppose $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}g\mathfrak{D}_e(K)$ from (iii), $\mathfrak{N}gecl(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$. That is $K \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K)$ by Theorem 2.21 implies $K \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}g\mathfrak{D}_e(K)$ by hypothesis. Therefore $K = \mathfrak{N}geint(K)$ and hence K is $\mathfrak{N}geo$ set.

(vii) $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}gecl(K) - \mathfrak{N}geint(K) = \mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K)) = \mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - K)$.

(viii) $\mathfrak{N}ge\mathfrak{F}r(U - K) = \mathfrak{N}gecl(U - K) - \mathfrak{N}geint(U - K) = (U - \mathfrak{N}geint(K)) - (U - \mathfrak{N}gecl(K)) = \mathfrak{N}gecl(K) - \mathfrak{N}geint(K) = \mathfrak{N}ge\mathfrak{F}r(K)$.

(ix) A subset K of U is $\mathfrak{N}gec$ iff $K = \mathfrak{N}gecl(K)$. Consider $\mathfrak{N}gecl(\mathfrak{N}ge\mathfrak{F}r(K)) = \mathfrak{N}gecl(\mathfrak{N}gecl(K) - \mathfrak{N}geint(K)) = \mathfrak{N}gecl(\mathfrak{N}gecl(K) \cap (U - \mathfrak{N}geint(K))) = \mathfrak{N}gecl(\mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - K)) \subseteq \mathfrak{N}gecl(\mathfrak{N}gecl(K)) \cap \mathfrak{N}gecl(\mathfrak{N}gecl(U - K)) = \mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - K) = \mathfrak{N}ge\mathfrak{F}r(K)$, by (ii), $\mathfrak{N}gecl(\mathfrak{N}\ ge\mathfrak{F}r(K)) \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$. But $\mathfrak{N}ge\mathfrak{F}r(K) \subseteq \mathfrak{N}gecl(\mathfrak{N}\ ge\mathfrak{F}r(K))$ is always true. Therefore $\mathfrak{N}gecl(\mathfrak{N}ge\mathfrak{F}r(K)) = \mathfrak{N}ge\mathfrak{F}r(K)$ and hence $\mathfrak{N}ge\mathfrak{F}r(K)$ is $\mathfrak{N}gec$ set.

(x) $K - \mathfrak{N}ge\mathfrak{F}r(K) = K \cap (U - \mathfrak{N}ge\mathfrak{F}r(K)) = K \cap (U - (\mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - K))) = K \cap ((U - \mathfrak{N}gecl(K)) \cup (U - \mathfrak{N}gecl(U - K))) = (K \cap (U - \mathfrak{N}gecl(K))) \cup (K \cap (U - \mathfrak{N}gecl(U - K))) = \phi \cup (K \cap \mathfrak{N}geint(K)) = \mathfrak{N}geint(K)$.

(xi) If K is both $\mathfrak{N}geo$ and $\mathfrak{N}gec$ set, then $K = \mathfrak{N}geint(K)$ and $K = \mathfrak{N}gecl(K)$. Now $\mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}gecl(K) - \mathfrak{N}geint(K) = K - K = \phi$. Conversely, $\mathfrak{N}ge\mathfrak{F}r(K) = \phi$ implies, $\mathfrak{N}gecl(K) - \mathfrak{N}geint(K) = \phi$ which implies, $\mathfrak{N}gecl(K) - \mathfrak{N}geint(K) \subseteq K$. That is, $\mathfrak{N}gecl(K) \subseteq K$. But, $K \subseteq \mathfrak{N}gecl(K)$ is always true. Therefore $K = \mathfrak{N}gecl(K)$. Hence K is $\mathfrak{N}gec$ set. Again $\mathfrak{N}ge\mathfrak{F}r(K) = \phi$ implies $\mathfrak{N}gecl(K) - \mathfrak{N}geint(K) = \phi$ which implies $\mathfrak{N}gecl(K) = \mathfrak{N}geint(K)$ implies $K \cup \mathfrak{N}g\mathfrak{D}_e(K) = \mathfrak{N}geint(K)$ which implies $K \subseteq \mathfrak{N}ge\ int(K)$. But $\mathfrak{N}geint(K) \subseteq K$ is always true. Therefore $K = \mathfrak{N}geint(K)$. Hence K is $\mathfrak{N}geo$ set.

(xii) Now, $\mathfrak{N}ge\mathfrak{F}r(\mathfrak{N}geint(K)) = \mathfrak{N}gecl(\mathfrak{N}geint(K)) - \mathfrak{N}geint(\mathfrak{N}geint(K)) \subseteq \mathfrak{N}gecl(K) - \mathfrak{N}geint(K)$ as $\mathfrak{N}geint(K) \subseteq K$. This implies $\mathfrak{N}ge\mathfrak{F}r(\mathfrak{N}geint(K)) \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(xiii) Consider $U - \mathfrak{N}ge\mathfrak{F}r(K) = U - (\mathfrak{N}gecl(K) - \mathfrak{N}geint(K)) = (U - \mathfrak{N}gecl(K)) \cup \mathfrak{N}geint(K) = \mathfrak{N}geint(U - K) \cup \mathfrak{N}geint(K)$.

(xiv) Now $\mathfrak{N}ge\mathfrak{F}r(\mathfrak{N}gecl(K)) = \mathfrak{N}gecl(\mathfrak{N}gecl(K)) - \mathfrak{N}$

$geint(\mathfrak{N}gecl(K)) = \mathfrak{N}gecl(\mathfrak{N}gecl(K)) \cap (U - \mathfrak{N}geint(\mathfrak{N}gecl(K))) = \mathfrak{N}gecl(K) \cap \mathfrak{N}gecl(U - \mathfrak{N}gecl(K))$

Also $K \subseteq \mathfrak{N}gecl(K) \Rightarrow U - \mathfrak{N}gecl(K) \subseteq U - K \Rightarrow \mathfrak{N}gecl(U - \mathfrak{N}gecl(K)) \subseteq \mathfrak{N}gecl(U - K)$ sub in (4), $\mathfrak{N}ge\mathfrak{F}r(\mathfrak{N}gecl(K)) \subseteq \mathfrak{N}gecl(K) - \mathfrak{N}gecl(U - K) = \mathfrak{N}ge\mathfrak{F}r(K)$. Thus $\mathfrak{N}ge\mathfrak{F}r(\mathfrak{N}gecl(K)) \subseteq \mathfrak{N}ge\mathfrak{F}r(K)$.

(xv) From (iii), $\mathfrak{N}gecl(K) = \mathfrak{N}geint(K) \cup \mathfrak{N}ge\mathfrak{F}r(K) \subseteq K \cup \mathfrak{N}ge\mathfrak{F}r(K)$ as $\mathfrak{N}geint(K) \subseteq K$. Also from (iii), $\mathfrak{N}ge\mathfrak{F}r(K) \subseteq \mathfrak{N}gecl(K)$ and $K \subseteq \mathfrak{N}gecl(K)$ is always true. Therefore $K \cup \mathfrak{N}ge\mathfrak{F}r(K) \subseteq \mathfrak{N}gecl(K)$. It follows that, $K \cup \mathfrak{N}ge\mathfrak{F}r(K) = \mathfrak{N}gecl(K)$.

The Theorem 4.2 is satisfied by $\mathfrak{N}g\mathscr{M}\mathfrak{F}r$ of a set K.

Conclusion

In this paper, we have studied many interesting notions on various forms of nano generalized e and nano generalized $\mathscr{M}$ open sets.

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