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Separation axioms in ideal bitopological spaces

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Abstract
In this paper, we introduce and study (i,j)-semi-$\mathscr{I}$-R_0 and (i,j)-semi-$\mathscr{I}$-R_1 spaces. Also we obtain several characterizations of this axioms.

Keywords
Ideal bitopological spaces, (i,j)-semi-$\mathscr{I}$-closed set, (i,j)-semi-$\mathscr{I}$-open set, (i,j)-semi-$\mathscr{I}$-closure, (i,j)-semi-$\mathscr{I}$-kernal.

AMS Subject Classification
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Contents

1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathasamy [10]. An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subset A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. Given a bitopological space (X,τ_1,τ_2) with an ideal $\mathscr{I}$ on X and if $\mathscr{P}(X)$ is the set of all subsets of X, a set operator $(.)_i^\star$: $\mathscr{P}(X) \to \mathscr{P}(X)$, called the local function [10] of A with respect to τ_i and $\mathscr{I}$, is defined as follows: for $A \subset X$, $A_i^\star(\tau_i,\mathscr{I}) = \{x \in X | U \cap A \notin \mathscr{I}$ for every $U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i | x \in U\}$. Observe additionally that τ_i-$\mathrm{Cl}^\star(A) = A \cup A_i^\star(\tau_i,\mathscr{I})$ defines a Kuratowski closure operator for $\tau^\star(\mathscr{I})$, when there is no chance of confusion, $A_i^\star(\mathscr{I})$ is denoted by $A_i^\star$ and τ_i-$\mathrm{Int}^\star(A)$ denotes the interior of A in $\tau_i^\star(\mathscr{I})$. In this paper, we introduce and study (i,j)-semi-$\mathscr{I}$-R_0 and (i,j)-semi-$\mathscr{I}$-R_1 spaces. Also we obtain several characterizations of this axioms.

2. Preliminaries

Let A be a subset of a bitopological space (X,τ_1,τ_2). We denote the closure of A and the interior of A with respect to τ_i by τ_i-$\mathrm{Cl}(A)$ and τ_i-$\mathrm{Int}(A)$, respectively.

Definition 2.1. *[1] A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is said to be (i,j)-semi-$\mathscr{I}$-open [1] if $A \subset \tau_j$-$\mathrm{Cl}^\star(\tau_i$-$\mathrm{Int}(A))$.*

The complement of an (i,j)-semi-$\mathscr{I}$-open set is called an (i,j)-semi-$\mathscr{I}$-closed set.

Definition 2.2. *[1] The intersection (resp. union) of all (i,j)-semi-$\mathscr{I}$-closed (resp. (i,j)-semi-$\mathscr{I}$-open) sets of X containing (resp. contained in) $A \subset X$ is called the (i,j)-semi-$\mathscr{I}$-closure (resp. (i,j)-semi-$\mathscr{I}$-interior) of A and is denoted by (i,j)-$s\mathscr{I}\,\mathrm{Cl}(A)$ (resp. (i,j)-$s\mathscr{I}\,\mathrm{Int}(A)$). The the intersection of all (i,j)-semi-$\mathscr{I}$-open sets of X containing A is called the (i,j)-semi-$\mathscr{I}$-kernal of A and is denoted by (i,j)-$s\mathscr{I}\,\mathrm{Ker}(A)$.*

Definition 2.3. *[7] An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is said to be*

1. *(i,j)-semi-$\mathscr{I}$-T_0 if for every pair of distinct points in X, there exists an (i,j)-semi-$\mathscr{I}$-open set of X containing one of the points but not the other.*
2. *(i,j)-semi-$\mathscr{I}$-T_1 if for every pair of distinct points x, y of X, there exists a pair of (i,j)-semi-$\mathscr{I}$-open sets one containing x but not y and the other containing y but not x.*

3. *(i,j)-semi-$\mathscr{I}$-T_2 if for every pair of distinct points x, y of X, there exists a pair of disjoint (i,j)-semi-$\mathscr{I}$-open sets, one containing x and the other containing y.*

3. On (i,j)-semi-$\mathscr{I}$-R_0 and (i,j)-semi-$\mathscr{I}$-R_1 spaces

Definition 3.1. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is said to be (i,j)-semi-$\mathscr{I}$-R_0 if for every (i,j)-semi-$\mathscr{I}$-open set of X contains the (i,j)-semi-$\mathscr{I}$-closure of each of its singletons.*

Definition 3.2. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is said to be (i,j)-semi-$\mathscr{I}$-symmetric if for each $x,y\in X$, $x\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ implies $y\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$.*

Theorem 3.3. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 if, and only if it is (i,j)-semi-$\mathscr{I}$-symmetric.*

Proof. Assume that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. Let $x\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ and U be any (i,j)-semi-$\mathscr{I}$-open set such that $y\in U$. Then by hypothesis, $x\in U$. Therefore, every (i,j)-semi-$\mathscr{I}$-open set which contains y contains x. Hence, $y\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Conversely, let U be an (i,j)-semi-$\mathscr{I}$-open set and $x\in U$. If $y\notin U$, then $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$, and thus by assumption, $y\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Therefore, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset U$, and hence, $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. □

Theorem 3.4. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_1 if, and only if $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_0 and (i,j)-semi-$\mathscr{I}$-R_0*

Proof. Let $x,y\in X$ and $x\neq y$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_0, we may assume without loss of generality that $x\in G\subset X\backslash\{y\}$ for some (i,j)-semi-$\mathscr{I}$-open set G. Thus, $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$, and by Theorem 3.3, $y\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Therefore, $X\backslash(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ is an (i,j)-semi-$\mathscr{I}$-open set containing y but not x. Hence, $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_1. The converse is clear. □

Proposition 3.5. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 space;*
2. *If for any $F\in(i,j)$-$S\mathscr{I}C(X)$, $x\notin F$, then $F\subset U$ and $x\notin U$ for some $U\in(i,j)$-$S\mathscr{I}O(X)$;*
3. *If for any $F\in(i,j)$-$S\mathscr{I}C(X)$ such that $x\notin F$, then $F\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})=\emptyset$;*
4. *If for any two distinct points $x,y\in X$, then either (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})=(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ or (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$.*

Proof. (1)⇒(2): Let $F\in(i,j)$-$S\mathscr{I}C(X)$ and $x\notin F$. Then by (1) (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset X\backslash F$. Set $U=X\backslash(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$, then $U\in(i,j)$-$S\mathscr{I}O(X)$ with $F\subset U$ and $x\notin U$.
(2)⇒(3): Let $F\in(i,j)$-$S\mathscr{I}C(X)$ such that $x\notin F$. Then by (2), there exists $U\in(i,j)$-$S\mathscr{I}O(X)$ such that $F\subset U$ and $x\notin U$. Since $U\in(i,j)$-$S\mathscr{I}O(X)$, $U\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})=\emptyset$ and $F\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})=\emptyset$.
(3)⇒(4): Suppose that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ for the distinct points $x,y\in X$. Then there exists $z\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ such that $z\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ (or $z\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ such that $z\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$). Then there exists $V\in(i,j)$-$S\mathscr{I}C(X,z)$ such that $y\notin V$, hence $x\in V$. Therefore, $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. By (3), we obtain (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$. The proof for the other case is similar.
(4)⇒(1): Let $V\in(i,j)$-$S\mathscr{I}O(X,x)$. For each $y\notin V$, we have $x\neq y$ and $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. This shows that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Hence (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$ for each $y\in X\setminus V$ and (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(\bigcup_{y\in X\setminus V}(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\}))=\emptyset$. Since $V\in(i,j)$-$S\mathscr{I}O(X)$ and $y\in X\setminus V$, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{y\})\subset X\setminus V$ and hence $X\setminus V=\bigcup_{y\in X\setminus V}(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Therefore, we obtain $(X\backslash V)\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})=\emptyset$ and hence (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset V$. Then $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. □

Theorem 3.6. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 if, and only if for any $x,y\in X$, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ implies (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$.*

Proof. Suppose that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 and x, $y\in X$ such that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. There exists $z\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ such that $z\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ (or $z\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ such that $z\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$). Since $z\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$, there exists $V\in(i,j)$-$S\mathscr{I}O(X,z)$ such that $y\notin V$. But $z\in(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ so $x\in V$. Then $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Hence $x\in X\setminus(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})\in(i,j)$-$S\mathscr{I}O(X)$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0, we have (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset X\setminus(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Hence (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$. The proof for otherwise is similar. Conversely, let $V\in(i,j)$-$S\mathscr{I}O(X,x)$. We will show that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset V$. Let $y\notin V$, that is, $y\in X\setminus V$. Then $x\neq y$ and $x\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. This shows that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. By assumption, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})=\emptyset$. Hence $y\notin(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ and therefore (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\subset V$. Hence $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. □

Theorem 3.7. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 if, and only if for any points x and y in X, (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$ implies (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\})\cap(i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})=\emptyset$.*

Proof. Suppose $(X,\tau_1,\tau_2,\mathscr{I})$ is an (i,j)-semi-$\mathscr{I}$-R_0 space. Then for any points x and y in X, if (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$, then (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})\neq(i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Assume that $z \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$. By $z \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$, $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{z\})$. Thus by Theorem 3.6, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{z\})$. Similarly, we have (i,j)-$s\mathscr{I}\operatorname{Cl}(\{y\}) = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{z\}) = ((i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$, a contradiction. Hence (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\}) = \emptyset$. Conversely, let (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \neq (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$ implies (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\}) = \emptyset$. Assume that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Then (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \neq (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$, and therefore by assumption, (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\}) = \emptyset$. Now if $z \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$, then $x \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{z\})$, and therefore, (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{z\}) \neq \emptyset$. By hypothesis, (i,j)-$s\mathscr{I}(\{x\}) = (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{z\})$. Thus $z \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ implies (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) = (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{z\}) = (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{y\})$, a contradiction. Therefore (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ implies that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \cap (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\}) = \emptyset$, and Theorem 3.6, $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. ☐

Theorem 3.8. *For an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$, the following statements are equivalent:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0.*
2. *For any nonempty subset A of X and $G \in (i,j)$-$S\mathscr{I}O(X)$ such that $A \cap G \neq \emptyset$, there exists $F \in (i,j)$-$S\mathscr{I}C(X)$ such that $A \cap F \neq \emptyset$ and $F \subset G$.*
3. *For any $G \in (i,j)$-$S\mathscr{I}O(X)$, $G = \cup\{F : F \in (i,j)$-$S\mathscr{I}C(X)$, $F \subset G\}$.*
4. *For any $F \in (i,j)$-$S\mathscr{I}C(X)$, $F = \cap\{G : G \in (i,j)$-$S\mathscr{I}O(X)$, $F \subset G\}$.*
5. *For any $x \in X$, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$.*

Proof. (1)⇒(2): Let A be a nonempty set of X and $G \in (i,j)$-$S\mathscr{I}O(X)$ such that $A \cap G \neq \emptyset$. Then there exists $x \in A \cap G$. Since $x \in G \in (i,j)$-$S\mathscr{I}O(X)$, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset G$. Set $F = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Then $F \in (i,j)$-$S\mathscr{I}C(X)$, $F \subset G$ and $A \cap F \neq \emptyset$.
(2)⇒(3): Let $G \in (i,j)$-$S\mathscr{I}O(X)$, then $G \supset \cup\{F : F \in (i,j)$-$S\mathscr{I}C(X)$, $F \subset G\}$. Let x be any point of G. Then there exists $F \in (i,j)$-$S\mathscr{I}C(X)$ such that $x \in F$ and $F \subset G$. Therefore, $x \in F \subset \cup\{F : F \in (i,j)$-$S\mathscr{I}C(X)$, $F \subset G\}$, and hence $G = \cup\{F : F \in (i,j)$-$S\mathscr{I}C(X)$, $F \subset G\}$.
(3)⇒(4): This is obvious.
(4)⇒(5): Let x be any point of X and $y \notin (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$. Then there exists $V \in (i,j)$-$S\mathscr{I}O(X,x)$ any $y \notin$ V; hence (i,j)-$s\mathscr{I}\operatorname{Cl}(\{y\}) \cap V = \emptyset$. By (4), $\cap\{G : G \in (i,j)$-$S\mathscr{I}O(X)$, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{y\}) \subset G\}$, and there exists $G \in (i,j)$-$S\mathscr{I}O(X)$ such that $x \notin G$ and (i,j)-$s\mathscr{I}\operatorname{Cl}(\{y\}) \subset G$. Therefore, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \cap G = \emptyset$ and $y \notin (i,j)$-$s\mathscr{I}\operatorname{Cl}((i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})) = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Consequently, we obtain (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$.
(5)⇒(1): Let $G \in (i,j)$-$S\mathscr{I}O(X,x)$. If $y \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$, then $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ and so $y \in G$. This implies that (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset G$. Therefore, $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset G$. This shows that $(X,\tau_1,\tau_2,\mathscr{I})$ is an (i,j)-semi-$\mathscr{I}$-R_0 space. ☐

Corollary 3.9. *For an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 if, and only if (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) =(i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$ for each $x \in X$.*

Proof. Suppose $(X,\tau_1,\tau_2,\mathscr{I})$ is an (i,j)-semi-$\mathscr{I}$-R_0 space. By Theorem 3.8, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$ for each $x \in X$. Let $y \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$. Then we have $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ and by Theorem 3.6 (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Therefore, $y \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ and hence (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. This shows that (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) =(i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$. The converse follows from Theorem 3.8. ☐

Theorem 3.10. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0.*
2. *$x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\}) \Leftrightarrow y \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ for any points x and y in X.*

Proof. (1)⇒(2): Assume that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 and $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Then (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$. Hence $y \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. The other part is similar.
(2)⇒(1): Let $x \in U \in (i,j)$-$S\mathscr{I}O(X,x)$. If $y \notin U$, then $x \notin (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ and hence $y \notin (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ (by (2)). Thus (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset U$. Hence $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. ☐

Theorem 3.11. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0.*
2. *If F is an (i,j)-semi-$\mathscr{I}$-closed subset of X, then $F = (i,j)$-$s\mathscr{I}\operatorname{Ker}(F)$.*
3. *If F is an (i,j)-semi-$\mathscr{I}$-closed subset of X and $x \in F$, then (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset F$.*
4. *If $x \in X$, then (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$.*

Proof. (1)⇒(2): Let F be an (i,j)-semi-$\mathscr{I}$-closed subset of X and $x \notin F$. Thus $X \backslash F \in (i,j)$-$S\mathscr{I}O(X,x)$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0, (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \subset X \setminus F$. Thus (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\}) \cap F = \emptyset$ and $x \notin (i,j)$-$s\mathscr{I}\operatorname{Ker}(F)$. Therefore, (i,j)-$s\mathscr{I}\operatorname{Ker}(F) = F$.
(2)⇒(3): If $A \subset B$, then (i,j)-$s\mathscr{I}\operatorname{Ker}(A) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(B)$. Then (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Ker}(F) = F$.
(3)⇒(4): Since $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ and (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})$ is (i,j)-semi-$\mathscr{I}$-closed, (i,j)-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$.
(4)⇒(1): If $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$, then $y \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\})$. Since $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$ and (i,j)-$s\mathscr{I}\operatorname{Cl}(\{x\})$ is (i,j)-semi-$\mathscr{I}$-closed, by (4), we obtain $y \in (i,j)$-$s\mathscr{I}\operatorname{Ker}(\{x\}) \subset (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. Then $x \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{y\})$ implies that $y \in (i,j)$-$s\mathscr{I}\operatorname{Cl}(\{x\})$. So $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. ☐

Definition 3.12. *A net $\{x_\alpha\}_{\alpha\in\Lambda}$ in an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is called (i,j)-semi-$\mathscr{I}$-convergent to a point x in X if for every $U \in (i,j)$-$S\mathscr{I}O(X,x)$, there exists $\alpha_0 \in \Lambda$ such that $x_\alpha \in U$ for each $\alpha \geq \alpha_0$.*

Lemma 3.13. *Let $(X,\tau_1,\tau_2,\mathscr{I})$ be an ideal bitopological space and let x and y any two points in X such that every net in X (i,j)-semi-$\mathscr{I}$-converging to y (i,j)-semi-$\mathscr{I}$-converges to x. Then $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$.*

Proof. Suppose $x_n = y$ for each $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a net in X that (i,j)-semi-$\mathscr{I}$-convergence to y. Thus by assumption, (i,j)-semi-$\mathscr{I}$-converges to x. So $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. □

Theorem 3.14. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0.*
2. *If $x, y \in X$, then $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$ if, and only if every net in X (i,j)-semi-$\mathscr{I}$-converging to y also (i,j)-semi-$\mathscr{I}$-converges to x.*

Proof. (1)⇒(2): Let $x, y \in X$ such that $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ (i,j)-semi-$\mathscr{I}$-converges to y. Since $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$, by Theorem 3.3, $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Conversely, let $x, y \in X$ such that every net in X (i,j)-semi-$\mathscr{I}$-converging to y (i,j)-semi-$\mathscr{I}$-converges to x. Then $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. By Theorem 3.10, $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$.
(2)⇒(1): Assume that x and y are any two points of X such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \cap (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \neq \emptyset$. Let $z \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \cap (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. There exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$ (i,j)-semi-$\mathscr{I}$-converges to z. Since $z \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$, $\{x_\alpha\}_{\alpha\in\Lambda}$ also (i,j)-semi-$\mathscr{I}$-converges to y. Hence by (2) $z \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Therefore (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{z\}) \subset (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ ($\star$). So $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{z\})$ gives (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{z\})$ ($\star\star$). Hence from ($\star$) and ($\star\star$), (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{z\})$. Similarly it can be shown that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{z\})$ by taking the net in (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. So (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. □

Definition 3.15. *An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is said to be (i,j)-semi-$\mathscr{I}$-R_1 if for each points x and y of X such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$, there exist disjoint (i,j)-semi-$\mathscr{I}$-open subsets of X, say, U and V such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset V$.*

Proposition 3.16. *Every (i,j)-semi-$\mathscr{I}$-R_1 space is (i,j)-semi-$\mathscr{I}$-R_0.*

Proof. Let $U \in (i,j)$-$S\mathscr{I}O(X,x)$. If $y \notin U$, then $x \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. So (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1, there exists an (i,j)-semi-$\mathscr{I}$-open set V_y such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset V_y$ and $x \notin V_y$, implies that $y \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$. Hence (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$. Then $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. □

Theorem 3.17. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_2,*
2. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1 and (i,j)-semi-$\mathscr{I}$-T_1,*
3. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1 and (i,j)-semi-$\mathscr{I}$-T_0.*

Proof. (1)⇒(2): Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_2, then it is (i,j)-semi-$\mathscr{I}$-T_1. If $x, y \in X$ such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$, then $x \neq y$ and there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Hence by Theorem 3.4, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = \{x\} \subset U$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) = \{y\} \subset V$. Hence $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1.
(2)⇒(3): Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_1, then it is (i,j)-semi-$\mathscr{I}$-T_0.
(3)⇒(1): Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1 and (i,j)-semi-$\mathscr{I}$-T_0, then by Proposition 3.16, $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 and (i,j)-semi-$\mathscr{I}$-T_0. Hence by Theorem 3.4, $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_1. Let $x, y \in X$ such that $x \neq y$. Then (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = \{x\} \neq \{y\} = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1, there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U and V such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = \{x\} \subset U$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) = \{y\} \subset V$. Hence we have $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_2 and thus by Theorem 3.3 (X,τ) is an (i,j)-semi-$\mathscr{I}$-R_0 space. □

Corollary 3.18. *For an (i,j)-semi-$\mathscr{I}$-R_1 space $(X,\tau_1,\tau_2,\mathscr{I})$, the following statements are equivalent:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_2.*
2. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_1.*
3. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_0.*

Theorem 3.19. *For an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1 if, and only if $x \in X \backslash (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ implies that x and y have disjoint (i,j)-semi-$\mathscr{I}$-neighbourhoods.*

Proof. Let $x \in X \backslash (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ and x and y have disjoint (i,j)-semi-$\mathscr{I}$-neighbourhoods. Conversely, first we show that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. Let U be an (i,j)-semi-$\mathscr{I}$-open set and $x \in U$. Suppose that $y \notin U$. Then, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \cap U = \emptyset$ and $x \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$ and $U_x \cap U_y = \emptyset$. Hence, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(U_x)$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(x) \cap U_y \subset (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{U_x\}) \cap U_y = \emptyset$. Therefore, $y \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$. Consequently, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0. Next, we show that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1. Suppose that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Assume that there exists $z \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$ such that $z \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$. Since $z \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$, $x \in V_z$. Since $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset V_z$, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset V_y$ and $V_z \cap V_y = \emptyset$. Then (X,τ_1,τ_2) is (i,j)-semi-$\mathscr{I}$-R_1 and thus by Theorem 3.3 (X,τ) is an (i,j)-semi-$\mathscr{I}$-R_0 space. □

Theorem 3.20. *The following statements are equivalent for an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$:*

1. *$(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1.*
2. *For each x, $y \in X$ one of the following holds:*
   (a) *If U is (i,j)-semi-$\mathscr{I}$-open, then $x \in U$ if, and only if $y \in U$.*
   (b) *there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$.*
3. *If x, $y \in X$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$, then there exist (i,j)-semi-$\mathscr{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.*

Proof. (1)⇒(2): Let $x, y \in X$. Then (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ or (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. If (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ and U is (i,j)-semi-$\mathscr{I}$-open, then $x \in U$ implies $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $y \in U$ implies $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset U$. Thus consider the case that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U and V such that $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\}) \subset V$.
(2)⇒(3): Let $x, y \in X$, (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then $x \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ or $y \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$, say $x \notin (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$. Then there exists an (i,j)-semi-$\mathscr{I}$-open set A such that $x \in$ A and $y \notin A$. Then by (2) there exist disjoint (i,j)-semi-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X \setminus V$ and $F_2 = X \setminus U$ are (i,j)-semi-$\mathscr{I}$-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.
(3)⇒ (1): We shall first show that $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 space. Let U be an (i,j)-semi-$\mathscr{I}$-open set such that $x \in U$. We claim that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \subset U$. For suppose $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \cap (X \backslash U)$. Then (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$ (if (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{x\}) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{y\})$, then $y \in U$) and hence by (3), there exist (i,j)-semi-$\mathscr{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Then $y \in F_2 \backslash F_1 = X \backslash F_1 \in (i,j)$-$S\mathscr{I}O(X)$ and $x \notin X \backslash F_1$, a contradicts the fact that $y \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{x\})$. Hence $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_0 space. Let $p, q \in X$ be such that (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{p\}) \neq (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(\{q\})$. Then by the given condition there exist (i,j)-semi-$\mathscr{I}$-closed sets H_1 and H_2 such that $p \in H_1$, $q \notin H_1$, $q \in H_2$, $p \notin H_2$ and $X = H_1 \cup H_2$. Thus $p \in H_1 \backslash H_2$ and $q \in H_2 \backslash H_1$, where $H_1 \backslash H_2$ and $H_2 \backslash H_1$ are disjoint (i,j)-semi-$\mathscr{I}$-open sets. Hence (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{p\}) \subset H_1 \backslash H_2$ and (i,j)-$s\mathscr{I}\,\mathrm{Cl}(\{q\}) \subset H_2 \backslash H_1$. Hence $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-R_1 space. and thus by Theorem 3.3 (X,τ) is an (i,j)-semi-$\mathscr{I}$-R_0 space. □

In view of Theorems 3.17 and 3.20, it now follows that

Theorem 3.21. *A bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-T_2 if, and only if for each x, $y \in X$ such that $x \neq y$, there exist (i,j)-semi-$\mathscr{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.*

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