



On cocoloring of corona of graphs

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Abstract

A k -cocoloring of a graph G is a partition of the vertex set into k subsets such that each set induces either a clique or an independent set in G . The cochromatic number $z(G)$ of a graph G is the least k such that G has a k -cocoloring of G . In this paper, we give exact bounds of the cochromatic number for the corona product of Path graph with $P_n, K_n, C_n, K_{1,n}$.

Keywords

Cocoloring, Cochromatic number, Corona.

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1. Introduction

In this paper, a graph $G = (V, E)$ is considered to be finite, simple and undirected. For definition not give in this paper, readers may kindly refer to [[4],[5]]. For $S \subset V, \langle \{S\} \rangle$ denotes the subgraph of G induced by S . A coloring of vertices of a graph $G = (V, E)$ is a partition $\{I_1, I_2, \dots, I_r\}$ of V such that for each $1 \leq j \leq r, I_j$ is an independent set. The chromatic number $\chi(G)$ the minimum size of such a partition. Clique cover number $\theta(G)$ of a graph G is the minimum size of a partition into cliques. The above two partition is based only one condition that either all the sets are independent or all the sets are cliques. Combining these two types, a new partition was introduced by Lesinak and Straight [[6]]. They call this as a cocoloring. A cocoloring of G is a partition of V into independent sets and cliques. In other words, if $\{I_1, I_2, \dots, I_r, C_1, C_2, \dots, C_s\}$ is a partition of V such that for each $i, 1 \leq i \leq r, I_i$ is an independent set and for each $j, 1 \leq j \leq s, C_j$ is a clique. The smallest size of a cocoloring is the cochromatic number $z(G)$. Research works were done on finding the bounds of the cochromatic number of various classes of graphs. In the case graph product, cocoloring of lexicographic product some classes of graph were studied under certain conditions. In graph product GpH will be taken

as Cartesian product $V(G) \times V(H)$ of the vertex sets of G and H . Corona is not exactly a graph product. Here $|V(G)|$ copies of the second graph H is taken and each vertex of G is adjacent to all the vertices of one of the copies of H . This graph construction was introduced by Harary and Frucht in 1970[[2]]. In this paper, we have studied the corona of any two graphs of the graph families of path, cycle, complete graph and star. One general upper and lower bound of a particular case and exact bounds of the products of graphs mentioned above were also obtained.

2. Preliminaries

In this section, we give some prior results.

Proposition 2.1 ([3]). For any graph $G, z(G) \leq \min\{\chi(G), \theta(G)\}$.

Proposition 2.2 ([3]).

i. For $n \geq 3, z(P_n) = 2$

ii. For $n \geq 1, z(K_n) = 1$

iii. For $n \geq 4, z(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

iv. For $m, n \geq 1, z(K_{m,n}) = 2$

3. Main Results

In this section, we have obtained (i) the lower and upper bound of cochromatic number of corona of a trivial graph with

any non trivial incomplete graph, (ii) an upper bound of $z(G)$ based on the chromatic number of G is obtained where G is corona of a complete graph with any non trivial incomplete connected graph. Some exact bounds of cochromatic number of corona of combinations of any two graphs of K_m, P_m, C_m and $K_{1,n}$ are obtained. Throughout this paper we follow the notations given below.

Let G and H be any two graphs. Then in $G \circ H$

- (i) G and H are of order m and n respectively.
- (ii) H^i denotes the i^{th} copy of H
- (iii) $V(G) = \{u_1, u_2, \dots, u_m\}, V(H) = \{v_1, v_2, \dots, v_n\}, v_i = V(H^i) = \{v_{i1}, v_{i2}, \dots, v_{in}\}$ where $1 \leq i \leq m$
- (iv) If I is an independent set and C is clique in H , then $I^i C^i$ denote the corresponding independent and clique in H respectively.
- (v) The set V and E respectively denote $V(G \circ H)$ and $E(G \circ H)$

observation 3.1. If I is an independent set in H , then $\bigcup_{j=1}^m I_j$ is an independent set in $G \circ H$.

Theorem 3.2. Let $H(\neq K_n)$ be any non-trivial connected graph. Then $z(H) \leq z(K_1 \circ H) \leq z(H) + 1$.

Proof. Since H is a non-trivial connected incomplete graph $z(H) \geq 2$.

Let $z(H) = k$. Then the minimum cocoloring is obtained by one of the following

- (i) k independent sets or
- (ii) k cliques
- (iii) l independent sets and r cliques such that $l + r = k$.

we discuss the above three cases.

Case (i):

The minimum cocoloring is obtained only by partitioning into k -independent sets. Then $z(H) = \chi(H)$.

Let $S = \{I_1, I_2, \dots, I_k\}$ be a minimum cocoloring then there can be atmost one I_i is singleton, otherwise union of two such I_i and I_j gives either $2K_1$ or K_2 . Hence $z(H)$ can be reduced to $k - 1$, a contradiction.

Subcase (i) : $|I_i| > 1 \forall i$

Since u_1 is adjacent to each v_j 's, $\langle I_i \cup \{u_1\} \rangle = K_1, |I_i|$, which is neither a null graph nor a clique. Hence u_1 cannot be combined with any I_i . Hence $S \cup \{u_1\}$ is a minimum cocoloring of $K_1 \circ H$.

$$\therefore z(K_1 \circ H) = z(H) + 1$$

Subcase (ii): $|I_i| = 1$ for some i .

Then $\langle I_i \cup \{u_i\} \rangle = K_2$. Hence $\{I_1, I_2, \dots, I_i \cup \{u_i\}, I_k\}$ is a coloring of $K_1 \circ H$

$$\therefore z(K_1 \circ H) = z(H).$$

Case (ii): The minimum cocoloring is obtained only by partitioning into k -cliques. Then $z(H) = \theta(H)$

Let $s = \{C_1, C_2, \dots, C_s\}$ be a minimum cocoloring. Clearly $C_i \cup \{u_i\}$ induces a clique of size $|C_i| + 1$.

Then $\{C_1, C_2, \dots, C_i \cup \{u_i\}, \dots, C_s\}$ is a minimum clique partition of $K_1 \circ H$.

$$z(K_1 \circ H) = z(H).$$

Case (iii): The minimum cocoloring is obtained by partitioning into l -independent sets and r cliques, where $l, r \geq 1$. As in case (ii), u can be combined with any one of the r -cliques. Here $z(K_1 \circ H) = z(H)$. □

Corollary 3.3. If $(H \neq K_n)$ is any non-trivial connected graph. Then $z(K_1 \circ H) = z(H) + 1$ if and only if $z(H) = \chi(H) < \theta(H)$ such that in every chromatic partition, each set is of size atleast 2.

Theorem 3.4. For $m, n \geq 1, z(K_m \circ K_n) = \min\{m, n + 1\}$

Proof. For $1 \leq j \leq n$. Let $I_j = \{v_{1j}, v_{2j}, \dots, v_{ij}, \dots, v_{mj}\}$. Clearly I_j contains exactly one vertex from each v_i and hence independent.

Claim: I_j is maximal.

For any $i, 1 \leq i \leq m, u_i v_{ij} \in E$. Hence no u_i can be added to I_j . Consider any v_{rs} . By notation $v_{rs} \in V_r$. I_j contains one element $v_{rj} \in V_r$. Hence no v_{rs} can be added to I_j . Hence I_j is maximal. As the factor graphs are complete, no other way an independent set can be formed. Hence I_j is maximum. Here a minimum cocoloring can be obtained in one of the following ways.

- (i) $\{u_i\} \cup V_i$ induces a clique of size $n + 1$. Hence V can be partitioned into m -cliques of size $n + 1$.
- (ii) $\{I_1, I_2, \dots, I_n, V(K_m)\}$ is a cocoloring containing n independent sets and a set inducing a clique.

$$\therefore z(K_m \circ K_n) = \min\{m, n + 1\}$$

□

Theorem 3.5. Let $H(\neq K_n)$ be any non-trivial connected graph with $z(H) = \chi(H)$ such that every minimum cocoloring contain only independent sets. Then for $m \geq 2, z(K_m \circ H) = z(H) + 1$

Proof. Let $z(H) = \chi(H) = k$ and I_1, I_2, \dots, I_k be a chromatic partition of H . Then $\{I_1^i, I_2^i, \dots, I_k^i\}$ is a chromatic partition of H^i . For $1 \leq r \leq k$

Let $J_r = \bigcup_{i=1}^m I_r^i$. Then J_r is an independent set in $K_m \circ H$, Whose union is $\bigcup_{i=1}^m V(H^i)$.

Here $\{J_1, J_2, \dots, J_k, V(K_m)\}$ is a cocoloring of $K_m \circ H$.

$$z(K_m \circ H) \leq z(H) + 1.$$

Suppose for some $j, |J_j| = 1$. Then $u_i \cup I_j$ induce a star. Then a cocoloring with respect to this partitioning gives a cocoloring with $k - 1$ independent sets and a clique.

$$z(K_m \circ H) \leq k - 1 + m.$$

$z(K_m \circ H) = k$ if and only if $m = 1$, a contradiction.

Hence any cocoloring containing clique has atleast $k - 1, m - 1$ elements.

$$\text{Hence } z(K_m \circ H) = z(H) + 1.$$

□



Corollary 3.6. *If H is any bipartite, $H \neq K_2$, then $z(K_m \circ H) = 3$.*

Theorem 3.7. *For $m \geq 2, n \geq 2$,*

$$z(P_m \circ P_n) = \begin{cases} 2, & \text{if } m = n = 2 \\ 3, & \text{otherwise} \end{cases}$$

Proof. For $m, n \geq 2, P_m \circ P_n$ is 3-chromatic.

$\therefore z(P_m \circ P_n) \leq 3$

If $m = n = 2$, then $P_m \circ P_n$ is obtained by joining $2K_3$ by an edge.

$\therefore z(P_m \circ P_n) = 2$.

Suppose $m \geq 3$ or $n \geq 3$. Removal of a maximal independent set gives a bipartite graph which is not K_2 . Removal of K_3 gives a 3-chromatic graph which is not K_3 . Hence $P_m \circ P_n$ cannot have a 2-cocoloring.

Hence $z(P_m \circ P_n) = 3$. □

Theorem 3.8. $z(P_m \circ C_n) = \begin{cases} 2, & \text{if } n \geq 3 \\ 3, & \text{if } n \text{ is even} \\ 4, & \text{if } n \text{ is odd} \end{cases}$

Proof. If $m = 2$ and $n = 3$. Then $P_m \circ C_n = K_2 \circ K_3$

By Theorem (3.4), $z(P_m \circ C_n) = 2$

For $m \geq 2$ or $n \geq 3$.

$$z(P_m \circ C_n) = \begin{cases} 3, & \text{if } n \text{ is even} \\ 4, & \text{if } n \text{ is odd} \end{cases}$$

By similar argument as in theorem (3.4), $z(P_m \circ C_n) \geq 3$.

Now, let $m \geq 2, n \geq 4$ and n is even then as $z(P_m \circ C_n) = 3$

$z(P_m \circ C_n) = 3$.
Suppose $m \geq 2, n \geq 4$ and n is odd. Suppose $P_m \circ C_n$ has a 3-cocoloring $\{I_1, I_2, I_3\}$. Clearly all I_i 's cannot be independent. To cover $\{u_i \cup V(c'_n)\}$ 3-cliques are needed. Hence all I_i 's cannot be cliques. Since $n \geq 5, |V(P_m \circ C_n)| \geq 10$.

Removal of $2K_3$, gives a graph which contain $2P_3$ as a subgraph, a contradiction.

Hence $P_m \circ C_n$ cannot have a 3-cocoloring with 2-cliques and one independent. Now removal of one clique leaves a 4-chromatic graph which is not K_4 which cannot be partite into two independent sets.

Hence $P_m \circ C_n$ cannot have a 3-cocoloring since $\chi(P_m \circ C_n) = 4$

$z(P_m \circ C_n) = 4$. □

Theorem 3.9. *For $m, n \geq 3, z(P_m \circ K_n) = \{m, n + 1\}$*

Proof. Since $\{u_i\} \cup K_n^i$ induce K_{n+1} and P_m is bipartite $\chi(P_m \circ K_n) = n + 1$. By similar discussion as in theorem(3.4), $P_m \circ K_n$ can be either partitioned into m -cliques of size $n + 1$ or into $n + 1$ independent sets of size m .

Hence $z(P_m \circ K_n) = \min\{m, n + 1\}$. □

Theorem 3.10. *For $m \geq 3, n \geq 2, \chi(C_m \circ P_n) = 3$*

Proof. For $m \geq 3, n \geq 2, \chi(C_m \circ P_n) = 3$.

Further $C_m \circ P_n$ has atleast 3 disjoint K_3 's. Hence $C_m \circ P_n$ cannot have 2-cocoloring. Hence the theorem. □

Theorem 3.11. $z(P_m \circ K_{1,n}) = 3$

Proof. In $P_m \circ K_{1,n}, \{u_i\} \cup K_{1,n}$ induces the complete 3-partite graph $K_{1,1,n}$. Since P_m is bipartite and $\chi(K_{1,1,n}) = 3$.

By similar argument as before, $z(P_m \circ K_{1,n}) = 3$. □

4. Conclusion

In this paper, a general lower and upper bound of cochromatic number of corona of a trivial graph with any non trivial incomplete graph is obtained. Also exact bounds of cochromatic number of corona of graphs G and H , where G and H are complete graph, Path, cycle or a star are obtained.

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