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On bipolar fuzzy rough continuous functions

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Abstract

The concept of bipolar fuzzy sets is an extension of fuzzy sets. Our aim is to define bipolar fuzzy rough topology by means of the lower and upper approximations on bipolar fuzzy sets, bipolar fuzzy rough image, bipolar fuzzy rough inverse image, bipolar fuzzy rough subspace and bipolar fuzzy rough continuous functions between these topologies. We construct some functions which are bipolar fuzzy rough continuous and further prove pasting lemma for bipolar fuzzy rough continuous mappings.

Keywords

Bipolar fuzzy rough image , bipolar fuzzy rough inverse image, bipolar fuzzy rough subspace, bipolar fuzzy rough continuous function.

AMS Subject Classification

03E72, 54A40.

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Contents

1. Introduction

Pawlak [11, 12] proposed the theory of rough sets. Keyun Qin and Pei [7] successfully compared fuzzy rough set models and fuzzy topologies on a finite universe. Mathew and John [3] established and developed topological structures on rough sets. Rough topology in terms of rough sets was introduced by Lellis Thivagar et al. [8]. In [4, 5, 10], the concept of fuzzy rough sets were studied by replacing crisp binary relations with fuzzy relations on the universe.

The concept of bipolar fuzzy sets was introduced by Zhang [13] and Muthuraj [9]. Anita shanthi et al.[2] proposed the notion of fuzzy rough continuous functions. They further introduced the concepts of Bipolar fuzzy rough set and bipolar fuzzy rough topology [1].

In this paper we aim to define bipolar fuzzy rough continuous function, construct some functions which are bipolar fuzzy rough continuous and further prove pasting lemma for bipolar fuzzy rough continuous mappings.

2. Bipolar fuzzy rough continuous function

In this section we define bipolar fuzzy rough image, bipolar fuzzy rough inverse image, bipolar fuzzy rough subspace, bipolar fuzzy rough continuous functions and also prove pasting lemma for bipolar fuzzy rough continuous mappings.

Definition 2.1. *Let $(U, BFR(A), \tau)$ and $(U^*, BFR(B), \tau)$ be two bipolar fuzzy rough topological spaces.*
$BFR(A) = (BF\underline{R}(A), BF\overline{R}(A)) \in \tau$ *and*
$BFR(B) = (BF\underline{R}(B), BF\overline{R}(B)) \in \tau'$.
If $f^- : BF\underline{R}^n(A) \to BF\underline{R}^n(B)$,
$f^+ : BF\underline{R}^p(A) \to BF\underline{R}^p(B)$,
$g^- : BF\overline{R}^n(A) \to BF\overline{R}^n(B)$,
$g^+ : BF\overline{R}^p(A) \to BF\overline{R}^p(B)$, *then a bipolar fuzzy rough mapping* $h = ((f^-, f^+), (g^-, g^+)) : BFR(A) \to BFR(B)$ *is defined as follows:*

$$h(BFR(A))(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} y, & \text{if } f^{-1}(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$h(BFR(A))$ is called bipolar fuzzy rough image of the bipolar fuzzy rough set $BFR(A)$.

Example 2.2. *Let $(U = \{x_1, x_2, x_3\}, BFR(A), \tau)$ and $(U^* = \{y_1, y_2, y_3\}, BFR(B), \tau')$ be two bipolar fuzzy rough topological spaces. f^-, f^+, g^-, g^+ are mappings defined as $((f^-, f^+)(x_i), (g^-, g^+)(x_i)) = y_i$.*

Consider $U = \{x_1, x_2, x_3\}$,
$A = \{x_1/(-0.6, 0.4), x_2/(-0.2, 0.3), x_3/(-0.4, 0.5)\}$ *and*

$$BF\mathbb{R} = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & (-1,1) & (-0.6,0.64) & (-0.6,0.54) \\ x_2 & (-0.6,0.64) & (-1,1) & (-0.62,0.54) \\ x_3 & (-0.6,0.54) & (-0.62,0.54) & (-1,1) \end{matrix}$$

The negative lower approximations are
$\mu_{FB\underline{R}^n(A)}(x_1) = -0.4$,
$\mu_{FB\underline{R}^n(A)}(x_2) = -0.4$,
$\mu_{FB\underline{R}^n(A)}(x_3) = -0.4$,
$BF\underline{R}^n(A) = \{x_1/-0.4, x_2/-0.4, x_3/-0.4\}$
The positive lower approximations are
$\mu_{FB\underline{R}^p(A)}(x_1) = 0.36$,
$\mu_{FB\underline{R}^p(A)}(x_2) = 0.3$,
$\mu_{FB\underline{R}^p(A)}(x_3) = 0.46$,
$BF\underline{R}^p(A) = \{x_1/0.36, x_2/0.3, x_3/0.46\}$
The negative upper approximations are
$\mu_{FB\overline{R}^n(A)}(x_1) = -0.6$,
$\mu_{FB\overline{R}^n(A)}(x_2) = -0.6$,
$\mu_{FB\overline{R}^n(A)}(x_3) = -0.6$,
$BF\overline{R}^n(A) = \{x_1/-0.6, x_2/-0.6, x_3/-0.6\}$
The positive upper approximations are
$\mu_{FB\overline{R}^p(A)}(x_1) = 0.5$,
$\mu_{FB\overline{R}^p(A)}(x_2) = 0.5$,
$\mu_{FB\overline{R}^p(A)}(x_3) = 0.5$,
$BF\overline{R}^p(A) = \{x_1/0.5, x_2/0.5, x_3/0.5\}$,
Hence, $\tau = \{\{x_1/(-1,1), x_2/(-0.6, 0.64), x_3/(-0.6, 0.54)\}, \{x_1/(0,0), x_2/(0,0), x_3/(0,0)\} \{x_1/(-0.4, 0.36), x_2/(-0.4, 0.3), x_3/(-0.4, 0.46)\}, \{x_1/(-0.6, 0.5), x_2/(-0.6, 0.5), x_3/(-0.6, 0.5)\}\}$.
$BFR(A) = \{\{x_1/(-0.4, 0.36), x_2/(-0.4, 0.3), x_3/(-0.4, 0.46)\}, \{x_1/(-0.6, 0.5), x_2/(-0.6, 0.5), x_3/(-0.6, 0.5)\}\}$.
Now

$$\begin{aligned} f^-(BF\underline{R}^n(A))(y_1) &= \vee_{x \in (f_1^-)^{-1}(y_1)} \mu_{BF\underline{R}(A)}(x) \\ &= \vee_{x \in x_3} \mu_{BF\underline{R}(A)}(x) \\ &= \mu_{BF\underline{R}(A)}(x_3) \\ &= -0.4. \end{aligned}$$

$$\begin{aligned} f^-(BF\underline{R}^n(A))(y_2) &= \vee_{x \in (f_1^-)^{-1}(y_2)} \mu_{BF\underline{R}(A)}(x) \\ &= -0.4. \end{aligned}$$

$$\begin{aligned} f^-(BF\underline{R}^n(A))(y_3) &= \vee_{x \in (f_1^-)^{-1}(y_3)} \mu_{BF\underline{R}(A)}(x) \\ &= \vee_{x \in \{x_1, x_2\}} \mu_{BF\underline{R}(A)}(x) \\ &= \mu_{BF\underline{R}(A)}(x_1) \vee \mu_{BF\underline{R}(A)}(x_2) \\ &= -0.4. \end{aligned}$$

Similarly, the other values are calculated.
Thus, $h(BFR(A)) = \{\{y_1/(-0.4.0.36), y_2/(-0.4, 0.3), y_3/(-0.4, 0.46)\}, \{y_1/(-0.6, 0.5), y_2/(-0.6, 0.5), y_3/(-0.6, 0.5)\}\}$ *is the bipolar fuzzy rough image of the bipolar fuzzy rough set* $BFR(A)$.

Definition 2.3. *Let* $(U, BFR(A), \tau)$ *and* $(U^*, BFR(B), \tau')$ *be two bipolar fuzzy rough topological spaces.* $h = ((f^-, f^+), (g^-, g^+) : BFR(A) \to BFR(B)$ *where*
$f^- : BF\underline{R}^n(A) \to BF\underline{R}^n(B)$,
$f^+ : BF\underline{R}^p(A) \to BF\underline{R}^p(B)$,
$g^- : BF\overline{R}^n(A) \to BF\overline{R}^n(B)$,
$g^+ : BF\overline{R}^p(A) \to BF\overline{R}^p(B)$ *are bipolar fuzzy rough mappings. Then for a bipolar fuzzy rough set* $BFR(B) \in \tau'$, $h^{-1}(BFR(B))$ *is a bipolar fuzzy rough set in* τ *obtained as follows:* $h^{-1}(BFR(B))(x) = BFR(B)(h(x))$. $h^{-1}(BFR(B))$ *is called the bipolar fuzzy rough inverse image of the bipolar fuzzy rough set* $BFR(B)$.

Example 2.4. *Let* $(U = \{x_1, x_2, x_3\}, BFR(A), \tau)$ *be defined as in Example 2.2 and* $(U^* = \{y_1, y_2, y_3\}, BFR(B), \tau')$ *be two bipolar fuzzy rough topological spaces.* $U^* = \{y_1, y_2, y_3\}$,
$B = \{y_1/(-0.2, 0.6), y_2/(-0.3, 0.4), y_3/(-0.5, 0.4)\}$ *and*

$$BF\mathbb{R} = \begin{matrix} & y_1 & y_2 & y_3 \\ y_1 & (-1,1) & (-0.3,0.6) & (-0.3,0.4) \\ y_2 & (-0.3,0.6) & (-1,1) & (-0.7,0.4) \\ y_3 & (-0.3,0.4) & (-0.7,0.4) & (-1,1) \end{matrix}$$

$BFR(B) = \{\{y_1/(-0.5, 0.4), y_2/(-0.3, 0.4), y_3/(-0.3, 0.4)\}, \{y_1/(-0.3, 0.6), y_2/(-0.3, 0.6), y_3/(-0.3, 0.4)\}\}$
Let f^-, f^+, g^-, g^+ *be functions defined as,*
$h(x_i) = ((f^-, f^+)(x_i), (g^-, g^+)(x_i)) = y_i, i = 1, 2, 3$. *Now,*
$(f^-)^{-1}(BF\underline{R}^n(B))(x_1) = BF\underline{R}^n(B) f^-(x_1)$
$= BF\underline{R}^n(B)\ y_1 = -0.5$.
Similarly, the other values are calculated.
$h^{-1}(BFR(B)) = \{\{x_1/(-0.5, 0.4), x_2/(-0.3, 0.4), x_3/(-0.3, 0.4)\}, \{x_1/(-0.3, 0.6), x_2/(-0.3, 0.6), x_3/(-0.3, 0.4)\}\}$.

Definition 2.5. *Let* $(U, BFR(A), \tau)$ *and* $(U^*, BFR(B), \tau')$ *be bipolar fuzzy rough topological spaces, where* $\tau = \{U, \phi, BFR(A)\}, \tau' = \{U^*, \phi, BFR(B)\}$ *and* $h = ((f^-, f^+), (g^-, g^+)) : BFR(A) \to BFR(B)$,
where $f^- : BF\underline{R}^n(A) \to BF\underline{R}^n(B)$,
$f^+ : BF\underline{R}^p(A) \to BF\underline{R}^p(B)$,
$g^- : BF\overline{R}^n(A) \to BF\overline{R}^n(B)$,
$g^+ : BF\overline{R}^p(A) \to BF\overline{R}^p(B)$ *are bipolar fuzzy rough mappings. Then h is said to be bipolar fuzzy rough continuous if the inverse image under h of any* $BFR(B) \in \tau'$ *is a bipolar fuzzy rough set* $BFR(A) \in \tau$,
i.e. $h^{-1}(BFR(B)) \in \tau$, *whenever* $BFR(B) \in \tau'$.

Example 2.6. *Let* $(U = \{x_1, x_2, x_3\}, BFR(A), \tau)$ *and* $(U^* = \{y_1, y_2, y_3\}, BFR(B), \tau')$ *be two bipolar fuzzy rough topological spaces. Consider* $U = \{x_1, x_2, x_3\}$,
the bipolar fuzzy subset A of U defined as
$A = \{x_1/(-0.15, 0.14), x_2/(-0.12, 0.17), x_3/(-0.19, 0.15)\}$
and the bipolar fuzzy relation $BF\mathbb{R}$ *defined on* $U \times U$ *as*

$$BF\mathbb{R} = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & (-1,1) & (-0.19,0.3) & (-0.19,0.23) \\ x_2 & (-0.19,0.3) & (-1,1) & (-0.2,0.23) \\ x_3 & (-0.19,0.23) & (-0.2,0.23) & (-1,1) \end{matrix}$$

$\tau = \{\{x_1/(-1,1), x_2/(-0.19, 0.3), x_3/(-0.19, 0.23)\}, \{x_1/(0,0), x_2/(0,0), x_3/(0,0)\}, \{x_1/(-0.19, 0.14), x_2/(-0.15, 0.17), x_3/(-0.19, 0.15)\}, \{x_1/(-0.19, 0.17), x_2/(-0.19, 0.17), x_3/(-0.19, 0.17)\}\}$.
$BFR(A) = \{\{x_1/(-0.19, 0.14), x_2/(-0.15, 0.17), x_3/(-0.19, 0.15)\}, \{x_1/(-0.19, 0.17), x_2/(-0.19, 0.17),$

$x_3/(-0.19,0.17)\}\}$.
Consider $U^* = \{y_1, y_2, y_3\}$,
$B = \{y_1/(-0.8,0.76), y_2/(-0.75,0.68), y_3/(-0.6,0.61)\}$

$$BF\mathbb{R} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} (-1,1) & (-0.75,0.8) & (-0.75,0.7) \\ (-0.75,0.8) & (-1,1) & (-0.3,0.7) \\ (-0.75,0.7) & (-0.3,0.7) & (-1,1) \end{pmatrix} \end{array}$$

$\tau' = \{\{y_1/(-1,1), y_2/(-0.75,0.8), y_3/(-0.75,0.7)\}$,
$\{y_1/(0,0), y_2/(0,0), y_3/(0,0)\}$,
$\{y_1/(-0.6,0.61), y_2/(-0.6,0.61), y_3/(-0.7,0.61)\}$,
$\{y_1/(-0.75,0.76), y_2/(-0.6,0.76), y_3/(-0.75,0.7)\}\}$.
$BFR(B) = \{\{y_1/(-0.6,0.61), y_2/(-0.6,0.61), y_3/(-0.7,0.61)\}, \{y_1/(-0.75,0.76), y_2/(-0.6,0.76), y_3/(-0.75,0.7)\}\}$.
f^-, f^+, g^-, g^+ *are functions defined as,*
$h(x_i) = ((f^-, f^+)(x_i), (g^-, g^+)(x_i)) = (\frac{y_i}{4}),\ i = 1,2,3.$
Now,
$(f^-)^{-1}(BF\underline{R}^n(B))(x_1) = BF\underline{R}^n(B) f^-(x_1)$
$= BF\underline{R}^n(B)(y_1/4) = -0.15.$
Similarly, the other values are calculated.
$h^{-1}(BFR(B)) = \{\{x_1/(-0.15,0.15), x_2/(-0.15,0.15), x_3/(-0.175,0.15)\}, \{x_1/(-0.18,0.19), x_2/(-0.15,0.19), x_3/(-0.18,0.175)\}\}$.
Hence, $h = ((f^-, f^+), (g^-, g^+)) : BFR(A) \to BFR(B)$ *is bipolar fuzzy rough continuous.*

Definition 2.7. *Let* $(U, BFR(A), \tau)$ *be a bipolar fuzzy rough topological space and* $BFR(B_s) \subset BFR(A)$. *Then the bipolar fuzzy rough topology* $\tau_{BFR(B_s)} = \{BFR(B_s) \cap BFR(O) | BFR(O) \in \tau\}$ *is called bipolar fuzzy rough subspace topology and* $(BFR(B_s), \tau_{BFR(B_s)})$ *is called bipolar fuzzy rough subspace of* $(U, BFR(A), \tau)$.

Example 2.8. *Consider,* $U = \{x_1, x_2, x_3\}$,
$A = \{x_1/(-0.07,0.1), x_2/(-0.28,0.24), x_3/(-0.28,0.36)\}$ *and*

$$BF\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} (-1,1) & (-0.7,0.94) & (-0.7,0.8) \\ (-0.7,0.94) & (-1,1) & (-0.8,0.8) \\ (-0.7,0.8) & (-0.8,0.8) & (-1,1) \end{pmatrix} \end{array}$$

$BFR(A) = \{x_1/(-0.28,0.1), x_2/(-0.2,0.1), x_3/(-0.2,0.2)\}$,
$\{x_1/(-0.7,0.36), x_2/(-0.7,0.36), x_3/(-0.7,0.36)\}$
Consider, $U = \{x_1, x_2, x_3\}$,
$B_s = \{x_1/(-0.23,0.1), x_2/(-0.21,0.21), x_3/(-0.31,0.32)\}$ *and*

$$BF\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} (-1,1) & (-0.69,0.44) & (-0.69,0.35) \\ (-0.69,0.44) & (-1,1) & (-0.81,0.35) \\ (-0.69,0.35) & (-0.81,0.35) & (-1,1) \end{pmatrix} \end{array}$$

$BFR(B_s) = \{x_1/(-0.31,0.1), x_2/(-0.23,0.21), x_3/(-0.23,0.32)\}, \{x_1/(-0.69,0.32), x_2/(-0.69,0.32), x_3/(-0.69,0.32)\}$
$BFR(B_s) \subset BFR(A)$.
$\tau_{BFR(B_s)} = \{U, \phi, BF\underline{R}(B_s), BF\overline{R}(B_s),\}$ *is the bipolar fuzzy rough subspace of* $(U, BFR(A), \tau)$.

Theorem 2.9. *Let* $(U, BFR(A), \tau)$ *and* $(U^*, BFR(B), \tau')$ *be two bipolar fuzzy rough topological spaces. Let* $h((f^-, f^+), (g^-, g^+)) : BFR(A) \to BFR(B)$, *where* $f^- : BF\underline{R}^n(A) \to BF\underline{R}^n(B)$, $f^+ : BF\underline{R}^p(A) \to BF\underline{R}^p(B)$, $g^- : BF\overline{R}^n(A) \to BF\overline{R}^n(B)$, $g^+ : BF\overline{R}^p(A) \to BF\overline{R}^p(B)$ *are bipolar fuzzy rough mappings. Then the following statements are equivalent:*
(i) The bipolar fuzzy rough function $h(f,g) : BFR(A) \to BFR(B)$ *is bipolar fuzzy rough continuous.*
(ii) The inverse image of every bipolar fuzzy rough closed set is bipolar fuzzy rough closed.
(iii) For each bipolar fuzzy rough point neighborhood of $h(xBFR(A))$ *under h is a bipolar fuzzy rough neighborhood of* $xBFR(A)$.
(iv) For each bipolar fuzzy rough point $xBFR(A)$ *in* $BFR(A)$ *and each bipolar fuzzy rough neighborhood* $BFR(B)$ *of* $h(xBFR(A))$, *there is a bipolar fuzzy rough neighborhood* $BFR(C)$ *of* $xBFR(A)$ *such that* $h(BFR(C)) \subseteq BFR(B)$.
(v) $h\overline{BFR(A)} \subseteq \overline{h(BFR(A))}$.

Proof. (i)⇒ (ii). Let $h(f,g) : BFR(A) \to BFR(B)$ be bipolar fuzzy rough continuous and $BFR(B) \in \tau'$. Then $h^{-1}(BFR(B)) \in \tau$. $BFR(B) \in \tau'$
$\Rightarrow [BFR(B)]^c$ is bipolar fuzzy rough closed in $(U, BFR(B), \tau')$. Again $h^{-1}[BFR(B)^c] = [h^{-1}(BFR(B))]^c$ and h being bipolar fuzzy rough continuous, $h^{-1}(BFR(B))$ is bipolar fuzzy rough open in $(U, BFR(A), \tau)$. Hence $[h^{-1}BFR(B)]^c$ is bipolar fuzzy rough closed in $(U, BFR(A), \tau)$. i.e.$h^{-1}BFR(B)^c$ is bipolar fuzzy rough closed in $(U, BFR(A), \tau)$.
(ii)⇒ (iii). Let $BFR(C)$ be a bipolar fuzzy rough neighborhood of $h(xBFR(A))$. Then there is a bipolar fuzzy rough open set $BFR(B) \in \tau'$ such that $h(xBFR(A)) \subseteq BFR(B) \subseteq BFR(C)$.
Now $xBFR(A) \in h^{-1}(h(xBFR(A))) \subseteq h^{-1}(BFR(B)) \subseteq h^{-1}(BFR(C))$.
i.e, $xBFR(A) \in h^{-1}(BFR(B)) \subseteq h^{-1}(BFR(C))$,
where $h^{-1}(BFR(B))$ is bipolar fuzzy rough open in $(U, BFR(A), \tau)$.
(iii)⇒ (iv). Let $xBFR(A) \in BFR(A)$ and $BFR(B)$ be bipolar fuzzy rough neighborhoods of $h(xBFR(A))$. Then $h^{-1}(BFR(B))$ is a bipolar fuzzy rough neighborhood of $xBFR(A)$. Thus there exists a bipolar fuzzy rough open set $BFR(C)$ in $BFR(A)$ such that,
$xBFR(A) \in BFR(C) \subseteq h^{-1}(BFR(B)).BFR(C) \subseteq h^{-1}(BFR(B))$
$\Rightarrow h(BFR(C)) \subseteq h(h^{-1}(BFR(B))) \subseteq BFR(B)$
i.e, $h(BFR(C)) \subseteq BFR(B)$.
(iv)⇒ (v). Since $\overline{h(BFR(A))}$ is bipolar fuzzy rough closed in $BFR(B)$, $h^{-1}(\overline{(h(BFR(A)))})$ is bipolar fuzzy rough closed in $BFR(A)$, Thus $h^{-1}(\overline{(h(BFR(A)))}) = \overline{h^{-1}(\overline{(h(BFR(A)))})}$.
Now, $BFR(A) \subseteq h^{-1}(h(BFR(A))) \subseteq h^{-1}(\overline{(h(BFR(A)))})$,
as $h(BFR(A)) \subseteq \overline{h(BFR(A))}$.
$\Rightarrow BFR(A) \subseteq h^{-1}(\overline{(h(BFR(A)))}) = \overline{h^{-1}(\overline{(h(BFR(A)))})}$
$\Rightarrow \overline{BFR(A)} \subseteq \overline{h^{-1}(\overline{(h(BFR(A)))})}$
$\Rightarrow \overline{BFR(A)} \subseteq h^{-1}(\overline{(h(BFR(A)))})$
$\Rightarrow h\overline{BFR(A)} \subseteq h(h^{-1}(\overline{h(BFR(A))})) \subseteq \overline{hBFR(A)}$.
i.e,$h(\overline{BFR(A)}) \subseteq \overline{h(BFR(A))}$.
(v)⇒ (i). Let $[BFR(B)]^c$ be bipolar fuzzy rough closed in

$BFR(B)$ and let $BFR(A) \in h^{-1}(BFR(B))$.
By assumption $h[\overline{h^{-1}(BFR(B))}] \subseteq \overline{h[h^{-1}(BFR(B))]}$.
i.e, $h[\overline{h^{-1}(BFR(B))}] \subset \overline{BFR(B)} = BFR(B)$
$\Rightarrow [\overline{h^{-1}(BFR(B))}] \subset [\overline{h^{-1}(BFR(B))}] \subset [h^{-1}(BFR(B))]$
$\Rightarrow [h^{-1}(BFR(B))] = [\overline{h^{-1}(BFR(B))}]$
Therefore, $h^{-1}[BFR(B)]$ is bipolar fuzzy rough closed in $BFR(A)$ whenever $[BFR(B)]^c$ is bipolar fuzzy rough closed in $BFR(B)$. Let $BFR(C)$ be bipolar fuzzy rough open set of $BFR(B)$.
$\Rightarrow [BFR(C)]^c$ is bipolar fuzzy rough closed in $BFR(B)$.
$\Rightarrow h^{-1}[BFR(C)]^c$ is bipolar fuzzy rough closed in $BFR(A)$.
$\Rightarrow h^{-1}(BFR(C))$ is bipolar fuzzy rough open in $BFR(A)$, whenever $BFR(C)$is bipolar fuzzy rough open in $BFR(B)$. Therefore h is bipolar fuzzy rough continuous. □

Theorem 2.10. *Let $(U, BFR(A), \tau)$ and $(U^*, BFR(B), \tau')$ be two bipolar fuzzy rough topological spaces. If $h : BFR(A) \to BFR(B)$ maps all of $BFR(A)$ into a single bipolar fuzzy rough point(constant) $xBFR(B)$ of $BFR(B)$. Then h is bipolar fuzzy rough continuous.*

Proof. Let $h : BFR(A) \to BFR(B)$ be bipolar fuzzy rough mapping such that $f(xBFR(A)) = xBFR(B)$ for every $xBFR(A) \in BFR(A)$. Consider $BFR(B)$ a bipolar fuzzy rough open set in $BFR(A)$.

$$f^{-1}(BFR(B)) = \begin{cases} \phi, & \text{if } xBFR(B) \notin BFR(B) \\ U, & \text{if } xBFR(B) \in BFR(B). \end{cases}$$

Then U and ϕ are bipolar fuzzy rough open in $BFR(A)$.
Therefore $f^{-1}(BFR(B))$ is bipolar fuzzy rough open in $BFR(A)$, whenever $BFR(C)$ is bipolar fuzzy rough open in $BFR(B)$. Therefore any constant function is bipolar fuzzy rough continuous. □

Theorem 2.11. *Let $(U, BFR(A), \tau)$ be a bipolar fuzzy rough topological space. If $BFR(B_s)$ is a bipolar fuzzy rough subspace of $BFR(A)$, the inclusion function $i : BFR(B_s) \to BFR(A)$ is bipolar fuzzy rough continuous.*

Proof. Let $BFR(B_s)$ be bipolar fuzzy rough subspace of $BFR(A)$ and $i : BFR(B_s) \to BFR(A)$. Since $BFR(B_s)$is a bipolar fuzzy rough subspace of $BFR(A)$, for any bipolar fuzzy rough open set $BFR(V)$ of $BFR(A), i^{-1}(BFR(V)) = BFR(V) \cap BFR(B_s)$ is bipolar fuzzy rough open in bipolar fuzzy rough subspace topology of $BFR(B_s)$. $i^{-1}(BFR(V))$ is bipolar fuzzy rough open in $BFR(B_s)$, whenever $BFR(V)$ is bipolar fuzzy rough open in $BFR(A)$. Therefore, every inclusion map is bipolar fuzzy rough continuous. □

Theorem 2.12. *Let $(U, BFR(A), \tau)$, $(U^*, BFR(B), \tau')$ and $(U^{**}, BFR(C), \tau'')$ be bipolar fuzzy rough topological spaces. If $h : BFR(A) \to BFR(B)$ and $j : BFR(B) \to BFR(C)$ are bipolar fuzzy rough continuous, then the map $j \circ h : BFR(A) \to BFR(C)$ is bipolar fuzzy rough continuous.*

Proof. Let $BFR(D)$ be bipolar fuzzy rough open in $BFR(C)$. As j is bipolar fuzzy rough continuous, $j^{-1}(BFR(D))$ is bipolar fuzzy rough open in $BFR(B)$. As h is bipolar fuzzy rough continuous,
$h^{-1}(j^{-1}(BFR(D))) = (j \circ h)^{-1}(BFR(D))$ is bipolar fuzzy rough open in $BFR(A)$, whenever $BFR(D)$ is bipolar fuzzy rough open in $BFR(C)$. Therefore, composition of two bipolar fuzzy rough continuous functions is bipolar fuzzy rough continuous. □

Theorem 2.13. *Let $(U, BFR(A), \tau)$ and $(U^*, BFR(B), \tau')$ be bipolar fuzzy rough topological spaces. If $h : BFR(A) \to BFR(B)$ is bipolar fuzzy rough continuous and if $BFR(B_s)$ is a subspace of $BFR(A)$, then the restricted function $h/BFR(B_s) : BFR(B_s) \to BFR(B)$ is bipolar fuzzy rough continuous.*

Proof. Let $i : BFR(B_s) \to BFR(A)$ be the inclusion map of $BFR(B_s)$ into $BFR(A)$. i being the inclusion map is bipolar fuzzy rough continuous. Given $h : BFR(A) \to BFR(B)$ is bipolar fuzzy rough continuous. Therefore it follows that $h \circ i : BFR(B_s) \to BFR(B)$ is bipolar fuzzy rough continuous. Hence $h/BFR(B_s) : BFR(B_s) \to BFR(B)$ is bipolar fuzzy rough continuous. □

Theorem 2.14. *Let $(U, BFR(A), \tau)$ and $(U^*, BFR(B), \tau')$ be bipolar fuzzy rough topological spaces. If $h : BFR(A) \to BFR(B)$ is bipolar fuzzy rough continuous and $BFR(B_s)$ is a bipolar fuzzy rough subspace of $BFR(B)$ containing the image set $h(BFR(A))$, then the function $j : BFR(A) \to BFR(B_s)$ obtained by restricting the range of h, is bipolar fuzzy rough continuous. If $BFR(C)$ has $BFR(B)$ as a subspace, then the function $k : BFR(A) \to BFR(C)$ obtained by expanding the range of h is bipolar fuzzy rough continuous.*

Proof. Let $h : BFR(A) \to BFR(B)$ be bipolar fuzzy rough continuous. If $h(BFR(A)) \subset BFR(B_s) \subset BFR(B)$, we show that the function $j : BFR(A) \to BFR(B_s)$ obtained from h is bipolar fuzzy rough continuous. Let $BFR(F)$ be bipolar fuzzy rough open in $\tau_{BFR(B_s)}$,
$BFR(F) = BFR(B_s) \cap BFR(E)$, for some bipolar fuzzy rough open set $BFR(E)$ of $(U^*, BFR(B), \tau')$. Since $h^{-1}(BFR(E))$ is bipolar fuzzy rough open in $(U, BFR(A), \tau)$, $h^{-1}(BFR(E)) = j^{-1}(BFR(F))$. Because $BFR(B_s)$ contains the entire image set $h(BFR(A))$. $j^{-1}(BFR(F))$ is bipolar fuzzy rough open in $(U, BFR(A), \tau)$ whenever $BFR(F)$ is bipolar fuzzy rough open in $BFR(B_s)$. Therefore $j : BFR(A) \to BFR(B_s)$ is bipolar fuzzy rough continuous. $h : BFR(A) \to BFR(B)$ is bipolar fuzzy rough continuous and the inclusion map $i : BFR(B) \to BFR(C)$ being the composition of two bipolar fuzzy rough continuous functions is bipolar fuzzy rough continuous. □

Theorem 2.15. *(Pasting lemma)*
Let $(U, BFR(A), \tau)$ and $(U^*, BFR(B), \tau')$ be bipolar fuzzy rough topological spaces. If $f : BFR(A) \to BFR(C)$ and $g : BFR(B) \to BFR(C)$ are bipolar fuzzy rough continuous mappings, then f and g combine to give a bipolar fuzzy rough

continuous function $h : (BFR(A), BFR(B)) \rightarrow BFR(C)$, *defined by setting*
h(BFR(A),BFR(B))(y)=

$$\begin{cases} f(BFR(A))(y) \text{ if } BFR(C) = f(BFR(A)) \\ g(BFR(B))(y) \text{ if } BFR(C) = g(BFR(B)). \end{cases}$$

Proof. Consider $BFR(C)$ a bipolar fuzzy rough open set in τ'. Then $h^{-1}(BFR(C)) = f^{-1}(BFR(C)) \cup g^{-1}(BFR(C))$. $f : BFR(A) \rightarrow BFR(C)$ is bipolar fuzzy rough continuous and $BFR(C)$ is bipolar fuzzy rough open in τ'. But $BFR(A)$ is bipolar fuzzy rough open in τ. Therefore $f^{-1}(BFR(C))$ is bipolar fuzzy rough open in τ. $g : BFR(B) \rightarrow BFR(C)$ is bipolar fuzzy rough continuous and $BFR(C)$ is bipolar fuzzy rough open in τ'. But $BFR(B)$ is bipolar fuzzy rough open in τ. Therefore $g^{-1}(BFR(C))$ is bipolar fuzzy rough open in τ. Therefore $h^{-1}(BFR(C)) = f^{-1}(BFR(C)) \cup g^{-1}(BFR(C))$ is bipolar fuzzy rough open in τ, whenever $BFR(C)$ is bipolar fuzzy rough open in τ'. Therefore $h : (BFR(A), BFR(B)) \rightarrow BFR(C)$ is bipolar fuzzy rough continuous. □

Example 2.16. *Let* $(U = \{x_1, x_2, x_3\}, BFR(A), BFR(B), \tau)$ *and* $(U^* = \{y_1, y_2, y_3\}, BFR(C), \tau')$ *be two bipolar fuzzy rough topological spaces.*
Consider $U = \{x_1, x_2, x_3\}$,
$A = \{x_1/(-0.3, 0.5), x_2/(-0.15, 0.25), x_3/(-0.25, 0.21)\}$
and

$$BF\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \begin{pmatrix} (-1,1) & (-0.35,0.6) & (-0.35,0.21) \\ (-0.35,0.6) & (-1,1) & (-0.4,0.21) \\ (-0.35,0.21) & (-0.4,0.21) & (-1,1) \end{pmatrix}$$

$BFR(A) = \{\{x_1/(-0.25, 0.4), x_2/(-0.3, 0.25), x_3/(-0.3, 0.21)\}, \{x_1/(-0.35, 0.5), x_2/(-0.35, 0.5), x_3/(-0.35, 0.21)\}\}$.
Consider $U = \{x_1, x_2, x_3\}$,
$B = \{x_1/(-0.2, 0.4), x_2/(-0.3, 0.3), x_3(-0.6, 0.6)\}$
and

$$BF\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \begin{pmatrix} (-1,1) & (-0.5,0.7) & (-0.5,0.6) \\ (-0.5,0.7) & (-1,1) & (-0.6,0.6) \\ (-0.5,0.6) & (-0.6,0.6) & (-1,1) \end{pmatrix}$$

$BFR(B) = \{\{x_1/(-0.5, 0.4), x_2/(-0.4, 0.3), x_3(-0.3, 0.4)\}, \{x_1/(-0.5, 0.6), x_2/(-0.5, 0.6), x_3/(-0.5, 0.6)\}\}$.
Consider $U^* = \{y_1, y_2, y_3\}$,
$C = \{y_1/(-0.2, 0.4), y_2/(-0.3, 0.5), y_3/(-0.4, 0.6)\}$
and

$$BF\mathbb{R} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ \end{array} \begin{pmatrix} (-1,1) & (-0.4,0.65) & (-0.4,0.5) \\ (-0.4,0.65) & (-1,1) & (-0.15,0.5) \\ (-0.4,0.5) & (-0.15,0.5) & (-1,1) \end{pmatrix}$$

$BFR(C) = \{\{y_1/(-0.4, 0.4), y_2/(-0.4, 0.4), y_3(-0.3, 0.3)\}, \{y_1/(-0.4, 0.5), y_2/(-0.4, 0.5), y_3/(-0.3, 0.5)\}\}$.
Let $h^- = ((f^-, g^-), h^+ = ((f^+, g^+)$ *be functions defined as,*

$h^-(x_i) = (f^-, g^-)(x_i) = (\frac{y_i - 1}{2})$,

$h^+(x_i) = (f^+, g^+)(x_i) = (\frac{y_i + 1}{2}), i = 1, 2, 3.$

Now, $(f^-)^{-1}(BF\underline{R}^n(C))(x_1) = BF\underline{R}^n(C) f^-(x_1)$
$= BF\underline{R}^n(C)(y_1 - 1/2) = -0.7$
$(f^-)^{-1}(BF\underline{R}^n(C))(x_2) = BF\underline{R}^n(C) f^-(x_2)$
$= BF\underline{R}^n(C)(y_2 - 1/2) = -0.7$
$(f^-)^{-1}(BF\underline{R}^n(C))(x_3) = BF\underline{R}^n(C) f^-(x_3)$
$= BF\underline{R}^n(C)(y_3 - 1/2) = -0.65.$
Similarly, the other values are calculated.
$((h^-)^{-1}, (h^+)^{-1}(BFR(C))$
$= \{\{x_1/(-0.7, 0.7), x_2/(-0.7, 0.7), x_3/(-0.65, 0.65)\}, \{x_1/(-0.7, 0.75), x_2/(-0.7, 0.75), x_3/(-0.65, 0.75)\}\}$.
Let $f^- = (\rho^-, \sigma^-), f^+ = (\rho^+, \sigma^+)$ *and where* $\rho^- : BF\underline{R}^n(A) \rightarrow BF\underline{R}^n(C), \rho^+ : BF\overline{R}^p(A) \rightarrow BF\overline{R}^p(C)$ *and*
$\sigma^- : BF\underline{R}^n(A) \rightarrow BF\underline{R}^n(C), \sigma^+ : BF\overline{R}^p(A) \rightarrow BF\overline{R}^p(C)$ *be functions defined as,*

$f^-(x_i) = (\rho^-, \sigma^-)(x_i) = (\frac{-1 - 2y_i}{2})$,

$f^+(x_i) = (\rho^+, \sigma^+(x_i) = (\frac{1 - 2y_i}{2}), i = 1, 2, 3.$

Now,
$(\rho^-)^{-1}(BF\underline{R}^n(C))(x_1) = BF\underline{R}^n(C)\rho^-(x_1)$
$= BF\underline{R}^n(C)(-1 - 2y_1/2) = -0.1$
$(\rho^-)^{-1}(BF\underline{R}^n(C))(x_2) = BF\underline{R}^n(C)\rho^-(x_2)$
$= BF\underline{R}^n(C)(-1 - 2y_2/2) = -0.1$
$(\rho^-)^{-1}(BF\underline{R}^n(C))(x_3) = BF\underline{R}^n(C)\rho^-(x_3)$
$= BF\underline{R}^n(C)(-1 - 2y_3/2) = -0.2.$
Similarly, the other values are calculated.
$((f^-)^{-1}, (f^+)^{-1})(BFR(C))$
$= \{\{x_1/(-0.1, 0.1), x_2/(-0.1, 0.1), x_3/(-0.2, 0.2)\}, \{x_1/(-0.1, 0), x_2/(-0.1, 0), x_3/(-0.2, 0)\}\}$
Next, $g^- = (\alpha^-, \beta^-), g^+ = (\alpha^+, \beta^+)$ *and where*
$\alpha^- : BF\underline{R}^n(B) \rightarrow BF\underline{R}^n(C)$,
$\alpha^+ : BF\overline{R}^p(B) \rightarrow BF\overline{R}^p(C)$ *and*
$\beta_- : BF\underline{R}^n(B) \rightarrow BF\underline{R}^n(C), \beta^+ : BF\overline{R}^p(B) \rightarrow BF\overline{R}^p(C)$ *be functions defined as,*

$g^-(x_i) = (\alpha^-, \beta^-)(x_i) = (\frac{y_i - 1}{2})$,

$g^+(x_i) = (\alpha^+, \beta^+)(x_i) = (\frac{y_i + 1}{2}), i = 1, 2, 3.$ *Now,*
$(\alpha^-)^{-1}(BF\underline{R}^n(C))(x_1) = BF\underline{R}^n(C)\alpha^-(x_1)$
$= BF\underline{R}^n(C)(y_1 - 1/2) = -0.7$
$(\alpha^-)^{-1}(BF\underline{R}^n(C))(x_2) = BF\underline{R}^n(C)\alpha^-(x_2)$
$= BF\underline{R}^n(C)(y_2 - 1/2) = -0.7$
$(\alpha^-)^{-1}(BF\underline{R}^n(C))(x_3) = BF\underline{R}^n(C)\alpha^-(x_3)$
$= BF\underline{R}^n(C)(y_3 - 1/2) = -0.65.$
$((g^-)^{-1}, (g^+)^{-1})(BFR(C))$
$= \{\{x_1/(-0.7, 0.7), x_2/(-0.7, 0.7), x_3/(-0.65, 0.65)\}, \{x_1/(-0.7, 0.75), x_2/(-0.7, 0.75), x_3/(-0.65, 0.75)\}\}$.
$((f^-)^{-1}, (f^+)^{-1})BFR(C) \cup ((g^-)^{-1}, (g^+)^{-1}BFR(C)$
$= \{\{x_1/(-0.7, 0.7), x_2/(-0.7, 0.7), x_3/(-0.65, 0.65)\}, \{x_1/(-0.7, 0.75), x_2/(-0.7, 0.75), x_3/(-0.65, 0.75)\}\}$.

References

[1] S. Anita Shanthi and M. Saranya, On Bipolar fuzzy rough connected spaces, *AIP Conference Proceedings,*

2177(2019), 020–002.

[2] S. Anita Shanthi, N. Thillaigovindan and R.Poovizhi , On Fuzzy Rough Continuous Functions, *The Journal of Fuzzy Mathematics,* 25(3)(2017), 1–18.

[3] Boby P.Mathew and Sunil Jacob John, On rough topological spaces, *International Journal of Mathematical Archive,* 3(9)(2012), 3413–3421.

[4] D.Dubois, H.Prade, Putting fuzzy sets and rough sets together, *Intelligent Decision Support, Kluwer Academic, Dordrecht,* (1992), 203–232.

[5] D.Dubois, H.Prade, Rough fuzzy set and fuzzy rough set, *International Journal of General Systems,* 17(1990), 191–209.

[6] George J. Klir, Tina A. Folger, *Fuzzy sets: Uncertainty and Information*, Prentice Hall of India, 2005.

[7] Keyun Qin and Zheng Pei, On the topological properties of fuzzy rough sets, *Fuzzy sets and Systems,* 151, (2005), 601-613.

[8] M.Lellis Thivagar, Carmel Richard and Nirmala Rebecca Paul, Mathematical Innovations of a Modern Topology in Medical Events, *International Journal of Information Science,* 2(4), (2012), 33-36.

[9] R.Muthuraj, M.Sridharan, Some properties of Bipolar fuzzy cosets of a bipolar fuzzy and Bipolar anti fuzzy HX subgroup, *IOSR Journal of Mathematics,* 9(6) , (2014), 128-137.

[10] S.Nanda, Fuzzy rough sets, *Fuzzy sets and systems,* 45,(1992), 157-160.

[11] Z.Pawlak, Rough sets, *International journal of Computer and Information Science,* 11, (1982) , 341-356.

[12] Z.Pawlak, Rough sets, *Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Boston,* (1991).

[13] W.R.Zhang, Bipolar Fuzzy sets, *Proceeding of FUZZ-IEEE,* (1998), 835–840.