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Fixed point theorems in partial S_b-metric spaces

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Abstract

Some remarks on the article "*A Fixed Point in Partial S_b-Metric spaces*, An. St. Univ. Ovidus Constanta, 24(3), 2016, 351-362" by *Nizar Souayah* have been expressed through this paper. All the results of the above-mentioned article have been rediscovered without using symmetric condition with necessary modifications and corrections. Also a Suzuki type fixed point theorem using F-contraction have been achieved in partial S_b-metric spaces.

Keywords

Partial S_b-metric space, Fixed point, F-contraction.

AMS Subject Classification

54H25; 47H10

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Contents

1. Introduction and Preliminaries

Present century saw various generalizations of metric spaces raised in several ways. For example, S. G. Matthews [12] defined partial metric space, Bakhtin [4] introduced b-metric spaces, S. Shukla [15] Partial-b metric spaces and generalization of many results related to fixed point theories have been studied in those spaces([3],[16],[17]). Nizar Souayah [9] introduced partial S_b metric space as an extension of partial b-metric spaces and studied few fixed point theorems. This paper is a modification of [9] as well as an extension of the study of partial S_b-metric spaces. Let's provide few definitions as ready references,

Definition 1.1. *[13] An S-metric on a nonempty X is a function $S : X^3 \longrightarrow [0,\infty)$ that satisfies the following conditions: for all $x,y,z,a \in X$,*

(s_1) $S(x,y,z) = 0 \Leftrightarrow x = y = z$;

(s_2) $S(x,y,,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$;

The pair (X,S) is called an S-metric space.

Definition 1.2. *[8] A mapping $S_p : X^3 \longrightarrow [0,\infty)$, where X is a non empty set, is said to be partial S-metric if whenever x, y, z, $t \in X$ the following conditions hold:*

(i) $x = y$ *if and only if* $S_p(x,x,x) = S_p(y,y,y) = S_p(x,x,y)$;

(ii) $S_p(x,x,x) \leq S_p(x,y,z)$;

(iii) $S_p(x,x,y) = S_p(y,y,x)$;

(iv) $S_p(x,y,z) \leq S_p(x,x,t) + S_p(y,y,t) + S_p(z,z,t) - S_p(t,t,t)$

The pair (X,S_p) is called partial S-metric space.

Definition 1.3. *[9] A mapping $S_b : X^3 \longrightarrow [0,\infty)$, where X is a non empty set, is said to be partial S_b-metric with coefficient $s \geq 1$ if whenever x, y, z, $t \in X$ the following conditions hold:*

(i) $x = y = z$ *iff* $S_b(x,y,z) = S_b(x,x,x) = S_b(y,y,y) = S_b(z,z,z)$;

(ii) $S_b(x,x,x) \leq S_b(x,y,z)$;

(iii) $S_b(x,x,y) = S_b(y,y,x)$;

(iv) $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)] - S_b(t,t,t)$

The pair (X,S_b) is called partial S_b-metric space with coefficient $s \geq 1$.

Definition 1.4. *[9] In a partial S_b-metric space (X,S_b) a sequence $\{x_n\}$ is said to be convergent to x if* $\lim_{n\to\infty} S_b(x_n,x_n,x) =$

$S_b(x,x,x)$. A Sequence $\{x_n\}$ is said to be a Cauchy sequence in X if $\lim_{n,m\to\infty} S_b(x_n,x_n,x_m)$ exists. A partial S_b-metric space (X,S_b) is said to be complete if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $\lim_{n\to\infty} S_b(x_n,x_n,x) = S_b(x,x,x) = \lim_{n,m\to\infty} S_b(x_n,x_n,x_m)$

2. Modification of the Results Appeared in [9]

Let's begin with the following example

Example 2.1. *Let $X = \{0,1,2,3\}$ and $S_b(x,y,z) =| x-y |^2 + | y-z |^2 + | z-x |^2 + x$. Define $T : X \longrightarrow X$ by $T0 = 0$, $T1 = 0$, $T2 = 1$, $T3 = 2$ which satisfies all the conditions of Theorem 2.1 [9]. Clearly 0 is the unique fixed point of T though S_b does not satisfy the partial symmetric condition ((iii) of Definition 1.3)[9] as seen in particular $S_b(1,1,2) \neq S_b(2,2,1)$*

Actually it is seen that all the results in the paper [9] can be proved without using partial symmetric condition. Just for simplicity of writing let's call the revised metric weak partial S_b-metric which is a generalization of S_b-metric. So,

Definition 2.2. *A mapping $S_b : X^3 \longrightarrow [0,\infty)$, where X is a non empty set, is said to be weak partial S_b-metric with coefficient $s \geq 1$ if the conditions (i), (ii) and (iv) of Definition 1.3 [9] hold.*

Example 2.3. *There is only one example in [9] (Example 1.5) which is NOT for that the author CLAIMED for; it is a weak partial S_b-metric space.*

Example 2.4. *Let $X = \{0,1,2,3\}$ and define $S_b : X^3 \longrightarrow \mathbb{R}^+$ by $S_b(x,y,z) =| x-y |^2 + | y-z |^2 + | z-x |^2 + x$. Then (X,S_b) is a weak partial S_b-metric space with coefficient $s = 2$ which is neither partial S_b-metric space nor an S metric space (since $S_b(1,1,1) \neq 0$) nor a partial S-metric space (since $S_b(0,0,3) > S_b(0,0,1)+S_b(0,0,1)+S_b(3,3,1)-S_b(1,1,1)$).*

It is noticed that in the Theorem 2.1 [9] (line 14 of page 355 and line 4 of page 356) author assumed $Tx_{n-1} = x_n$ and $Tx_n = x_{n+1}$ respectively though he defined $F^k x_0 = x_k$ for all $k \in \mathbb{N}$, where x_0 is an arbitrary point of X and $T^{n_0} \equiv F$ for some $n_0 \in \mathbb{N}$ (line 8 of page 354) which is absurd.

Now few lines back the Example 2.1 shows that it is not necessary for the space in Theorem 2.1 [9] to be a partial S_b-metric space it may be weak partial S_b-metric space to ensure existence and uniqueness of fixed point for such mappings. The following theorem proves the fact in general.

Theorem 2.5. *Let (X,S_b) be a complete weak partial S_b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition*

$$S_b(Tx,Ty,Tz) \leq \lambda S_b(x,y,z)\ \forall x,y,z \in X,\ \lambda \in [0,1). \quad (2.1)$$

Then T has a unique fixed point $u \in X$ with $S_b(u,u,u) = 0$.

Proof. First we show that the fixed point of T is unique and if u be a fixed point of T then $S_b(u,u,u) = 0$. Let u,v be two distinct fixed point of T. i.e., $Tu = u$ and $Tv = v$. Let if possible $S_b(u,u,u) > 0$. Then from equation (2.1),

$S_b(u,u,u) = S_b(Tu,Tu,Tu) \leq \lambda S_b(u,u,u) < S_b(u,u,u)$, a contradiction. Hence $S_b(u,u,u) = 0$. Similarly $S_b(v,v,v) = 0$. Now

$S_b(u,u,v) = S_b(Tu,Tu,Tv) \leq \lambda S_b(u,u,v) < S_b(u,u,v)$.
Hence $S_b(u,u,v) = 0 \Rightarrow u = v$. Therefore T has a unique fixed point.

Since $\lambda \in [0,1)$, we can choose $n_0 \in \mathbb{N}$ such that for a given $0 < \varepsilon < 1$, we have $\lambda^{n_0} < \frac{\varepsilon}{8s}$. Let $T^{n_0} \equiv F$ and $F^k x_0 = x_k\ \forall k \in \mathbb{N}$, where $x_0 \in X$. Then for all $x,y,z \in X$,

$$S_b(Fx,Fy,Fz) = S_b(T^{n_0}x,T^{n_0}y,T^{n_0}z) \leq \lambda^{n_0} S_b(x,y,z) \quad (2.2)$$

Using inequality (2.2) for any $k \in \mathbb{N}$, we have

$\max\{S_b(x_{k+1},x_{k+1},x_k), S_b(x_k,x_k,x_{k+1})\} \longrightarrow 0$ as $k \longrightarrow \infty$.
So we can choose $l \in \mathbb{N}$ such that

$\max\{S_b(x_{l+1},x_{l+1},x_l), S_b(x_l,x_l,x_{l+1})\} < \frac{\varepsilon}{8s}$.
Let us define a relation ρ on X by

$y\rho x \Leftrightarrow \max\{S_b(x,x,y), S_b(y,y,x)\} - S_b(x,x,x) < \frac{\varepsilon}{2}$.
Let $A = \{y \in X : y\rho x_l\}$. Since $x_l \rho x_l$, $A \neq \phi$. Let $x_z \in A$. Then $\max\{S_b(x_l,x_l,x_z), S_b(x_z,x_z,x_l)\} - S_b(x_l,x_l,x_l) < \frac{\varepsilon}{2}$.
Using equation (2.2)

$$S_b(Fx_z,Fx_z,Fx_l) < \frac{\varepsilon}{8s}[1 + S_b(x_l,x_l,x_l)].$$

Therefore

$$\begin{aligned} S_b(x_l,x_l,Fx_z) &\leq s[2S_b(x_l,x_l,Fx_l) + S_b(Fx_z,Fx_z,Fx_l)] \\ &\quad - S_b(Fx_l,Fx_l,Fx_l) \\ &< \frac{\varepsilon}{2} + S_b(x_l,x_l,x_l). \end{aligned}$$

Similarly, $S_b(Fx_z,Fx_z,x_l) < \frac{\varepsilon}{2} + S_b(x_l,x_l,x_l)$. Hence $Fx_z\ \rho\ x_l$ and consequently $Fx_z \in A$. Since $x_l \in A$ therefore $Fx_l \in A$. Repeating this above process $F^n x_l \in A\ \forall n \in \mathbb{N}$. i.e., $x_m \in A$ $\forall m \geq l$. Let $m > n \geq l$ and $n = l+i$. Then

$$\begin{aligned} S_b(x_n,x_n,x_m) &= S_b(Fx_{n-1},Fx_{n-1},Fx_{m-1}) \\ &\leq \lambda^{in_0} S_b(x_{n-i},x_{n-i},x_{m-i}) \\ &< S_b(x_l,x_l,x_{m-i}) \\ &< \frac{\varepsilon}{2} + S_b(x_l,x_l,x_l) < \varepsilon. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,S_b). By completeness of (X,S_b) there exists $u \in X$ such that

$$\lim_{n\to\infty} S_b(x_n,x_n,u) = \lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = S_b(u,u,u) = 0 \quad (2.3)$$

Now we show that u is a fixed point of T. First,
$S_b(u,u,x_n) \leq s[2S_b(u,u,u) + S_b(x_n,x_n,u)] - S_b(u,u,u)$

$$\text{Passing limits we have } \lim_{n\to\infty} S_b(u,u,x_n) = 0 \quad (2.4)$$

For all $n \in \mathbb{N}$,

$$\begin{aligned} S_b(u,u,Fu) &\leq s[2S_b(u,u,x_{n+1}) + S_b(Fu,Fu,x_{n+1})] \\ &\quad - S_b(x_{n+1},x_{n+1},x_{n+1}) \\ &\leq s[2S_b(u,u,x_{n+1}) + \lambda^{n_0} S_b(u,u,x_n)] \end{aligned}$$

Using equation (2.3) and (2.4) we have $S_b(u,u,Fu) = 0$. Also from equation (2.1) $S_b(Fu,Fu,Fu) = 0$. Hence $Fu = u$. i.e., $T^{n_0}u = u$. Since $\{T^n u\}$ is a Cauchy sequence with $\lim_{n,m\to\infty} S_b(u_n,u_n,u_m) = 0$, we have $Tu = u$. □

Example 2.6. *Let $X = \{0,1,2,3\}$ and $S_b(x,y,z) = [\max\{x,y\}]^2 + |\max\{x,y\} - z|^2$ as in the Example 1.5 in [9]. Then (X,S_b) is a complete weak partial S_b-metric space which is not partial S_b-metric space as $S_b(1,1,2) \neq S_b(2,2,1)$. Define $T : X \longrightarrow X$ by $T0 = 0$, $T1 = 0$, $T2 = 1$, $T3 = 2$. Then T satisfies the condition of Theorem 2.5 and T has a unique fixed point namely* 0.

Now let's look into the Theorem 2.2 [9]. The proof of this theorem is confusing because the statement allows λ to be any real number in $[\frac{1}{4},\frac{1}{3})$ and $s = 2$ but then $1 - 2s\lambda \leq 0$ and $1 - 3s\lambda < 0$ which does not allow the transition from line number 5 to 6 of page 358 [9]. Here is a variant of the Theorem 2.2 [9] as follows:

Theorem 2.7. *Let (X,S_b) be a complete weak partial S_b-metric space with coefficient s such that $2s > 3$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition*

$$S_b(Tx,Ty,Tz) \leq \lambda[S_b(x,x,Tx) + S_b(y,y,Ty) + S_b(z,z,Tz)] \tag{2.5}$$

for all $x,y,z \in X$, where $\lambda \in [0,\frac{1}{2s})$. Then T has a unique fixed point $u \in X$ with $S_b(u,u,u) = 0$.

Proof. Define a sequence $x_{n+1} = Tx_n$ $\forall n \in \mathbb{N}$. Using the contraction principle (2.5) $\lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = 0$. i.e., $\{x_n\}$ is a Cauchy sequence in (X,S_b). By completeness of (X,S_b) there exists $u \in X$ such that

$$\lim_{n\to\infty} S_b(x_n,x_n,u) = \lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = S_b(u,u,u) = 0 \tag{2.6}$$

Now, $S_b(u,u,x_n) \leq s[2S_b(u,u,u) + S_b(x_n,x_n,u)] - S_b(u,u,u)$. Taking limit and using (2.6) we have $\lim_{n\to\infty} S_b(u,u,x_n) = 0$.
Claim: u is a fixed point of T. For

$$\begin{aligned} S_b(u,u,Tu) &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,x_{n+1})] \\ &= s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,Tx_n)] \\ &\leq s[2S_b(u,u,x_{n+1}) + \lambda(2S_b(u,u,Tu) + \\ &\quad S_b(x_n,x_n,x_{n+1}))] \end{aligned}$$

Taking limit $S_b(u,u,Tu) \leq 2s\lambda S_b(u,u,Tu) < S_b(u,u,Tu)$, a contradiction. Hence $S_b(u,u,Tu) = 0$. Also from (2.5) $S_b(Tu,Tu,Tu) \leq 3\lambda S_b(u,u,Tu)$ and consequently $S_b(Tu,Tu,Tu) = 0$. Hence $Tu = u$. The uniqueness of the fixed point u follows from the contraction principle. □

Theorem 2.8. *Let (X,S_b) be a complete weak partial S_b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition for all $x,y,z \in X$,*

$$\begin{aligned} S_b(Tx,Ty,Tz) \leq \lambda \max\{&S_b(x,y,z), S_b(x,x,Tx), \\ &S_b(y,y,Ty), S_b(z,z,Tz)\} \end{aligned} \tag{2.7}$$

where $\lambda \in [0,\frac{1}{2s})$. Then T has a unique fixed point $u \in X$ with $S_b(u,u,u) = 0$.

Proof. For existence of fixed point let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ $\forall n \in \mathbb{N}$. Now for all $n \in \mathbb{N}$ from (2.7) we obtain

$$S_b(x_n,x_n,x_{n+1}) \leq \lambda^n S_b(x_0,x_0,x_1) \tag{2.8}$$

Now for all $n \in \mathbb{N}$,
$S_b(Tx_n,Tx_n,Tx_{n-1})$
$\leq \lambda \max\{S_b(x_n,x_n,x_{n-1}), S_b(x_n,x_n,Tx_n), S_b(x_{n-1},x_{n-1},Tx_{n-1})\}$
$= \lambda \max\{S_b(x_n,x_n,x_{n-1}), S_b(x_n,x_n,x_{n+1}), S_b(x_{n-1},x_{n-1},x_n)\}$.

If $\max\{S_b(x_n,x_n,x_{n-1}), S_b(x_n,x_n,x_{n+1}), S_b(x_{n-1},x_{n-1},x_n)\} = S_b(x_n,x_n,x_{n-1})$
Then $S_b(x_{n+1},x_{n+1},x_n) \leq \lambda S_b(x_n,x_n,x_{n-1})$

$$\Rightarrow S_b(x_{n+1},x_{n+1},x_n) \leq \lambda^n S_b(x_1,x_1,x_0) \tag{2.9}$$

If $\max\{S_b(x_n,x_n,x_{n-1}), S_b(x_n,x_n,x_{n+1}), S_b(x_{n-1},x_{n-1},x_n)\} = S_b(x_n,x_n,x_{n+1})$ then using (2.8) we have

$$S_b(x_{n+1},x_{n+1},x_n) \leq \lambda^{n+1} S_b(x_0,x_0,x_1) \tag{2.10}$$

Similarly for the rest case

$$S_b(x_{n+1},x_{n+1},x_n) \leq \lambda^n S_b(x_0,x_0,x_1) \tag{2.11}$$

Using (2.10) and (2.11) we have for $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} S_b(x_n,x_n,x_m) \leq 2s\lambda^n \frac{1}{1-2s\lambda} \max\{&S_b(x_0,x_0,x_1), \\ &S_b(x_1,x_1,x_0)\}. \end{aligned}$$

Passing through limits we have $\lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X,S_b). Since (X,S_b) is complete there exists $u \in X$ such that

$$\lim_{n\to\infty} S_b(x_n,x_n,u) = \lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = S_b(u,u,u) = 0 \tag{2.12}$$

It follows from (2.12)

$$\lim_{n\to\infty} S_b(u,u,x_n) = 0. \tag{2.13}$$

Now we will show u is a fixed point of T. For all $n \in \mathbb{N}$,

$$\begin{aligned} S_b(u,u,Tu) &\leq s[2S_b(u,u,x_{n+1}) + S_b(Tu,Tu,Tx_n)] \\ &\quad - S_b(x_{n+1},x_{n+1},x_{n+1}) \\ &\leq s[2S_b(u,u,x_{n+1}) + \lambda \max\{S_b(u,u,x_n), \\ &\quad S_b(u,u,Tu), S_b(x_n,x_n,x_{n+1})\}] \end{aligned}$$

$\Rightarrow S_b(u,u,Tu) \leq \lambda s S_b(u,u,Tu) < S_b(u,u,Tu)$, a contradiction.
$\Rightarrow S_b(u,u,Tu) = 0$.

$$\begin{aligned} S_b(Tu,Tu,Tu) &\leq 3sS_b(Tu,Tu,Tx_n) \\ &\leq 3s\lambda \max\{S_b(u,u,x_n), S_b(u,u,Tu), \\ &\quad S_b(x_n,x_n,x_{n+1})\} \end{aligned}$$

Passing through limits we have $S_b(Tu,Tu,Tu) = 0$. Hence $Tu = u$. Uniqueness of the fixed point directly follows from the contraction principle. □

Example 2.9. *Let $X = \{0,1,2,3\}$ and $A = \{(x,y,z) : x,y,z \in \{0,2\}\} \setminus (0,0,0)$. Define $S_b : X^3 \longrightarrow \mathbb{R}^+$ by*

$$\begin{aligned} S_b(x,y,z) &= 5\mid x-y \mid^2 + 5 \mid y-z \mid^2 + 5 \mid z-x \mid^2 + x^4 \\ &\qquad if (x,y,z) \notin A \\ &= 2 \;\; if\, (x,y,z) \in A. \end{aligned}$$

Then (X,S_b) is a complete weak partial S_b-metric space with coefficient $s = 2$. Define $T : X \longrightarrow X$ by
$T0 = 0$, $T1 = 2$, $T2 = 0$, $T3 = 0$
T satisfies all the conditions of Theorem 2.8 and T has a fixed point namely 0. But, since $S_b(1,1,2) \neq S_b(2,2,1)$ Theorem 2.3 of [9] is not applicable.

3. Fixed point Theorem using F-contraction

As in [18], Let (X,d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be F-contraction if there exists a $\tau > 0$ such that

$\forall x,y \in X, d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))$
where $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a mapping satisfying the following conditions:
(1) F is strictly increasing, (2) For each sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ and (3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In 2016 Piri and Kumam [3] describe a large class of function by taking an additional condition that F is continuous on $(0,\infty)$ and neglecting condition (2) and (3). Let $\mathfrak{F}$ be the family of all functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

(F_1) F is srtictly increasing.

(F_2) F is continuous on $(0,\infty)$.

and $\mathfrak{U}$ be the set of all function $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that ψ is continuous and $\psi(t) = 0$ if and only if $t = 0$.

Definition 3.1. *A mapping $f : (X,S_b) \longrightarrow (Y,S_b)$ is said to be continuous at a point x if for every sequence $\{x_n\}$ in X convergent to x, then $\lim_{n\to\infty} f(x_n) = f(x)$ and a function f is continuous on X if every f is continuous at every point $x \in X$.*

Lemma 3.2. *If a sequence $\{x_n\}$ in (X,S_b) converges to two different limits x and y with $S_b(x,x,x) = 0$ and $S_b(y,y,y) = 0$ then $x = y$. Moreover if $\{y_n\}$ be a sequence in X with $S_b(x_n,x_n,y_n) = 0$ then $\{y_n\}$ also converges to x.*

Lemma 3.3. *[11] Let (X,S_b) be a partial S_b-metric space with the coefficient $s \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X converges to x and y respectively. Then $\frac{1}{s^2}S_b(x,x,y) - \frac{2}{s}S_b(x,x,x) - 2S_b(y,y,y) \leq \lim_{n\to\infty} inf S_b(x_n,x_n,y_n)$ $\leq \lim_{n\to\infty} sup S_b(x_n,x_n,y_n) \leq 2sS_b(x,x,x) + 2s^2S_b(y,y,y)$ $+ s^2S_b(x,x,y)$. Moreover for all $z \in X$, $\frac{1}{s}S_b(x,x,z) - 2S_b(x,x,x) \leq \lim_{n\to\infty} inf S_b(x_n,x_n,z)$ $\leq \lim_{n\to\infty} sup S_b(x_n,x_n,z) \leq sS_b(x,x,z) + 2sS_b(x,x,x)$.*

Theorem 3.4. *Let (X,S_b) be a complete partial S_b-metric space with coefficient $s \geq 1$. Let S, T be mappings on X satisfying the following:*

(a) S is continuous;

(b) $T(X)$ subset of $S(X)$;

(c) S and T commute, i.e., $ST = TS \quad \forall x \in X$;

and for all $x,y \in X$ with $Sx \neq Tx$ or $Sy \neq Ty$
$\frac{1}{3s}S_b(Sx,Sx,Tx) \leq S_b(Sx,Sx,Sy)$
$\Rightarrow F(s^4S_b(Tx,Tx,Ty)) \leq F(G_{S,T}(x,y)) - \psi(G_{S,T}(x,y))$,
where $G_{S,T}(x,y) = \max\{S_b(Sx,Sx,Sy), S_b(Sx,Sx,Tx)$,
$S_b(Sy,Sy,Ty), \frac{S_b(Sx,Sx,Ty)+S_b(Sy,Sy,Tx)}{4s^3}\}$, $F \in \mathfrak{F}$ and $\psi \in \mathfrak{U}$.
Then there exists a common fixed point of S and T.

Proof. By condition (b), we can define a mapping I on X satisfying $SIu = Tu \quad \forall u \in X$ and I commutes with S. Let $x_0 = x$ and $x_n = I^n x_0 \; \forall n \in \mathbb{N}$. Then $x_{n+1} = Ix_n$ and $Sx_{n+1} = Tx_n \; \forall n \in \mathbb{N}$. Let for all $n \in \mathbb{N}$, $S_b(Sx_n,Sx_n,Sx_{n+1}) \neq 0$. Now

$$\begin{aligned} \frac{1}{3s}S_b(Sx_n,Sx_n,Tx_n) &= \frac{1}{3s}S_b(Sx_n,Sx_n,Sx_{n+1}) \\ &< S_b(Sx_n,Sx_n,Sx_{n+1}). \end{aligned}$$

So by the hypothesis

$$\begin{aligned} F(S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})) &= F(S_b(Tx_n,Tx_n,Tx_{n+1})) \\ &\leq F(G_{S,T}(x_n,x_{n+1})) \\ &\quad -\psi(G_{S,T}(x_n,x_{n+1})) \end{aligned}$$

Now $max\{S_b(Sx_n,Sx_n,Sx_{n+1}), S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\}$

$$\begin{aligned} &\leq G_{S,T}(x_n,x_{n+1}) \\ &= max\{S_b(Sx_n,Sx_n,Sx_{n+1}), S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2}), \\ &\quad \frac{S_b(Sx_n,Sx_n,Sx_{n+2}) + S_b(Sx_{n+1},Sx_{n+1},Sx_{n+1})}{4s^3}\} \\ &= max\{S_b(Sx_n,Sx_n,Sx_{n+1}), S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\} \end{aligned}$$

So we have

$$\begin{aligned} &F(S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2}) \leq F(\max\{S_b(Sx_n,Sx_n,Sx_{n+1}), \\ &S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\}) - \psi(\max\{S_b(Sx_n,Sx_n,Sx_{n+1}), \\ &S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\}). \qquad (3.1) \end{aligned}$$

$\max\{S_b(Sx_n,Sx_n,Sx_{n+1}),S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\}$
$=S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})$ leads to a contradiction. Therefore from equation (3.1)

$$F(S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2}))\leq F(S_b(Sx_n,Sx_n,Sx_{n+1})) -\psi(S_b(Sx_n,Sx_n,Sx_{n+1})).$$

Using monotonocity of F we have $S_b(Sx_{n+1},Sx_{n+1},Sx_{n+2})\leq S_b(Sx_n,Sx_n,Sx_{n+1})$, for all $n\in\mathbb{N}$. So, $\{S_b(Sx_n,Sx_n,Sx_{n+1})\}$ is a non increasing sequence. So there exists a $\alpha>0$ such that $\lim_{n\to\infty}S_b(Sx_n,Sx_n,Sx_{n+1})=\alpha$.
So from (3.1) we have $F(\alpha)\leq F(\alpha)-\psi(\alpha)\Rightarrow\psi(\alpha)=0$ $\Rightarrow\alpha=0$. i.e.,

$$\lim_{n\to\infty}S_b(Sx_n,Sx_n,Sx_{n+1})=0 \tag{3.2}$$

Now we will show that $\lim_{n,m\to\infty}S_b(Sx_n,Sx_n,Sx_m)=0$. Suppose the contrary, i.e., $\lim_{n,m\to\infty}S_b(Sx_n,Sx_n,Sx_m)\neq 0$. Let $\varepsilon>0$ and $\{p_n\}$ and $\{q_n\}$ be two sequence of natural numbers such that for all $n\in\mathbb{N}$, $p(n)>q(n)>n$,

$$\left.\begin{array}{rl} S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)})\geq & \varepsilon,\\ S_b(Sx_{p(n)-1},Sx_{p(n)-1},Sx_{q(n)})< & \varepsilon \end{array}\right\} \tag{3.3}$$

Observe

$$\begin{aligned} \varepsilon &\leq S_b(Sx_{q(n)},Sx_{q(n)},Sx_{p(n)-1})\\ &\leq s[2S_b(Sx_{q(n)},Sx_{q(n)},Sx_{q(n)})\\ &\quad +S_b(Sx_{p(n)-1},Sx_{p(n)-1},Sx_{q(n)})]\\ &< s\varepsilon. \end{aligned}$$

$$\begin{aligned} \varepsilon &\leq S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)})\\ &\leq s[2S_b(Sx_{p(n)},Sx_{p(n)},Sx_{p(n)-1})\\ &\quad +S_b(Sx_{q(n)},Sx_{q(n)},Sx_{p(n)-1})]\\ &\leq s^2\varepsilon. \end{aligned}$$

So,

$$\varepsilon\leq\lim_{n\to\infty}sup\ S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)})\leq s^2\varepsilon \tag{3.4}$$

In a similar way using (3.4)

$$\frac{\varepsilon}{s}\leq\lim_{n\to\infty}sup\ S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)+1})\leq 2s^3\varepsilon \tag{3.5}$$

Similarly we can show that

$$\frac{\varepsilon}{s}\leq\lim_{n\to\infty}sup\ S_b(Sx_{q(n)},Sx_{q(n)},Sx_{p(n)+1})\leq 2s^4\varepsilon \tag{3.6}$$

$$\lim_{n\to\infty}sup\ S_b(Sx_{p(n)+1},Sx_{p(n)+1},Sx_{q(n)+1})\geq\frac{\varepsilon}{s^2} \tag{3.7}$$

For each positive integer $n\geq N$

$$\begin{aligned} \frac{1}{3s}S_b(Sx_{p(n)},Sx_{p(n)},Tx_{p(n)}) &< S_b(Sx_{p(n)},Sx_{p(n)},Sx_{p(n)+1})\\ &< \varepsilon\\ &< S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)}) \end{aligned}$$

Then for all $n\geq N$, by the hypothesis we have
$F(s^4S_b(Tx_{p(n)},Tx_{p(n)},Tx_{q(n)})$

$$\leq F(G_{S,T}(x_{p(n)},x_{q(n)}))-\psi(G_{S,T}(x_{p(n)},x_{q(n)})) \tag{3.8}$$

Now $S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)})$

$$\begin{aligned} &\leq G_{S,T}(x_{p(n)},x_{q(n)})\\ &= max\{S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)}),\\ &\quad S_b(Sx_{p(n)},Sx_{p(n)},Sx_{p(n)+1}),\\ &\quad S_b(Sx_{q(n)},Sx_{q(n)},Sx_{q(n)+1}),\\ &\quad \frac{S_b(Sx_{p(n)},Sx_{p(n)},Sx_{q(n)+1})}{4s^3}\\ &\quad +\frac{S_b(Sx_{q(n)},Sx_{q(n)},Sx_{p(n)+1})}{4s^3}\} \end{aligned}$$

$\Rightarrow\varepsilon\leq\lim_{n\to\infty}sup\ G_{S,T}(x_{p(n)},x_{q(n)})\leq s^2\varepsilon$ (using (3.2)-(3.7)).
$\Rightarrow F(s^2\varepsilon)\leq F(s^2\varepsilon)-\psi(\varepsilon)$, a contradiction.
Hence, $\lim_{n,m\to\infty}S_b(Sx_n,Sx_n,Sx_m)=0$. Hence $\{Sx_n\}$ is a Cauchy sequence in (X,S_b). Similarly, $\lim_{n,m\to\infty}S_b(SSx_n,SSx_n,SSx_m)=0$. By completeness of (X,S_b) there exists $z\in X$ such that $\{Sx_n\}$ converges to z. Since S is continuous, $\{SSx_n\}$ convergent to Sz. Now we will prove $Sz=z$. We consider two cases. In the first case when the set

$$\{n:S_b(Sx_n,Sx_n,Tx_n)>S_b(Sx_n,Sx_n,SSx_n)\}$$

is infinite, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $S_b(Sx_{n_j},Sx_{n_j},Tx_{n_j})>S_b(Sx_{n_j},Sx_{n_j},SSx_{n_j})$.

$$\begin{aligned} \lim_{j\to\infty}S_b(SSx_{n_j},SSx_{n_j},Sx_{n_j}) &\leq \lim_{j\to\infty}S_b(Sx_{n_j},Sx_{n_j},Tx_{n_j})\\ &= 0. \end{aligned}$$

So by Lemma 3.2 we have $Sz=z$.

In the second case when the set $\{n:S_b(Sx_n,Sx_n,Tx_n)>S_b(Sx_n,Sx_n,SSx_n)\}$ is finite, there exists $M_1\in\mathbb{N}$ such that $S_b(Sx_n,Sx_n,Tx_n)\leq S_b(Sx_n,Sx_n,SSx_n)$ $\forall n\geq M_1$. So by the assumption
$\frac{1}{3s}S_b(Sx_n,Sx_n,Tx_n)\leq S_b(Sx_n,Sx_n,Tx_n)\leq S_b(Sx_n,Sx_n,SSx_n)$ $\forall n\geq M_1$. So from the hypothesis

$$\begin{aligned} F(s^4S_b(Tx_n,Tx_n,TSx_n) &\leq F(G_{S,T}(x_n,Sx_n))\\ &- \psi(G_{S,T}(x_n,Sx_n)) \end{aligned} \tag{3.9}$$

Now $S_b(Sx_n,Sx_n,SSx_n)$

$$\begin{aligned} &\leq G_{S,T}(x_n,Sx_n)\\ &= \max\{S_b(Sx_n,Sx_n,SSx_n),S_b(Sx_n,Sx_n,Tx_n),\\ &\quad S_b(SSx_n,SSx_n,TSx_n),\\ &\quad \frac{S_b(Sx_n,Sx_n,TSx_n)+S_b(SSx_n,SSx_n,Tx_n)}{4s^3}\} \end{aligned}$$

$\Rightarrow \frac{1}{s^2}S_b(z,z,Sz) \leq \lim_{n\to\infty} sup\ G_{S,T}(x_n,Sx_n) \leq s^2 S_b(z,z,Sz)$.
From (3.9) we have

$$\begin{aligned} F(s^4\frac{1}{s^2}S_b(z,z,Sz)) &\leq F(s^2S_b(z,z,Sz)) \\ &\quad -\psi(\frac{1}{s^2}S_b(z,z,Sz)) \end{aligned}$$

$\Rightarrow S_b(z,z,Sz) = 0 \Rightarrow z = Sz$.
Now we will show that z is a fixed point of T. First prove for all $n \in \mathbb{N}$

$$\frac{1}{3s}S_b(SSx_n,SSx_n,TSx_n) \leq S_b(SSx_n,SSx_n,z) \qquad (3.10)$$

or

$$\frac{1}{3s}S_b(STx_n,STx_n,T^2x_n) \leq S_b(STx_n,STx_n,z) \qquad (3.11)$$

holds. Contrapositively let there exists $m \in \mathbb{N}$ such that $\frac{1}{3s}S_b(SSx_m,SSx_m,TSx_m) > S_b(SSx_m,SSx_m,z)$ and $\frac{1}{3s}S_b(STx_m,STx_m,T^2x_m) > S_b(STx_m,STx_m,z)$
Then,

$$\begin{aligned} 3sS_b(SSx_m,SSx_m,z) &< S_b(SSx_m,SSx_m,TSx_m) \\ &\leq s[2S_b(SSx_m,SSx_m,z) \\ &\quad +S_b(TSx_m,TSx_m,z)] \end{aligned}$$

$\Rightarrow S_b(SSx_m,SSx_m,z) < S_b(TSx_m,TSx_m,z)$. Now

$$\begin{aligned} S_b(STx_m,STx_m,T^2x_m) &= S_b(SSx_{m+1},SSx_{m+1},SSx_{m+2}) \\ &\leq S_b(SSx_m,SSx_m,SSx_{m+1}) \\ &\leq s[2S_b(SSx_m,SSx_m,z) \\ &\quad +S_b(SSx_{m+1},SSx_{m+1},z)] \\ &< 3sS_b(TSx_m,TSx_m,z) \\ &< S_b(STx_m,STx_m,T^2x_m) \end{aligned}$$

This is a contradiction. So (3.10) or (3.11) holds. From (3.10) $F(s^4S_b(TSx_n,TSx_n,Tz)) \leq F(G_{S,T}(Sx_n,z)) - \psi(G_{S,T}(Sx_n,z))$ $\Rightarrow S_b(z,z,Tz) = 0 \Rightarrow Tz = z$. Similarly from (3.11) we have $Tz = z$. □

Example 3.5. *Let $X = \{0,1,2,3\}$. Let $A = \{(1,1,0),(0,0,1),(1,0,0)\}$, $B = \{(x,x,y) : \forall x,y \in X\} \setminus A \setminus (0,0,0)$ and $C = \{(x,x,x) : \forall x \neq 0 \in X\}$.*

$$\begin{aligned} S_b(x,y,z) &= \frac{1}{500} \quad if\ x=y=z=0 \\ &= \frac{1}{16} \quad if\ (x,y,z) \in A \\ &= \frac{3}{2} \quad if (x,y,z) \in B \\ &= \frac{1}{20} \quad if (x,y,z) \in C \\ &= 4 \quad otherwise. \end{aligned}$$

Then S_b is a partial-S-b metric on X with coefficient 2. Let $F(x) = logx$ and $\psi(t) = \frac{0.01t}{10+t}$. we define $S,T : X \longrightarrow X$ by $S = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $ST = TA$ and S is continuous. Also $T(X) \subset S(X)$. S and T satisfies the assumption of Theorem 3.3 and 0 is the common fixed point of S and T.

4. Conclusion

All the results of the article [9] were done using symmetric condition, whereas the same results have been produced in the present article in less condition, i.e., without using symmetric condition with necessary modifications and corrections. In this article we define weak partial metric space and established a fixed point theorem using F- contraction which can be studied further more for more characterization of completeness of this space.

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