Malaya Journal of Matematik, Vol. 8, No. 1, 159-162, 2020
https://doi.org/10.26637/MJM0801/0026

# Equal eccentric domination in graphs

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**Abstract**
A subset $S$ of $V$ in a graph $G=(V,E)$ is called an equal eccentric dominating set(eed-set) if $S$ is a dominating set and $\forall y \in V-S$, $\exists$ at least one equal eccentric vertex $x$ of $y$ in $S$. In this paper, equal eccentric vertex, equal eccentric set, equal eccentric dominating set and equal eccentric domination numbers are defined. The equal eccentric domination numbers of various standard graphs are obtained and the bounds on equal eccentric domination numbers are also obtained and theorems related to this concepts are stated and proved.

**Keywords**
Graph, Domination number, Eccentricity, Eccentric domination number, Equal eccentric domination number.

**MSC[2010] Subject Classification**
05C07, 05C69, 05C12.

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**Article History**: Received **17** October **2019**; Accepted **11** December **2019**


## Contents



## 1. Introduction

In 1960, the domination in graphs were started. O.Ore [8] defined the terms dominating set and domination number of a graph in 1962. Haynes et. al., [5] discussed many dominating parameters in the year 1998. Alikhani et. al., [1] determines many domination numbers for some standard graphs. The radius, diameter and eccentricity of a vertex of a graphs are defined by Harary [3],[4]. Janakiraman et. al., [6] defined eccentric domination in graphs in the year 2010. In 2011, eccentric domination numbers are determined in trees by Bhanumathi and Muthammai [2] and various bounds of eccentric domination in graphs are also determined by the same authors. Detour eccentric domination number in graphs are discussed by Mohamed Ismayil and Priyadharshini [7] in 2019.

In this paper, the equal eccentric dominating set and its numbers are defined. The equal eccentric domination numbers are obtained for various standard graphs. The bounds on equal eccentric domination number are obtained for various graphs.

## 2. Preliminaries

In this section, some preliminary definitions which are very useful to understand this paper clearly and statement of the theorems proved already by different authors which are mentioned in the bibliography. For undefined graph theoretical terminologies one can refer Harary [4], Buckley and Harary [3]. The graph $G(V,E)$ is simply denoted by $G$ and $G$ is a connected graph unless otherwise stated in this paper.

**Definition 2.1.** *Let $G(V,E)$ be a finite, simple and undirected graph with vertex set $V$ of order $|V|=n$ and the edge set $E$ of size $|E|=m$. Let $x,y$ be any two vertices in $V$, then $N(x)=\{y \mid (x,y)\in E\}$ is called the neighbourhood of $x$ and $N[x]=N(x)\cup\{x\}$ is called the closed neighbourhood of $x$.*

**Definition 2.2** ([4], p.35)**.** *Let $G$ be a connected graph and $x,y$ are any two vertices in $G$. The length of the shortest path between two vertices $x$ and $y$ is called distance of $x$ and $y$ and is denoted by $d(x,y)$. The eccentricity of a vertex $x$ is $e(x)=max\{d(x,y)|y\in V\}$. The minimum eccentricity of a vertex is called the radius of $G$ and is denoted by $r(G)$. The maximum eccentricity of a vertex is called the diameter of $G$ and is denoted by $d(G)$. If $e(x)=r(G)$ then $x$ is called centre* *of G and if $e(x) = d(G)$ then x is called peripheral vertex or boundary of G.*

**Definition 2.3** ([6],p.55)**.** *For a connected graph, $r(G) \leq d(G) \leq 2.r(G)$*

**Definition 2.4** ([6],p.55)**.** *For any vertex $x \in V$, the eccentric set $E(x)$ of a vertex x is defined by*

$$E(x) = \{y \in V | d(x,y) = e(x)\}$$

**Definition 2.5** ([1],p.1880)**.** *Let $x, y \in V$ be any two vertices in a graph G. The vertex x dominates y if $d(x,y) = 1$. A set $D \subseteq V$ is called a dominating set in G if $N[D] = V$. If $D' \subset D$ is not a dominating set then the cardinality of D is called domination number and is denoted by the symbol $\gamma$.*

**Definition 2.6** ([6],p.56)**.** *A subset S of V in a graph G is an eccentric dominating set(ed-set) if S a is dominating set and $\forall y \in V - S$, $\exists$at least one eccentric vertex x of y in S. If $S' \subset S$ is not an eccentric dominating set then the eccentric dominating set S is minimal. The minimum cardinality among all the ed-set is called eccentric domination number and is denoted by the symbol $\gamma_{ed}$.*

Let $x$ and $y$ be any two vertices in a graph $G$ and if $e(x) = e(y)$ then $x$ and $y$ have the same eccentric values (or $x$ and $y$ are equal eccentric vertices) in a graph $G$. Using this concept, the equal eccentric set, equal eccentric dominating set and equal eccentric domination numbers are defined.

## 3. Equal eccentric domination

In this section, equal eccentric set and equal eccentric dominating sets are defined with suitable examples. Some observations related to equal eccentric dominating sets and its numbers are discussed. Theorems on equal eccentric dominating sets are stated and proved.

**Definition 3.1.** *Let $G = (V,E)$ be a graph and $x, y \in V$. If $e(x) = e(y)$ then y is an equal eccentric vertex of x. The equal eccentric set of a vertex y is defined by*

$$E_e(y) = \{x \in V | e(x) = e(y)\}$$

*The set $S \subseteq V$ is called an equal eccentric set if $\forall y \in V - S$. $\exists$ at least one equal eccentric vertex x of $y \in S$ such that $e(x) = e(y)$.*

**Definition 3.2.** *A subset S of V in a graph $G = (V,E)$ is called an equal eccentric dominating set(eed-set) if S is a dominating set and $\forall y \in V - S$, $\exists$ at least one equal eccentric vertex x of y in S.*

**Definition 3.3.** *An equal eccentric dominating set S is minimal if there does not exist any equal eccentric dominating set $S'$ such that $S' \subset S$. The minimum cardinality taken over all eed-set is called equal eccentric domination number and is denoted by the symbol $\gamma_{eed}$. The maximum cardinality of a minimal eed-set is called upper equal eccentric domination number and is denoted by the symbol $\Gamma_{eed}$.*

**Remark 3.4.** *A $\gamma_{eed}$-set is called an equal eccentric dominating set with minimum cardinality. Similarly $\gamma_{ed}$-set and $\gamma$-set.*

**Observation 3.5.** .

1. *For every connected graph G, $\gamma \leq \gamma_{eed} \leq \gamma_{ed}$.*
2. *For any graph, $1 \leq \gamma_{eed} \leq n$, where n is the order of G.*
3. *Let S be an eed-set, if $E_e(x) = \phi$ for some $x \in V$ then $x \in S$.*
4. *A connected graph G with diameter $d = 2$ then $\gamma_{eed} = 2$ otherwise $\gamma_{eed} \geq 2$.*
5. *For any graph, $\gamma_{eed} \leq \Gamma_{eed}$.*
6. *Every super set of an eed-set is also an eed-set.*
7. *The complement of a minimum eed-set need not be an eed-set.*

**Example 3.6.** *Consider the graph given in Figure 1.*

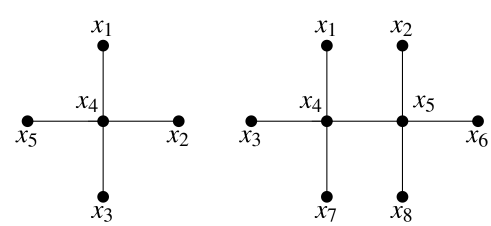


**Figure 1**

*In Figure 1(i), $\gamma_{ed}$-set $= \{x_1,x_4\} = \{x_2,x_4\} = \{x_3,x_4\} = \{x_4,x_5\} \Rightarrow \gamma_{ed} = 2$ also these are all $\gamma_{eed}$-sets. Hence $\gamma_{ed} = \gamma_{eed} = 2$, but $\gamma$-set $= \{x_4\} \Rightarrow \gamma = 1$. Hence $\gamma = 1 \leq \gamma_{eed} = 2 \leq n = 5$. Every eed-set contains the vertex $\{x_4\}$ and hence $E_e(x_4) = \phi$. The complement of $\{x_1,x_4\}$ is $\{x_2,x_3,x_5\}$ which is not an eed-set.*

*From the Figure 1(ii), $\gamma_{ed}$-set $= \{x_1,x_2,x_4,x_5\} \Rightarrow \gamma_{ed} = 4$ and $\gamma_{eed}$-set $= \{x_2,x_4,x_5\} \Rightarrow \gamma_{eed} = 3$. Hence $\gamma_{eed} < \gamma_{ed}$. $\{x_2,x_4,x_6,x_8\}$ is a minimal equal eccentric dominating set with maximum cardinality $\Rightarrow \Gamma_{eed} = 4$. Hence $\gamma_{eed} \leq \Gamma_{eed}$*

**Remark 3.7.** *Every eccentric dominating set is an equal eccentric dominating set but converse need not be true.*

**Example 3.8.** *Consider the Figure 1(ii) $\{x_1,x_2,x_4,x_5\}$ is an eccentric dominating set as well as equal eccentric dominating set and $\{x_2,x_4,x_5\}$ is an equal eccentric dominating set but not an eccentric dominating set.*

**Theorem 3.9.** *An equal eccentric dominating set S in a graph G is minimum if and only if every vertex $x \in S$ satisfies one of the following conditions:*
*(i) $N(x) \cap S = \phi$ or $E_e(x) = \phi$.*
*(ii) $N(y) \cap S = \{x\}$ or $E_e(y) \cap S = \{x\}$ for some $y \in V - S$.*

*Proof.* Assume that $S$ is a minimal equal eccentric dominating set of a graph $G$. Then for every vertex $x \in S$, $S-\{x\}$ is not an equal eccentric dominating set. Then there exists some $y \in (V-S) \cup \{x\}$ which is not dominated by any vertex in $S-\{x\}$ or $y$ has no equal eccentric vertex in $S-\{x\}$.
Case(i): Suppose $y = x$ then $x$ is an isolate of $S$ or $x$ has no equal eccentric vertex in $S$. In this case $N(x) \cap S = \phi$ or $E_e(x) = \phi$.
Case(ii): suppose $y \in V-S$.

(a) If $y$ is not dominated by $S-\{x\}$, but dominated by $S$, then $y$ is adjacent to only $x$ in $S$, that is $N(y) \cap S = \{x\}$.

(b) Suppose $y$ has no equal eccentric vertex in $S-\{x\}$ but $y$ has equal eccentric vertex in $S$. Then $x$ is the only equal eccentric vertex of $y$ in $S$, that is $Ee(y) \cap S = \{x\}$.

Conversely, suppose that $S$ is an equal eccentric dominating set and for each $x \in S$ one of the conditions holds. Suppose S is not an equal eccentric dominating set then there exists a vertex $x \in D$ such that $D-\{x\}$ is an equal eccentric dominating set. Hence $x$ is adjacent to at least one vertex $y$ in $S-\{x\}$ and x has an equal eccentric vertex in $S-\{x\}$. Condition (i) doest not hold.
If $S-\{x\}$ is an equal eccentric dominating set, then every vertex $y$ in $(V-S) \cup \{x\}$ is adjacent to at least one vertex in $S-\{x\}$ and $y$ has an equal eccentric vertex in $S-\{x\}$. Therefore condition (ii) does not hold. This contradicts to our assumption that for each $x \in S$ one of the conditions holds. □

**Theorem 3.10.** . *Every dominating set of a connected regular graph is an equal eccentric dominating set.*

*Proof.* Let $G(V,E)$ be a connected regular graph. Then every singleton subset of V has same eccentricity. Therefore every vertex of V is equal eccentric to all other vertices. Every vertex of V is an equal eccentric set, it is also an element of dominating set. Hence every dominating set of a regular graph is an equal eccentric dominating set. □

**Theorem 3.11.** *If $x$ is a unique central vertex then $x$ is an element of every eed-set.*

*Proof.* Let $x$ be a unique central vertex then $E_e(x) = \phi$. Hence by the Observation 3.5(3) , $x$ is an element of every eed-set. □

**Observation 3.12.** *Every eed-set containing at least a central vertex and a peripheral vertex.*

**Example 3.13.** *In the figure 1(i) the central vertex $x_4$ and one of the peripheral vertex $x_1$ is in the $\gamma_{eed}$-set. In the figure 1(ii) one of the central vertex $x_4$ and one of the peripheral vertex $x_2$ are in the $\gamma_{eed}$-set.*

**Theorem 3.14.** *If G is a disconnected graph then every dominating set is an equal eccentric dominating set.*

*Proof.* If $x$ and $y$ are any two vertices in two different components then $d(x,y) = \infty$. Then the $deg(y_i) = \infty$ for all $y_i \in V$. Hence every vertex of $V$ is equal eccentric to all other vertices. Therefore every dominating set is an equal eccentric dominating set.

Converse not true since by theorem 3.10 every dominating set is an equal eccentric dominating set in a connected regular graph. □

## 4. Equal eccentric domination number of some standard graphs

In this section, the equal eccentric domination numbers of some well known graphs are established in the following theorem.

**Theorem 4.1.** .

| | | | |
|---|---|---|---|
| *(1)* | $\gamma_{eed}(K_n)$ | $= 1$, | *where n is order of the graph.* |
| *(2)* | $\gamma_{eed}(P_n)$ | $= \lceil \frac{n}{2} \rceil$, | *where $\lceil \frac{n}{2} \rceil$ is the least integer greater then $\frac{n}{2}$.* |
| *(3)* | $\gamma_{eed}(K_{(1,n)})$ | $= 2$ , | *for $n \geq 2$.* |
| *(4)* | $\gamma_{eed}(K_{(m,n)})$ | $= 2$ , | *for $m = n \neq 1$.* |
| *(5)* | $\gamma_{eed}(W_n)$ | $= 2$ , | *for $n \geq 4$.* |
| *(6)* | $\gamma_{eed}(K_n - \{e\})$ | $= 2$ , | *for $n \geq 2$.* |

*Proof.* (1) Every vertex of $K_n$ is a dominating set as well as equal eccentric set. Hence every singleton set of $K_n$ is a minimum equal eccentric dominating set. Thus $\gamma_{eed}(K_n) = 1$.
(2) Let the vertex set of $P_n$ be $V = \{x_1, x_2, x_3 \ldots x_n\}$.
Case(i): If $n$ is even and $\{x_1, x_3, x_5 \ldots x_{n-1}\}$ is one of a minimum equal eccentric dominating set. Hence $\gamma_{eed}(P_n) = \frac{n}{2}$.
Case(ii): If $n$ is odd and $V = \{x_1, x_3, x_5 \ldots x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \ldots, x_{n-1}\}$ is one of a minimum equal eccentric dominating set. Hence $\gamma_{eed}(P_n) = \frac{n+1}{2}$. From both the cases $\gamma_{eed}(P_n) = \lceil \frac{n}{2} \rceil$.
(3) Let $x$ be a central vertex of $K_{(1,n)}$. Then $\{x\}$ is a dominating set, but not an eccentric dominating set(since eccentricity of $x$ is one and all other vertices two). Let $y$ be any one of the vertex with eccentricity two. Hence $\{x,y\}$ is a minimum equal eccentric dominating set. Thus $\gamma_{eed}(K_{(1,n)}) = 2$.
(4) Every vertex of a bipartite graph has equal eccentricity two. Therefore every singleton set of $K_{(m,n)}$ is an equal eccentric set but not a dominating set. Let $V = V_1 \cup V_2$, $x \in V_1$ and $y \in V_2$. Then $\{x,y\}$ is a dominating set and also equal eccentric set. Therefore $\{x,y\}$ is one of the minimum equal eccentric dominating set. Hence $\gamma_{eed}(K_{(m,n)}) = 2$.
(5) In a wheel graph, central vertex (say) $x$ dominates all other vertices but $\{x\}$ is not an equal eccentric set. Every vertex (say) $y$ with eccentricity two is an element of equal eccentric set. Hence $\{x,y\}$ is a minimum equal eccentric dominating set. Thus $\gamma_{eed}(W_n)) = 2$.
(6) If $e = (x,y)$ is a deleted edge from $K_n$ then $x$ does not dominates $y$. $\{x,y\}$ is a dominating set of $K_n - \{e\}$ and these vertices have same eccentricity (two). Therefore $\{x,y\}$ is not an equal eccentric dominating set. Let $w$ be a vertex with eccentricity one, $w$ is an equal eccentric vertex of all the vertices except $x$ and $y$. Hence $\{x,w\}$ and $\{y,w\}$ are some minimum equal eccentric dominating sets. Hence $\gamma_{eed}(K_n - \{e\}) = 2$. □

**Theorem 4.2.** *If G is a connected regular graph, then $\gamma_{eed}(G) = \gamma(G)$.*

*Proof.* By a theorem 3.10, every dominating set is an equal eccentric dominating set in a connected regular graph. Hence $\gamma_{eed}(G) = \gamma(G)$.

The converse of the above corollary is not true. For example, $\gamma(P_4) = 2 = \gamma_{eed}(P_4)$ but $P_4$ is not regular. □

**Observation 4.3.** *Cyclic graph $C_n$ is a regular graph, then $\gamma_{eed}(C_n) = \gamma(G) = \lceil \frac{n}{3} \rceil$.*

**Theorem 4.4.** *If G is of radius one and diameter two, then $\gamma_{eed} = 2$.*

*Proof.* Let $x$ be a vertex with radius one and diameter of $G$ be two. Then $x$ is a central vertex and $\{x\}$ is a dominating set but not an equal eccentric set. Let $y$ be a vertex with diameter two. Then $y$ is an equal eccentric vertex of all the vertices except $x$. Hence $\{x, y\}$ is an equal eccentric dominating set. □

**Theorem 4.5.** *In a graph G, if $r(G) = d(G)$, then $\gamma_{eed} = \gamma$.*

*Proof.* If $r(G) = d(G)$ then G is a regular graph. By corollary 4.2 $\gamma_{eed} = \gamma$ □

## 5. Bounds on equal eccentric domination number

In this section, bounds on equal eccentric domination numbers for standard graphs are obtained.

**Theorem 5.1.** *1. If G is a connected graph then $1 \leq \gamma_{eed} \leq \lceil \frac{n}{2} \rceil$.*
*2. If G is a disconnected graph then $x \leq \gamma_{eed} \leq n$, where x is a number of components in G.*

*Proof.* (1) From theorem 4.1*(i)* and 4.1*(ii)* we obtain $1 \leq \gamma_{eed} \leq \lceil \frac{n}{2} \rceil$.
(2) Let $G$ be a disconnected graph. Then vertices from different components are same eccentricity. The set $\{x_i | x_i \in V_x 1 \leq x \leq n\}$ is a minimum dominating set and equal eccentric set and hence every dominating set is an equal eccentric dominating set. Therefore $\gamma_{eed} = \gamma \leq n$. Suppose G is totally disconnected then $\gamma_{eed} = \gamma = n$. From both the cases $x \leq \gamma \leq n$. □

**Theorem 5.2.** *If $T$ is a tree with n vertices, then $\gamma_{eed} \leq n - t + 1$, where t is the number of pendent vertices in T.*

*Proof.* Every pendent vertex of $T$ is adjacent to a non pendent vertex and the set of all non pendent vertex is a dominating set which implies that $\gamma \leq n - t$. The eccentricity of some pendent vertex and the eccentricity of some non pendent are same. The pendent vertex (which is a peripheral vertex) of a tree is not an equal the eccentric vertex of any non pendent vertices. Include any one such vertex in the dominating set, then the dominating set becomes an equal eccentric dominating set. Hence $\gamma_{eed} \leq n - t + 1$. □

## 6. Conclusion

In this paper, the equal eccentric sets and equal eccentric dominating sets in graphs are studied. Equal eccentric domination numbers are discussed with suitable examples. The equal eccentric numbers of some standard graphs are observed. The bounds for equal eccentric domination numbers are obtained. Further connected equal eccentric domination and independent equal eccentric domination can be developed using this concept.

## References


[1] S. Alikhani et. al., On the domination number of some graphs, *International Mathematical Forum*, 3(38)(2008), 1879–1884.

[2] M. Bhanumathi, S. Muthammai, On eccentric domination in trees, *International Journal of Engineering Science, Advanced computing and Bio Technology*, 2(1)(2011), 38–46.

[3] F. Buckly, F. Harary , *Distance in Graphs*, Addition-Wesley, Publishing company, 1990.

[4] F. Harary, *Graph Theory*, Addition Wesley, Publishing company, 1972.

[5] T.W. Haynes et. al., *Domination in Graphs*, Marcel Dekker, Inc., 1998.

[6] T.N. Janakiraman et. al., Eccentric domination in graphs, *International Journal of Engineering Science, Advanced computing and Bio Technology* 1(2)(2010), 55–70.

[7] A. Mohamed Ismayil and R. Priyadharshini, Detour eccentric domination in graphs, *Bulletin of Pure and Applied Sciences*, 38E(1)(2019), 342–347.

[8] Ore, O. Theory of Graphs. Amer. Math. Soc. Colloq. Publ., 38, Amer. Math. Soc., Providence, RI (1962).



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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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