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Approximating positive solutions of nonlinear BVPs of ordinary second order hybrid differential equations

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Abstract. In this paper we prove the existence and approximation of solution for a nonlinear two point boundary value problem of ordinary second order hybrid differential equations with Dirichlet boundary conditions via construction of an algorithm. The nonlinearity present on the right hand side of the differential equation is assumed to be Carathoèodory and the proof is based on a Dhage iteration method based on a hybrid fxed point theorem of Dhage (2014) in an ordered Banach algebra.

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1. Introduction

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_+$ the set of all nonnegative reals. Given a closed and bounded interval $J = [a, b] \subset \mathbb{R}$, $a < b$, consider the nonlinear two point hybrid boundary value problem (in short HBVP) of ordinary hybrid differential equation,

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' &= g(t,x(t)) \quad \text{a.e. } t \in J, \\ x(a) = &0 = x(b), \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and $f : J \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

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When $f \equiv 1$ on $I \times \mathbb{R}$, the HBVP (1.1) reduces to the well-known nonlinear two point BVP

$$\left.\begin{aligned} -x''(t) &= g(t, x(t)) \quad \text{a.e. } t \in J, \\ x(a) &= 0 = x(b), \end{aligned}\right\} \tag{1.2}$$

which is studied earlier extensively in the literature (see Bailey *et al.* [1]).

Definition 1.1. *A function $x \in AC^1(J, \mathbb{R})$ is said to be a* lower solution *of the HBVP (1.1) if*

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t, x(t))}\right)'' &\leq g(t, x(t)) \quad a.e. \ t \in J, \\ x(a) \leq \ 0 \ &\geq x(b), \end{aligned}\right\} \tag{1.3}$$

where, $AC^1(J, \mathbb{R})$ is the space of functions $x \in C(J, \mathbb{R})$ whose first derivative exists and is absolutely continuous on I. Similarly, $x \in AC^1(J, \mathbb{R})$ is called an upper solution *of (1.1) on J if the reversed inequalities hold in (1.3). If equalities hold in (1.3), we say that x is a* solution *of (1.1) on J.*

Notice that the differential equation (1.1) is a hybrid nonlinear differential equation with a quadratic perturbation of second type. The details of classification of different types of perturbations of the differential are given in Dhage [4]. The existence of the solution to the problem (1.1) may be proved by using hybrid fixed point theorems of Dhage in a Banach algebra as did in Dhage [2, 3], Dhage and Dhage [9], Dhage and Dhage [11] and Dhage and Imdad [13]. The existence and approximation result for the PBVP and the BVP (1.2) is already proved respectively in Dhage and Dhage [10] and Dhage [7] via a new Dhage iteration method developed in [5, 6]. In the present paper, we shall extend above Dhage iteration method to the HBVP (1.1) and study the existence and approximation of positive solutions of under certain hybrid conditions on the nonlinearities f and g from algebra, analysis and topology.

2. Auxiliary Results

We need the following definitions in what follows.

Definition 2.1 (Dhage [2–5])**.** *An upper-semicontinuous and nondecreasing real function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the condition $\psi(0) = 0$ is called a $\mathcal{D}$-function on $\mathbb{R}_+$. The class of all $\mathcal{D}$-functions is denoted by $\mathfrak{D}$.*

A few examples of the $\mathcal{D}$ functions on $\mathbb{R}_+$ appear in Dhage and Dhage [10] and references therein.

Definition 2.2. *A function $\beta : J \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t, x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto \beta(t, x)$ is continuous for each $t \in J$.

The following lemma is often used in the study of nonlinear differential equations (see Dhage [3], Dhage and Imdad [13] and references therein).

Lemma 2.3 (Carathéodory)**.** *Let $\beta : J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the map $(t, x) \mapsto \beta(t, x)$ is jointly measurable. In particular the map $t \mapsto \beta(t, x(t))$ is measurable on J for each $x \in C(J, \mathbb{R})$.*

We need the following hypotheses in the sequel.

(H_1) f defines a continuous bounded function $f : J \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ with bound M_f.

(H$_2$) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$0 \leq f(t,x) - f(t,y) \leq \psi_f\big(x-y\big)$$

for all $\in J$ and $x, y \in \mathbb{R}$ with $x \geq y$. Moreover, $\dfrac{(b-a)^2}{8} M_g \psi_f(r) < r, \;\; r > 0$.

(H$_3$) The function g is Carathéodory on $J \times \mathbb{R}$ into $\mathbb{R}_+$.

(H$_4$) g is bounded on $J \times \mathbb{R}$ with bound M_g.

(H$_5$) $g(t,x)$ is nondecreasing in x for each $t \in J$.

(LS) The HBVP (1.1) and (1.3) has a lower solution $u \in AC^1(J,\mathbb{R})$.

(US) The HBVP (1.1) and (1.3) has an upper solution $v \in AC^1(J,\mathbb{R})$.

Lemma 2.4. *Given any function $h \in L^1(J,\mathbb{R})$, the HBVP*

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' &= h(t) \quad a.e. \;\; t \in J, \\ x(a) = &0 = x(b), \end{aligned}\right\} \tag{2.1}$$

is equivalent to the quadratic hybrid integral equation (in short QHIE)

$$x(t) = \big[f(t,x(t))\big] \left(\int_a^b G(t,s)h(s)\,ds\right), \;\; t \in J, \tag{2.2}$$

where $G(t,s)$ is the Green's function associated with the homogeneous boundary value problem

$$\left.\begin{aligned} -x''(t) = 0 \;\; a.e. \;\; t \in J, \\ x(a) = 0 = x(b). \end{aligned}\right\} \tag{2.3}$$

Notice that the function x given by (2.2) belongs to the class $C(J,\mathbb{R})$. Clearly, $G(t,s)$ is continuous and nonnegative on $J \times J$ and satisfies the inequalities

$$0 \leq G(t,s) \leq \frac{b-a}{4} \quad \text{and} \quad \int_a^b G(t,s)\,ds \leq \frac{(b-a)^2}{8}.$$

The proof of our main result will be based on the **Dhage monotone iteration principle** or **Dhage monotone iteration method** contained in a applicable hybrid fixed point theorem in the partially ordered Banach algebras. A non-empty closed convex subset K of the Banach algebra E is called a cone if it satisfies i) $K + K \subset K$, ii) $\lambda K \subseteq K$ for $\lambda > 0$ and iii) $\{-K\} \cap K = \{0\}$. We define a partial order $\preceq$ in E by the relation $x \preceq y \iff y - x \in K$. The cone K is called positive if iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplicative composition in E. In what follows we assume that the cone K in a partially ordered Banach algebra (E,K) is always positive. Then the following results are known in the literature.

Lemma 2.5 (Dhage [8])**.** *Every ordered Banach space (E,K) is regular.*

Lemma 2.6 (Dhage [8])**.** *Every partially compact subset S of an ordered Banach space (E,K) is a Janhavi set in E.*

Theorem 2.7 (Dhage [5, 6])**.** *Let $\big(E, K, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra and let every chain C in E be a Janhavi set. Suppose that $\mathcal{A}, \mathcal{B} : E \to K$ are two monotone nondecreasing operators such that* *(a)* *$\mathcal{A}$ is partially bounded and partial $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*

(b) *$\mathcal{B}$ is partially continuous and uniformly partially compact,*

(c) *$M_{\mathcal{B}}\, \psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M_{\mathcal{B}} = \sup\{\|\mathcal{B}(C)\| \, : C$ is a chain in $E\}$, and*

(d) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0\, \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\, \mathcal{B}x_0$.*

Then the hybrid operator equation $\mathcal{A}x\, \mathcal{B}x = x$ has a solution x^* in K and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\, \mathcal{B}x_n$ converges monotonically to x^*.

The details of Dhage monotone iteration principle or method and related definitions of Janhavi set and uniformly partially compact operator along with some applications may be found in Dhage [5–8] and the references therein.

3. Existence and Approximation Result

Let $C_+(J,\mathbb{R})$ denote the space of all nonnegative-valued functions of $C(J,\mathbb{R})$. We assume that the space $C(J,\mathbb{R})$ is endowed with the norm $\|\cdot\|$ and the multiplication "$\cdot$" defined by

$$\|x\| = \max_{t\in J} |x(t)| \quad \text{and } (x \cdot y)(t) = x(t)y(t) \;\; t \in J. \tag{3.1}$$

We define a partial order $\preceq$ in E with the help of the cone K in E defined by

$$K = \big\{x \in E \mid x(t) \geq 0 \;\; \text{for all} \;\; t \in J\big\} = C_+(J,\mathbb{R}), \tag{3.2}$$

which is obviously a positive cone in $C(J,\mathbb{R})$. Thus, we have $x \preceq y \iff y - x \in K$.

Clearly, $C(J,\mathbb{R})$ is a partially ordered Banach algebra with respect to above supremum norm, multiplication and the partially order relation in $C(J,\mathbb{R})$. A solution ξ^* of the HBVP (1.1) is *positive* if it is in the class of function space $C_+(J,\mathbb{R})$.

Theorem 3.1. *Suppose that hypotheses (H$_1$)-(H$_5$) and (LS) hold. Then the BVP (1.1) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= u(t), \;\; t \in J, \\ x_{n+1}(t) &= \big[f(t,x_n(t))\big]\left(\int_a^b G(t,s)g(s,x_n(t))\,ds\right), \;\; t \in J, \end{aligned}\right\} \tag{3.3}$$

converges monotone nondecreasingly to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemmas 2.5 and 2.6, E is regular and every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\preceq$ so that every compact chain C is a Janhavi set in E.

Now by Lemma 2.3, the BVP (1.1) is equivalent to the QHIE

$$x(t) = \big[f(t,x(t))\big]\left(\int_a^b G(t,s)g(s,x(t))\,ds\right), \; t \in J. \tag{3.4}$$

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)), \;\; t \in J, \tag{3.5}$$

and

$$\mathcal{B}x(t) = \int_a^b G(t,s)g(s,x(t))\,ds, \ \ t \in J. \tag{3.6}$$

From hypotheses (H_1) and (H_3), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} \,:\, E \to K$. Now the QHIE (3.4) is equivalent to the quadratic hybrid operator equation

$$\mathcal{A}x(t)\,\mathcal{B}x(t) = x(t), \ t \in J. \tag{3.7}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, from the hypothesis (H_2) it follows that

$$\mathcal{A}x(t) = f(t,x(t)) \geq f(t,y(t)) = \mathcal{A}x(t)$$

for all $t \in J$. Hence $\mathcal{A}x \succeq \mathcal{A}(y)$ and that $\mathcal{A}$ is nondecreasing on E. Similarly, we have by hypothesis (H_5),

$$\mathcal{B}x(t) = \int_{t_0}^{t_1} G(t,s)g(s,x(s))d \geq \int_{t_0}^{t_1} G(t,s)g(s,y(s))\,ds = \mathcal{B}y(t)$$

for all $t \in I$. This implies that $\mathcal{B}x \succeq \mathcal{B}y$ whenever $x \succeq y$. Thus, $\mathcal{B}$ is also nondecreasing operator on E.

Step II: *Next we show that $\mathcal{A}$ is partially bounded and partial $\mathcal{D}$- Lipschitz on E.*

Now, for any $x \in E$, one has

$$\|\mathcal{A}x\| = \sup_{t \in J} |f(t,x(t))| \leq M_f$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ for all $x \in E$ and so $\mathcal{A}$ is bounded and so partially bounded on E. Nxt let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_2),

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \varphi_f(|x(t) - y(t)|) \leq \varphi_f(\|x - y\|)$$

for all $t \in J$. Taking the supremum over t, we get

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi_f(\|x - y\|)$$

for all $x, y \in E$, $x \succeq y$. This shows that $\mathcal{A}$ is a partial $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-function φ_f.

Step III: *$\mathcal{B}$ is a partially continuous and partially compact on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Since the f is continuous, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_a^b G(t,s)g(s,x_n(s))\,ds \\ &= \int_a^b G(t,s)\left[\lim_{n\to\infty} g(s,x_n(s))\right] ds = \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J. Next, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now for any $t_1, t_2 \in J$, one obtains

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \leq M_g \int_a^b |G(t_1,s) - G(t_2,s)|\,ds \tag{3.8}$$

Since the function $t \to G(t,s)$ is continuous on compact Jit is uniformly continuous there. Consequently the function $t \to G(t,s)$ is uniformly continuous on J. Therefore, we have

$$|G(t_1,s) - G(t_2,s)| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly on J. As a result, we have that

$$|\mathcal{B}x_n(t_1) - \mathcal{B}x_n(t_2)| \to 0 \quad \text{as } t_1 \to t_2,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Next, we show that $\mathcal{B}$ is a uniformly partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H$_2$)

$$|y(t)| = |\mathcal{B}x(t)| \le \int_a^b G(t,s)|g(s,x(s))|\,ds \le \frac{(b-a)^2}{8} M_g,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| = \|\mathcal{B}x\| \le \dfrac{(b-a)^2}{8} M_g$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step IV: *$\mathcal{A}$ and $\mathcal{B}$ satisfy the growth inequality $M_B\,\psi_A(r) < r$, $r > 0$.*

Now, it can be shown $\|\mathcal{B}(C)\| \le \dfrac{(b-a)^2}{8} M_g = M_{\mathcal{B}}$ for all chain C in E. Therefore, we obtain

$$M_{\mathcal{B}}\psi_{\mathcal{A}}(r) = \frac{(b-a)^2}{8} M_g \psi_f(r) < r$$

for all $r > 0$ and so the hypothesis (c) of Theorem 2.7 is satisfied.

Step VI: *The function u satisfies the operator inequality $u \preceq \mathcal{A}u\,\mathcal{B}u$.*

By hypothesis (LS), the HBVP (1.1) has a lower solution u defined on J. Then, we have

$$\left.\begin{aligned} -\left(\frac{u(t)}{f(t,u(t))}\right)'' &\le g(t,u(t)) \quad \text{a.e. } t \in J, \\ \frac{u(a)}{f(a,u(a))} \;\le 0\; &\ge\; \frac{u(b)}{f(b,u(b))}. \end{aligned}\right\} \tag{3.9}$$

By using this, the maximum principle [15] and the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, it can be shown that the function $u \in C(J,\mathbb{R})$ satisfies the relation $u \preceq \mathcal{A}u\,\mathcal{B}u$ on J.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7 and so the quadratic hybrid operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$ with initial term $x_0 = u$ converges monotone nondecreasingly to x^*. Therefore, the QHIE (3.4) and consequently the HBVP (1.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) with $x_0 = u$, converges monotone nondecreasingly to x^*. This completes the proof. ■

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (LS) with (US). The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. In this case the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.3) with $x_0(t) = v(t),\ t \in [0, T]$, converges monotone nonincrasingly to the solution x^* of he HIVP (1.1) on J. Again, the existence and approximation result, Theorem 3.1 includes similar result for the positive solution of the HBVP (1.2) as a special case.

Remark 3.3. We note that if the HBVP (1.1) has a lower solution $u \in AC^1(J, \mathbb{R})$ as well as an upper solution $v \in AC^1(J, \mathbb{R})$ such that $u \preceq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and y^* and these solutions satisfy the inequality

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the HBVP (1.1) in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set of elements in $C(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because of the order cone K defined by (3.2) is a closed convex subset of $C(J, \mathbb{R})$. However, we have not used any property of the cone K in the main existence results of this paper. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [8].

4. An Example

Example 4.1. Given a closed interval $J = [-1, 1]$ in $\mathbb{R}$, consider the nonlinear HBVP

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t, x(t))}\right)'' &= \tanh x(t) + 1 \quad \text{a.e. } t \in J, \\ x(-1) &= 0 = x(1), \end{aligned}\right\} \tag{4.1}$$

where the function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is defined by

$$f(t, x) = \begin{cases} 1, & x \leq 0, \\ 1 + \dfrac{x}{1 + x}, & x > 0. \end{cases}$$

Then the function f satisfies the hypotheses (H_1)-(H_2) with $M_f = 2$ and $\psi_f(r) = \dfrac{r}{1 + \xi^2}$, $0 \leq \xi \leq r$. Here $g(t, x) = \tanh x + 1$ and satisfies the hypotheses (H_3)-(H_5) with $M_g = 2$. Now the HBVP (4.1) is equivalent to the QHIE

$$x(t) = \big[f(t, x(t))\big] \left(\int_{-1}^{1} k(t, s)\big[\tanh x(s) + 1\big]\, ds\right), \quad t \in [-1, 1],$$

where k is a Green's function associated with the homogeneous BVP

$$-x''(t) = 0, \ \ t \in [-1, 1], \quad x(-1) = 0 = x(1), \tag{4.2}$$

defined by

$$k(t, s) = \begin{cases} \dfrac{(1 - t)(1 + s)}{2}, & -1 \leq s \leq t \leq 1, \\[2ex] \dfrac{(1 + t)(1 - s)}{2}, & -1 \leq t \leq s \leq 1, \end{cases} \tag{4.3}$$

which is continuous and nonnegative on $J \times J$. It can be verified that the function $u \in C(J, \mathbb{R})$ defined by $u(t) = -\int_{-1}^{1} k(t,s)\,ds$ and $v(t) = 4\int_{-1}^{1} k(t,s)\,ds$ are respectively the lower and upper solutions of the QHBVP (4.1) on $[-1,1]$. Hence, by an application of Theorem 3.1, the HBVP (4.1) has a positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = -\int_{-1}^{1} k(t,s)\,ds \;\; t \in [-1,1],$$

$$x_{n+1}(t) = \big[f(t, x_n(t))\big]\left(\int_{-1}^{1} k(t,s)\big[\tanh x_n(s) + 1\big]\,ds\right), \;\; t \in [-1,1],$$

converges monotone nondecreasingly to x^*. Similarly, the sequence $\{y_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$y_0(t) = 4\int_{-1}^{1} k(t,s)\,ds, \;\; t \in [-1,1],$$

$$y_{n+1}(t) = \big[f(t, y_n(t))\big]\left(\int_{-1}^{1} k(t,s)\big[\tanh y_n(s) + 1\big]\,ds\right), \;\; t \in [-1,1],$$

converges monotone nonincreasingly to the positive solution y^* of the QHBVP (4.1) on $[-1,1]$. .

5. The Conclusion

Finally in the conclusion, we mention that the existence and approximation results for the BVP (1.1) and (1.2) on J may also be obtained by using other iteration methods already known in the literature. In case of well-known Picard iteration method, the nonlinearity f is required to satisfy a certain so called strong Lipschitz condition whereas in our Theorem 3.1, it is not a requirement. Similarly, in case of monotone iterative technique for the BVP (1.1) and (1.2), we need to have the existence of both comparable lower as well as upper solutions along with a cumbersome comparison result for getting theoretic approximation of the solution (see [14] and references therein). However, here in the present approach of this paper we get rid of above stringent conditions and still obtain the existence of and approximation of solution in an easy straight forward way. Again, in the case of existence result via generalized iteration method developed by Heikkilä and Lakshmikantham [14] (see also Dhage and Heikkilä [12] and references therein), we also need the existence of both comparable upper as well as lower solutions together with some other conditions such as integrability of the nonlinearity f, notwithstanding it does not yield any algorithm for the solution. Furthermore, the conclusion of the upper and lower solutions method is a by-product of our monotone iteration method as mentioned in Remark 3.3. Therefore, in view of above observations, we conclude that our Dhage iteration method of this paper is an elegant, relatively better and more powerful than all the above mentioned frequently used iteration methods for nonlinear problems because it provides the additional information of algorithm along with the monotonic characterization of the convergence of the sequence of iterations to the approximate solution of the BVP (1.1) and (1.2) defined on J under weaker conditions.

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