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Dom-chromatic number of certain cycle related graphs

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Abstract
For a given χ - colouring of a graph G, a dominating set $S \subseteq V(G)$ is said to be a *dom-colouring set* if it contains atleast one vertex of each colour class of G. The *dom chromatic number* of a graph G is the minimal cardinality taken over all its dom-colouring sets and is denoted by $\gamma_{dc}(G)$. In this paper we introduce algorithms to obtain the dom-colouring and dom-chromatic number of few cycle related graphs.

Keywords
Chromatic number, Dominating set, Domination number, Dom-colouring set, Dom-chromatic number.

AMS Subject Classification
05C69, 05C15, 05C38.

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Contents

1. Introduction

Graph colouring and domination are two major areas in graph theory that have been well studied. These concepts also give rise to a number of practical applications in real life. In recent years several graph-theoretic parameters that combine the concepts of domination and colouring have been investigated by several authors. One such parameter is the concept of *dom-colouring* which was introduced by T.N. Janakiraman and M. Poobalaranjani in the year 2010[3].

A subset $D \subseteq V$ of G is called the *dominating set*, if each vertex of $V - D$, is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets of a graph G . The minimum dominating set of a graph G is called the γ*-set* of G. For a given χ - colouring of a graph G, a dominating set $S \subseteq V(G)$ is said to be a *dom-colouring set* if it contains atleast one vertex of each colour class of G. The *dom-chromatic number* of a graph G is the minimal cardinality taken over all its dom-colouring sets and is denoted by $\gamma_{dc}(G)$[2].

2. Preliminaries

Definition 2.1. *A Flower graph Fl_n is the graph obtained from a helm graph H_n by joining each pendant vertex to the central vertex of the helm[1].*

Definition 2.2. *A shell graph is defined as a cycle C_n with $n-3$ chords sharing a common end point called the apex. Shell graphs are denoted by $C(n, n-3)$[6].*

Definition 2.3. *A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. A Shell-Butterfly graph is a double shell in which each shell has any order with exactly two pendant edges at the apex[4].*

For definitions of wheel graph, crown graph, helm graph and closed helm graph one can refer to the Gallian survey on graph labeling[5].

3. Main results

3.1 Wheel graph

To facilitate our proof, we call the vertex adjacent to all vertices of the cycle in the wheel graph as the universal vertex.

Dom-colouring algorithm for Wheel graphs W_n, $n \geq 4$

Input: Wheel graph W_n.

Algorithm

Step 1: Label the vertices of the cycle C_{n-1} in the clockwise direction, namely with labels $1, 2, \ldots, n-1$.
Step 2: Label the universal vertex adjacent to all vertices of the cycle as n.
Step 3: Colour vertex n with colour a. Since this vertex is adjacent to every other vertex of the graph, colour a cannot be used in the further colouring of the graph.
Step 4: Colour the vertices of C_{n-1} in W_n using the following procedure.
Case 1: If $n-1$ is odd in C_{n-1}.
Colour the first $n-2$ vertices of C_{n-1} with colours b, c alternatively in clockwise direction. Since $n-1$ is odd, when it comes to colouring the last vertex of C_{n-1}, namely the vertex $n-1$, it will be adjacent to all the colours used so far, namely, a, b, c. Thus colour the vertex $n-1$ with colour d. See Figure 1.

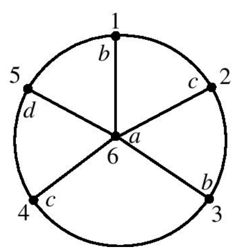

Figure 1. Dom-colouring of W_6

Case 2: If $n-1$ is even in C_{n-1}.
Colour the $n-1$ vertices of C_{n-1} with colours b, c alternatively in clockwise direction. See Figure 2.

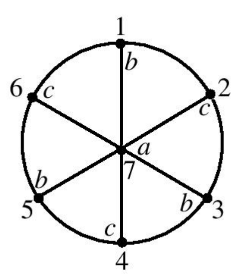

Figure 2. Dom-colouring of W_7

Output: We obtain the dom-colouring of W_n.

For any wheel graph W_n obtained by applying the above algorithm, the set $\{n\}$ is a set of minimum cardinality that dominates every other vertex of the graph. The set $\{1, 2, n\}$ (n being odd) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set. The set $\{1, 2, n-1, n\}$ (n being even) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set.

The minimum dominating and dom-colouring sets obtained for W_n is:

When n is odd

(i) Minimum dominating set of W_n is $\{n\}$.

(ii) Minimum dom-colouring set of W_n is $\{1, 2, n\}$.

(iii) $\gamma_{dc}(W_n) = 3$.

Where n is even

(i) Minimum dominating set of W_n is $\{n\}$.

(ii) Minimum dom-colouring set of W_n is $\{1, 2, n-1, n\}$.

(iii) $\gamma_{dc}(W_n) = 4$.

3.2 Crown graph

Dom-colouring algorithm for crown graphs R_n, $n \geq 3$.

Input: Crown graph R_n.

Algorithm

Step 1: Label the vertices of the cycle C_n in clockwise direction, namely as $1, 2, \ldots, n$.
Step 2: Label the pendant vertex attached to vertex 1 by $n+1$, pendant vertex attached to the vertex 2 by $n+2$, and so on. Finally label the pendant vertex attached to the vertex n by $n+n$.
Step 3: Colour the vertices of R_n using the following procedure.
Case 1: When n is odd.
Colour the first $n-1$ vertices, namely $1, 2, \ldots, n-1$ of C_n with colours a and b alternatively in clockwise direction. Then colour the last vertex of C_n namely n, with colour c, since it will be adjacent to vertices having colours a and b. While colouring the pendant vertices, colour the vertices $2n, n+1, n+2, n+3, \ldots, n+n-2$ with colours a and b alternatively in clockwise direction. Finally colour the last pendant vertex $n+n-1$ with colour c. See Figure 3.

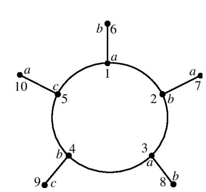

Figure 3. Dom-colouring of R_5

Case 2: When n is even.
Colour the first n vertices, namely $1,2,\dots,n$ of C_n with colours a and b alternatively in clockwise direction. While colouring the pendant vertices, colour the vertices $2n, n+1, n+2, n+3,\dots,2n-1$ with colours a and b alternatively in clockwise direction. See Figure 4.

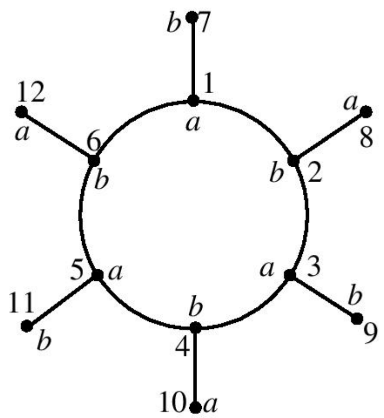

Figure 4. Dom-colouring of R_6

Output: We obtain the dom-colouring of R_n.
For any crown graph R_n obtained by applying the above algorithm, the set $\{1,2,\dots,n\}$ is a set of minimum cardinality that dominates every other vertex of the graph. Also the above set has at least one vertex from each of the colour classes of R_n. Hence it satisfies all properties required to form a dom-colouring set.

The minimum dominating and dom-colouring sets obtained for R_n is:

(i) The minimum dominating and dom-colouring sets of R_n is $\{1,2,\dots,n\}$.

(ii) $\gamma_{dc}(R_n) = n$.

3.3 Helm graph

Dom-colouring algorithm for Helm graphs H_n, $n \geq 4$.

Input: Helm graph H_n.

Algorithm

Step 1: Label the vertices of the cycle C_{n-1} in clockwise direction by $1,2,\dots,n-1$. Label the pendant vertex attached to vertex 1 by n, pendant vertex attached to vertex 2 by $n+1$, and so on. Then label the pendant vertex attached to vertex $n-1$ by $n+n-2$. Finally label the vertex attached to all vertices of the cycle C_{n-1} by $n+n-1$.
Step 2: Colour the vertices of the graph H_n as follows.
Case 1: If $n-1$ is odd in C_{n-1}.
Colour the vertices $1,2,\dots,n-2$ of the cycle C_{n-1} with colours a and b alternatively in clockwise direction. Colour the last vertex of the cycle namely $n-1$ with colour c, since it will be adjacent to vertices having colours a and b. Now colour the pendant vertex attached to the vertex $n-1$ with colour a, pendant vertex attached to 1 by colour b, pendant vertex attached to 2 by colour a, pendant vertex attached to 3 by colour b, and so on, and finally colour the pendant vertex attached to $n-3$ with colour b, that is, colour the pendant vertices $n+n-2, n, n+1, n+2, n+3,\dots,n+n-4$ with colours a and b alternatively in clockwise direction. Then colour the pendant vertex attached to vertex $n-2$, namely the vertex $n+n-3$ with colour c. Lastly colour the vertex $n+n-1$ with colour d as it will be adjacent to vertices having colours a,b and c. See Figure 5.

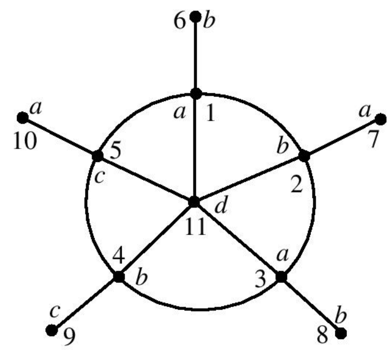

Figure 5. Dom-colouring of H_6

Case 2: If $n-1$ is even in C_{n-1}.
Colour the vertices $1,2,\dots,n-1$ of the cycle C_{n-1} with colours a and b alternatively in clockwise direction. Then colour the pendant vertices $n+n-2, n, n+1, n+2, n+3,\dots,n+n-3$ with colours a and b alternatively in clockwise direction. Finally colour the vertex $n+n-1$ with colour c as it will be adjacent to vertices of colours a and b. See Figure 6.

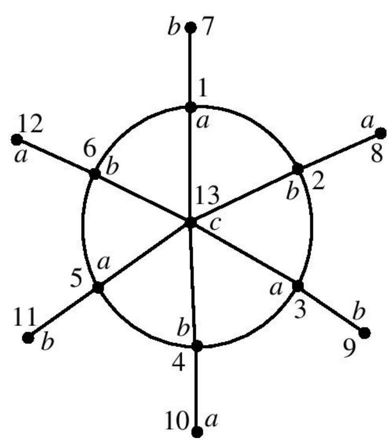

Figure 6. Dom-colouring of H_7

Output: We obtain the dom-colouring of H_n.

For any helm graph H_n obtained by applying the above algorithm, the set $\{1,2,\dots,n-1\}$ is the set of minimum cardinality that dominates every other vertex of the graph. The set $\{1,2,\dots,n\}$ is a dominating set having at least one vertex of each colour class of H_n. Also the above set is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set.

The minimum dominating and dom-colouring sets obtained for H_n is:

(i) The minimum dominating set of H_n is given by $\{1,2,\ldots,n-1\}$.

(ii) The minimum dom-colouring set of H_n is given by $\{1,2,\ldots,n-1,n+n-1\}$

(iii) $\gamma_{dc}(H_n) = n$.

3.4 Closed Helm graph

A closed helm graph $C(H_n)$ with $2n-1$ vertices, $n \geq 4$ differ from the helm graph H_n, $n \geq 4$ by an additional set of $n-1$ edges, given by $(n+i, n+i+1)$, for $0 \leq i \leq n-3$ and the edge $(n+n-2, n)$. The dom-colouring of $C(H_n)$ coincides with the dom-colouring of H_n. Moreover the results obtained for H_n can be applied to the graph $C(H_n)$ as well.

The minimum dominating set of $C(H_6)$ is $\{1,2,3,4,5\}$ and its minimum dom-colouring set is $\{1,2,3,4,5,11\}$. Thus $\gamma_{dc}(C(H_6)) = 6$. See Figure 7.

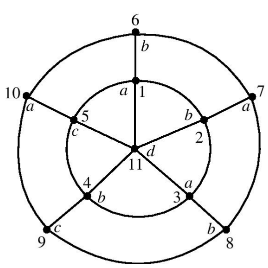

Figure 7. Dom-colouring of CH_6

The minimum dominating set of $C(H_7)$ is $\{1,2,3,4,5,6\}$ and its minimum dom-colouring set is $\{1,2,3,4,5,6,13\}$. Thus $\gamma_{dc}(C(H_7)) = 7$. See Figure 8.

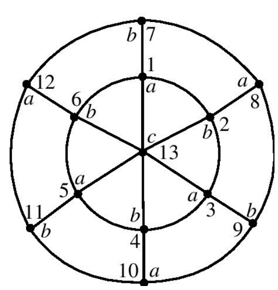

Figure 8. Dom-colouring of CH_7

3.5 Flower graph

A Flower graph is a helm graph with an additional set of $n-1$ edges connecting the pendant vertices to the central vertex and thus the dom-colouring algorithm of helm graphs applies to flower graphs as well.

Also for any flower graph Fl_n with $2n-1$ vertices, $n \geq 4$ obtained by applying the algorithm used for H_n, the set $\{2n-1\}$ is a set of minimum cardinality that dominates every other vertex of the graph. The set $\{1,2,2n-1\}$ (n being odd) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set. The set $\{1,2,n-1,2n-1\}$ (n being even) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set.

The minimum dominating and dom-colouring sets for Fl_n is given by:

(i) The minimum dominating set of Fl_n is $\{2n-1\}$.

(ii) The minimum dom-colouring set of Fl_n is

$$\begin{cases} \{1,2,2n\text{-}1\} & \text{when } n \text{ is odd.} \\ \{1,2,n\text{-}1,2n\text{-}1\} & \text{when } n \text{ is even.} \end{cases}$$

(iii)

$$\gamma_{dc}(Fl_n) = \begin{cases} 3 & \text{when } n \text{ is odd.} \\ 4 & \text{when } n \text{ is even.} \end{cases}$$

The minimum dominating set of Fl_6 is $\{11\}$ and its minimum dom-colouring set is $\{1,2,5,11\}$. Thus $\gamma_{dc}(Fl_6) = 4$. See Figure 9.

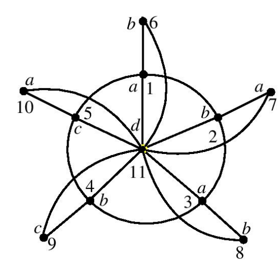

Figure 9. Dom-colouring of Fl_6

The minimum dominating set of Fl_7 is $\{13\}$ and its minimum dom-colouring set is $\{1,2,13\}$. Thus $\gamma_{dc}(Fl_7) = 3$. See Figure 10.

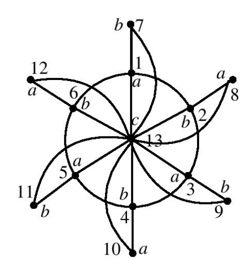

Figure 10. Dom-colouring of Fl_7

3.6 Shell graph

Dom-colouring algorithm for shell graphs $C(n,n-3)$, $n \geq 6$.
Input: Shell graph $C(n,n-3)$.
Algorithm
Step 1: Label the vertices of the cycle C_n as $1,2,\ldots,n$ in clockwise direction.
Step 2: Fix vertex 1 to be the apex of the graph and construct $n-3$ chords sharing a common vertex with the apex.
Step 3: Colour the vertices of $C(n,n-3)$ as follows:
Colour vertex 1 with colour a. Consider the successive $\lfloor \frac{n-2}{2} \rfloor$ vertices to the left and right of the apex and colour these vertices with colours b and c alternatively downwards. Finally colour the remaining vertices using the following procedure:
Case 1: When n is even.
In this case, the vertex $\frac{n}{2}+1$ will be left uncoloured. Thus colour it with colour b or c depending on whether the vertices adjacent to it are coloured c or b respectively. See Figure 11.

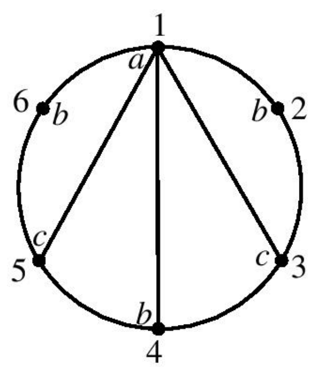

Figure 11. Dom-colouring of $C(6,3)$

Case 2: When n is odd.
In this case, the vertices $\frac{n+1}{2}+1$ and $\frac{n+1}{2}$ will be left uncoloured and thus colour these vertices with colours b and d or c and d depending on whether the vertices adjacent to them are coloured c or b respectively, such that the vertex $\frac{n+1}{2}$ has colour d. See Figure 12.

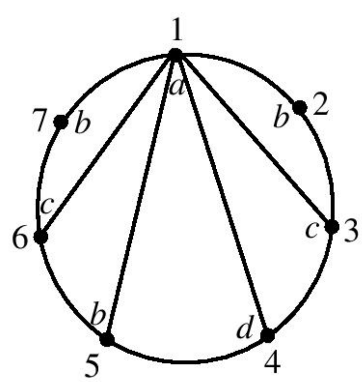

Figure 12. Dom-colouring of $C(7,4)$

Output: We obtain the dom-colouring of $C(n,n-3)$.
For any shell graph $C(n,n-3)$ obtained by applying the above algorithm, the set $\{1\}$ is a set of minimum cardinality that dominates every other vertex of the graph. The set $\{1,2,3,\frac{n+1}{2}\}$ (n being odd) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set. The set $\{1,2,3\}$ (n being even) is a dominating set having exactly a vertex from each of its colour classes. Also it is the set having minimum cardinality that satisfies the properties required to form a dom-colouring set.
The minimum dominating and dom-colouring sets obtained for $C(n,n-3)$ is:

(i) The minimum dominating set of $C(n,n-3)$ is $\{1\}$.

(ii) The minimum dom-colouring set of $C(n,n-3)$ is

$$\begin{cases} \{1,2,3\} & \text{when } n \text{ is even.} \\ \{1,2,3,\frac{n+1}{2}\} & \text{when } n \text{ is odd.} \end{cases}$$

(iii) $\gamma_{dc}(C(n,n-3)) =$

$$\begin{cases} 3 & \text{when } n \text{ is even.} \\ 4 & \text{when } n \text{ is odd.} \end{cases}$$

3.7 Shell-Butterfly graph

Dom-colouring algorithm for Shell-Butterfly graphs
Input: Shell-Butterfly graph with shell orders m and l where $m,n \geq 3$.
Algorithm
Step 1: Label the vertices of the shell of order m with labels $1,2,\ldots,m$ upwards.
Step 2: Similarly label the vertices of the shell of order l with labels $m+1,m+2,\ldots,m+l$ downwards.
Step3: Label the vertices of the pendant edges as $m+l+1,m+l+2$. Finally label the central apex as $m+l+3$.
Step 4: Colour the vertices of the Shell-Butterfly graph as follows:
Colour the vertex $m+l+3$ with colour a. Colour the vertices $m+l+1,m+l+2$ with colour b. Colours the vertices $1,2,\ldots,m$ with colours c and b alternatively. Similarly colour the vertices $m+1,m+2,\ldots,m+l$ with colours b and c alternatively. See Figure 13.

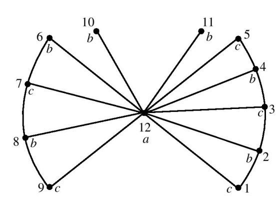

Figure 13. Dom-colouring of Shell-Butterfly graph with shell orders 5 and 4

Output: We obtain the dom-colouring of the Shell-Butterfly graph.
For any shell butterfly graph obtained by applying the above algorithm, the set $\{m+l+3\}$ is a set of minimum cardinalty that dominates every other vertex of the graph. The set $\{1,2,m+l+3\}$ is a dominating set and has at least one vertex of each of its colour classes. Also the above set is a set having minimum cardinalty that satisfies the properties required to form a dom-colouring set.
The minimum dominating and dom-colouring sets obtained for all Shell-Butterfly graphs is:

(i) It's minimum dominating set is $\{m+l+3\}$.

(ii) It's minimum dom-colouring set is $\{1,2,m+l+3\}$.

(iii) The dom-chromatic number γ_{dc} of all shell-butterfly graphs is 3.

References

[1] R. Arundhadhi and V. Ilayarani, Total Colouring of Closed Helm, Flower and Bistar graph family, *International Journal of Scientific and Research Publications*, 7(7)(2017), 616–621.

[2] B. Chaluvaraju and C. AppajiGowda, Dom-chromatic number of a graph, *Malaya Journal of Mathematik*, 4(1)(2016), 1–7.

[3] T.N Janakiraman and M. Poobalaranjani, Dom-chromatic sets in Bi-partite graphs, International Journal of Engineering Science, *Advanced Computing and Bio-Technology*, 1(2)(2010), 80–95.

[4] J. Jeba Jesintha and K. Ezhilarasi Hilda, Shell-Butterfly graphs are Graceful, *International Journal of Pure and Applied Mathematics*, 101(6)(2015), 949–956.

[5] Joseph A. Gallian, A Dynamic survey of Graph Labeling, *The Electronic Journal of Combinatorics*, 21, 2018.

[6] S. Meena, M. Renugha and M. Sivasakthi, Cordial labeling for different types of shell graphs, *International Journal of Scientific and Engineering Research*, 6(9)(2015), 1282–1288.

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